APMA 0350 – MIDTERM 1 – SOLUTIONS

1. **Integrating Factors**

STEP 1: Standard Form

$$y' - \left(\frac{2}{t}\right) y = t^3 \cos(t)$$

STEP 2: Integrating Factor

$$e^{\int -\frac{2}{t} dt} = e^{-2\ln(t)} = \left(e^{\ln(t)}\right)^{-2} = t^{-2}$$

Here we used $t > 0$ so $|t| = t$

STEP 3: Multiply by t^{-2}

$$\begin{aligned}
t^{-2}y' - t^{-2}\left(\frac{2}{t}\right) y =& t^{-2}t^3 \cos(t) \\
\left(t^{-2}y\right)' =& t\cos(t) \\
t^{-2}y = \int t\cos(t)dt = t\sin(t) - \int \sin(t)dt =& t\sin(t) + \cos(t) + C \\
y =& t^2\left(t\sin(t) + \cos(t) + C\right) \\
y =& t^3\sin(t) + t^2\cos(t) + Ct^2
\end{aligned}$$

STEP 4: Initial Condition

$$\begin{aligned}
y(\pi) =& 0 \\
\pi^3\sin(\pi) + \pi^2\cos(\pi) + C\pi^2 =& 0 \\
0 - \pi^2 + C\pi^2 =& 0 \\
C =& 1
\end{aligned}$$

STEP 5: Solution

$$y = t^3\sin(t) + t^2\cos(t) + t^2$$

2. Separation of variables

STEP 1:

$$\begin{aligned}\frac{dy}{dt} &= \frac{\sin^{-1}(t)}{y\sqrt{1-t^2}} \\ ydy &= \frac{\sin^{-1}(t)}{\sqrt{1-t^2}}dt \\ \int ydy &= \int \frac{\sin^{-1}(t)}{\sqrt{1-t^2}}dt\end{aligned}$$

For the integral on the right, we use $u = \sin^{-1}(t)$ so $du = \frac{1}{\sqrt{1-t^2}}dt$

$$\begin{aligned}\int ydy &= \int udu \\ \frac{1}{2}y^2 &= \frac{1}{2}u^2 + C = \frac{1}{2}\left(\sin^{-1}(t)\right)^2 + C \\ y^2 &= \left(\sin^{-1}(t)\right)^2 + \underbrace{2C}_{C} \\ y &= \pm\sqrt{\left(\sin^{-1}(t)\right)^2 + C}\end{aligned}$$

But because of the initial condition $y(0) = -2$ we have to choose

$$y = -\sqrt{\left(\sin^{-1}(t)\right)^2 + C}$$

STEP 2:

$$\begin{aligned}y(0) &= -2 \\ -\sqrt{\left(\sin^{-1}(0)\right)^2 + C} &= -2 \\ \sqrt{C} &= 2 \\ C &= 4\end{aligned}$$

STEP 3: **Solution**

$$y = -\sqrt{\left(\sin^{-1}(t)\right)^2 + 4}$$

3. Exact Equations

STEP 1: Multiply the ODE by ye^x

$$ye^x\left(\frac{\sin(y)}{y}-2e^{-x}\sin(x)\right)dx+ye^x\left(\frac{\cos(y)+2e^{-x}\cos(x)}{y}\right)dy=0$$
$$\left(e^x\sin(y)-2y\sin(x)\right)dx+\left(e^x\cos(y)+2\cos(x)\right)dy=0$$

Hence $P=e^x\sin(y)-2y\sin(x)$ and $Q=e^x\cos(y)+2\cos(x)$

STEP 2: Check exact

$$P_y=\left(e^x\sin(y)-2y\sin(x)\right)_y=e^x\cos(y)-2\sin(x)$$
$$Q_x=\left(e^x\cos(y)+2\cos(x)\right)_x=e^x\cos(y)-2\sin(x)\checkmark$$

STEP 3:

$$f_x=e^x\sin(y)-2y\sin(x)\Rightarrow f=\int e^x\sin(y)-2y\sin(x)dx=e^x\sin(y)+2y\cos(x)+g(y)$$
$$f_y=e^x\cos(y)+2\cos(x)\Rightarrow f=\int e^x\cos(y)+2\cos(x)dy=e^x\sin(y)+2y\sin(x)+h(x)$$

Comparing, we get

$$f(x,y)=e^x\sin(y)+2y\cos(x)$$

STEP 4: General Solution:

$$e^x\sin(y)+2y\cos(x)=C$$

STEP 5: Initial Condition

$$\begin{aligned} y(0)&=\pi\\ e^0\sin(\pi)+2\pi\cos(0)&=C\\ 2\pi&=C\\ C&=2\pi \end{aligned}$$

$$e^x\sin(y)+2y\cos(x)=2\pi$$

4.

Note:

$$r^2 - 9r + 20 = (r-4)(r-5) = 0 \Rightarrow r = 4 \text{ or } r = 5$$

$$r^2 - 4r + 13 = 0 \Rightarrow (r-2)^2 - 4 + 13 = 0 \Rightarrow (r-2)^2 + 9 = 0 \Rightarrow r = 2 \pm 3i$$

Therefore the roots of the auxiliary equation are

$$\begin{aligned}
r =& -2 \\
r =& 0 \text{ (repeated 3 times)} \\
r =& 1 \text{ (repeated 4 times)} \\
r =& 2 \\
r =& 4 \text{ (repeated twice)} \\
r =& 5 \text{ (repeated twice)} \\
r =& \pm 3i \\
r =& 2 \pm 3i \text{ (repeated twice)}
\end{aligned}$$

And so the general solution is

$$\begin{aligned}
y =& Ae^{-2t} \\
&+ B + Ct + Dt^2 \\
&+ Ee^t + Fte^t + Gt^2e^t + Ht^3e^t \\
&+ Ie^{2t} \\
&+ Je^{4t} + Kte^{4t} \\
&+ Le^{5t} + Mte^{5t} \\
&+ N\cos(3t) + O\sin(3t) \\
&+ Pe^{2t}\cos(3t) + Qe^{2t}\sin(3t) + Rte^{2t}\cos(3t) + Ste^{2t}\sin(3t)
\end{aligned}$$

5. **STEP 1:** Differential Equation: For some constant k we have

$$\begin{cases} h'(t) = & \dfrac{k}{(h(t))^2} \\ h(0) = & 1 \end{cases}$$

STEP 2: Separation of variables:

$$\begin{aligned} \frac{dh}{dt} =& \frac{k}{h^2} \\ h^2\,dh =& kdt \\ \int h^2\,dh =& \int kdt \\ \frac{1}{3}h^3 =& kt + C \\ h^3 =& 3kt + \underbrace{3C}_{C} \\ h^3 =& 3kt + C \\ h(t) =& \sqrt[3]{3kt + C} \end{aligned}$$

Note: Strictly speaking you are not allowed to change the $3k$ to k because it already appears in the differential equation, but for the purposes of this problem it would be ok to do so

STEP 3: Initial Conditions

$$\begin{aligned} h(0) =& 1 \\ \sqrt[3]{3k0 + C} =& 1 \\ \sqrt[3]{C} =& 1 \\ C =& 1^3 = 1 \end{aligned}$$

$$h(t) = \sqrt[3]{3kt + 1}$$

STEP 3: Solve for k

$$\begin{aligned}h(1) =& 4\\ \sqrt[3]{3k(1)+1} =& 4\\ \sqrt[3]{3k+1} =& 4\\ 3k+1 =& 4^3 = 64\\ 3k =& 63\\ k =& 21\end{aligned}$$

$$h(t) = \sqrt[3]{3(21)t+1} = \sqrt[3]{63t+1}$$

STEP 4: Answer: $t = 20$ seconds corresponds to $t = \frac{1}{3}$ hours

$$h\left(\frac{1}{3}\right) = \sqrt[3]{63\left(\frac{1}{3}\right)+1} = \sqrt[3]{21+1} = \sqrt[3]{22} \text{ m}$$