APMA 1941G Homework 4 Solutions
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Problem 1

Suppose ϕ has a global max at 0 with $\phi(0) = 0, \phi'(0) = 0, \phi''(0) = -1$, and $\phi'''(0) = 0$. Let $a(x) \equiv 1$ and consider

$$I[\epsilon] = \int_{\mathbb{R}} e^{\frac{\phi(x)}{\epsilon}} dx.$$

Show that

$$I[\epsilon] \sim \sqrt{2\pi\epsilon} + \frac{\sqrt{2\pi}}{8}\phi''''(0)\epsilon^{3/2} + o(\epsilon^{3/2}),$$

assuming that

$$(L_0(a))(0) = a(0) = \sqrt{\frac{2\pi}{|\phi''(0)|}} = \sqrt{2\pi}$$

and that

$$C_2 = \int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}.$$

Proof. Since the desired asymptotic expansion features an $o(\epsilon^{3/2})$ term, per the Laplace Method, we only need to calculate through $k = 2$, so we only need to calculate $(L_2 a)(0)$ and $(L_4 a)(0)$ (our indexing is by $2k$). Then, as discussed in class,

$$I[\epsilon] \sim L_0(a)(0)\sqrt{\epsilon} + L_2(a)(0)\epsilon + L_4(a)(0)\epsilon^{3/2} + o(\epsilon^{3/2}) \tag{1}$$

where

$$L_2(a)(0) = (\tilde{a})'(0)C_1 \text{ and } L_4(a)(0) = \frac{(\tilde{a})''(0)}{2}C_2$$

and, as in class (using the same notation),

$$\tilde{a}(y) = a(\psi(y))\eta(\psi(y))\psi'(y).$$

Since $a(x) \equiv 1$, we can simplify

$$\tilde{a}(y) = \eta(\psi(y))\psi'(y).$$

The first derivative of $\tilde{a}$ is given by

$$\tilde{a}'(y) = \eta'(\psi(y))(\psi'(y))^2 + \psi''(y)\eta(\psi(y)) \tag{2}$$

and the second derivative is given by

$$\tilde{a}''(y) = \eta'(\psi(y))2\psi'(y)\psi''(y) + (\psi'(y))^2\eta''(\psi(y))\psi'(y) + \psi''(y)\eta'(\psi(y))\psi'(y) + \eta(\psi(y))\psi'''(y). \tag{3}$$

Recall that we derived in class that $\psi(0) = 0$, $\eta(0) = 1$, and $\psi'(0) = \frac{1}{\sqrt{|\phi''(0)|}}$. Since by assumption, $\phi''(0) = -1$, then $\psi'(0) = 1$. Similar to what we did in class, we determine $\psi''(y)$ by further differentiating $\phi'(\psi(y))\psi'(y) = -y$. Differentiating this one time, as in class, we obtain.

$$\phi''(\psi(y))(\psi'(y))^2 + \phi'(\psi(y))\psi''(y) = -1.$$

Now differentiating this once further, we obtain

$$\phi'''(\psi(y))(\psi'(y))^3 + 2\psi'(y)\psi''(y)\phi''(\psi(y)) + \phi'(\psi(y))\psi'''(y) + \psi''(y)\phi''(\psi(y))\psi'(y) = 0. \tag{4}$$

Evaluating at $y = 0$ and noting that $\psi(0) = 0$, $\phi''(0) = -1$, and $\phi'''(0) = 0$, we obtain the equation

$$0 + 2 \cdot 1 \cdot \psi''(0) \cdot -1 + 0 + \psi''(0) \cdot -1 \cdot 1 = 0.$$

We may then conclude that $\psi''(0) = 0$. The only other information we need in order to evaluate $\tilde{a}'(0)$, per (2), is about $\eta'(\psi(0)) = \eta'(0)$. Yet, since $\eta = 1$ in $(-\delta, \delta)$, we know that $\eta'(0) = \eta''(0) = 0$. We can then evaluate

$$\tilde{a}'(0) = \eta'(0) + 0 = 0.$$

With this computed, we turn to $\tilde{a}''(0)$. From (3), what we have already computed, and our assumptions:

$$\tilde{a}''(0) = 0 + \eta''(0) + \psi'''(0) = \psi'''(0).$$

We then need to find $\psi'''(0)$. We can further differentiate (4) in order to find $\psi'''(0)$. In the following, ellipses signify terms that we can immediately see will be zero when evaluated at zero (for the sake of brevity):

$$\begin{aligned}\phi''''(\psi(y))(\psi'(y))^4 + 3(\psi'(y))^2\psi''(y)\phi'''(\psi(y)) + 2\psi''(y)... + 2\psi'(y)\phi''(\psi(y))\psi'''(y)\\ +\phi'(\psi(y))... + \psi'''(y)\cdot\phi''(\psi(y))\cdot\psi'(y) + \psi''(y)... + \phi''(\psi(y))\psi'(y)\psi'''(y) = 0.\end{aligned}$$

At zero, since $\psi(0) = 0$, $\psi'(0) = 1$, and $\psi''(0) = 0$, we obtain

$$\phi'''(0) + 0 + 0 + 2\cdot -1\cdot\psi'''(y) + 0 + \psi'''(y)\cdot -1\cdot 1 + 0 - \psi'''(y) = 0$$

so

$$\psi'''(0) = \frac{\phi''''(0)}{4}.$$

We then have that

$$\tilde{a}''(0) = \frac{\phi''''(0)}{4}.$$

Using our calculations for $\tilde{a}'(0)$ and $\tilde{a}''(0)$, we may now compute that

$$L_2(a)(0) = 0 \text{ and } L_4(a)(0) = \frac{\phi''''(0)}{4\cdot 2}C_2 = \frac{\sqrt{2\pi}}{8}\phi''''(0).$$

Hence, substituting into (1), we obtain

$$I[\epsilon] \sim \sqrt{2\pi\epsilon} + \frac{\sqrt{2\pi}}{8}\phi''''(0)\epsilon^{3/2} + o(\epsilon^{3/2})$$

as claimed.

Problem 2

Suppose that $H = H(x)$ is a double-well potential function, that is, a smooth, even function, with a local minimum at $x = \pm 1$ with $H(\pm 1) = 0$ and a local maximum at $x = 0$ with $H(0) = 1$. Laplace's method says that if ϕ has a global maximum at $x = 1$, and for any smooth a, then

$$I[\epsilon] = \int_0^\infty a(x)e^{\frac{\phi(x)}{\epsilon}}dx \sim a(1)e^{\frac{\phi(1)}{\epsilon}}\sqrt{\frac{2\pi\epsilon}{|\phi''(1)|}} + o(\sqrt{\epsilon}). \quad (5)$$

(a)

Let Z^ϵ be a 'normalizing' constant such that

$$Z^\epsilon \int_{\mathbb{R}} e^{-\frac{H(x)}{\epsilon}}dx = 1. \quad (6)$$

Use Laplace's method to show that

$$\frac{1}{Z^\epsilon} \sim \frac{2\sqrt{2\pi\epsilon}}{\sqrt{H''(1)}} + o(\sqrt{\epsilon})$$

For the sake of simplicity, we hereafter ignore the remainder term and assume that

$$\frac{1}{Z^\epsilon} = \frac{2\sqrt{2\pi\epsilon}}{\sqrt{H''(1)}}$$

Proof. First, by the symmetry of H about zero, (6) may equivalently be written

$$Z^\epsilon \cdot 2\int_0^\infty e^{-\frac{H(x)}{\epsilon}} = 1.$$

We can re-arrange this to obtain

$$\frac{1}{Z^\epsilon} = 2\int_0^\infty e^{-\frac{H(x)}{\epsilon}}dx.$$

We see that we can apply (5), with $a(x) \equiv 1$ and $\phi(x) = -H(x)$. We can do this because as we can see from the schematic diagram given in the problem set as well the description of H, the local minimum at 1 is actually a global minimum, so by assigning $\phi(x) = -H(x)$, we obtain a ϕ that has a global maximum at 1 as required (also at -1, since these have the same value). Accordingly, we compute the right-hand side of (5). We have $a(1) = 1$. By assumption, $\phi(1) = -H(1) = 0$. We immediately obtain

$$\frac{1}{Z^\epsilon} \sim \frac{2\sqrt{2\pi\epsilon}}{\sqrt{H''(1)}} + o(\sqrt{\epsilon})$$

as claimed.

(b)

Let $\sigma^\epsilon(x) = Z^\epsilon e^{-\frac{H(x)}{\epsilon}}$ and $\tau^\epsilon = \frac{1}{\epsilon}e^{-\frac{1}{\epsilon}}$ and let $\delta = \delta(\epsilon)$ be chosen such that $\delta \to 0$ and $\frac{\delta}{\sqrt{\epsilon}} \to \infty$ as $\epsilon \to 0$. Use a second-order Taylor expansion of H around 0 (ignore the higher-order terms) as well as the fact that $\int_{\mathbb{R}} e^{-\frac{x^2}{2}}dx = \sqrt{2\pi}$ to show that, as $\epsilon \to 0$

$$\lim_{\epsilon \to 0}\int_{-\delta}^{\delta} \frac{\tau^\epsilon}{\sigma^\epsilon(x)}dx = \frac{2}{\kappa} \quad (7)$$

where $\kappa = \frac{\sqrt{|H''(0)|H''(1)}}{2\pi}$.

Proof. We want to show that

$$\lim_{\epsilon \to 0}\int_{-\delta}^{\delta} \frac{1}{\epsilon}\frac{e^{-\frac{1}{\epsilon}}}{Z^\epsilon e^{-\frac{H(x)}{\epsilon}}}dx = \frac{2 \cdot 2\pi}{\sqrt{|H''(0)|H''(1)}},$$

or, written in more compact notation,

$$\lim_{\epsilon\to 0}\int_{-\delta}^{\delta}\frac{1}{\epsilon Z^\epsilon}e^{-\frac{1-H(x)}{\epsilon}}dx = \frac{2\cdot 2\pi}{\sqrt{|H''(0)|H''(1)}}.$$

Using a second-order Taylor expansion, $H(x)\approx H(0)+H'(0)x+\frac{H''(0)}{2}x^2$. Plugging this expansion and what we found in 2(a) into the left-hand side of (7), we have

$$\lim_{\epsilon\to 0}\int_{-\delta}^{\delta}\frac{1}{\epsilon}\frac{2\sqrt{2\pi\epsilon}}{\sqrt{H''(1)}}e^{-\frac{1-(H(0)+H'(0)x+\frac{1}{2}H''(0)x^2)}{\epsilon}}dx.$$

Pulling out the constants and recall we have a local max at 0 so actually $H'(0)=0$, and by assumption $H(0)=1$, this then becomes

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\frac{2\sqrt{2\pi\epsilon}}{\sqrt{H''(1)}}\int_{-\delta}^{\delta}e^{-\frac{\frac{1}{2}H''(0)x^2}{\epsilon}}dx$$

Now we want to evaluate the integral

$$\lim_{\epsilon\to 0}\frac{2\sqrt{2\pi}}{\sqrt{H''(1)\epsilon}}\int_{-\delta}^{\delta}e^{-\frac{H''(0)}{\epsilon}\frac{1}{2}x^2}dx.$$

Let $u=\frac{\sqrt{|H''0|}}{\sqrt{\epsilon}}x$ so that $u^2=\frac{H''(0)}{\epsilon}x^2$. Then $\frac{\sqrt{\epsilon}}{\sqrt{|H''0|}}du=dx$, so we have after using u-substitution

$$\lim_{\epsilon\to 0}\frac{2\sqrt{2\pi}}{\sqrt{H''(1)\epsilon}}\left[\sqrt{2\pi}\frac{\sqrt{\epsilon}}{\sqrt{|H''0|}}\right]_{x=-\delta}^{x=\delta}.$$

Notice that when we evaluate the integral, u appears nowhere, so it does not matter that we did not convert the bounds when we did u-substitution, and we have arrived at

$$\lim_{\epsilon\to 0}\int_{-\delta}^{\delta}\frac{\tau^\epsilon}{\sigma^\epsilon(x)}dx = \lim_{\epsilon\to 0}\frac{2\sqrt{2\pi}}{\sqrt{H''(1)\epsilon}}\sqrt{2\pi}\frac{\sqrt{\epsilon}}{\sqrt{|H''0|}}.$$

Simplifying the right hand side, we have

$$\lim_{\epsilon\to 0}\frac{2\cdot 2\pi}{\sqrt{H''(1)|H''(0)|}}.$$

Evaluating this limit, since ϵ actually appears nowhere, we obtain

$$\lim_{\epsilon\to 0}\int_{-\delta}^{\delta}\frac{\tau^\epsilon}{\sigma^\epsilon(x)}dx = \lim_{\epsilon\to 0}\frac{2\cdot 2\pi}{\sqrt{H''(1)|H''(0)|}} = \frac{2\cdot 2\pi}{\sqrt{H''(1)|H''(0)|}} = \frac{2}{\kappa},$$

as claimed.