APMA 1941G Homework 5 Solutions
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Problem 1

This problem refers to Example 1, "Rapidly Oscillating Coefficients."

(a)

In lecture, we defined

$$\overline{a} = \int_0^1 a + 2w'a + wa'dy \tag{1}$$

where $w = w(y)$ is the solution of

$$\begin{cases} -(aw')' &= a' \\ w(0) &= w(1) \end{cases} \tag{2}$$

Show that

$$\overline{a} = \left(\int_0^1 \frac{1}{a(y)} dy \right)^{-1}. \tag{3}$$

Proof. Following the provided hint, we first solve (2) by anti-differentiating and by assuming that $w'(0) = 0$ without loss of generality. We have:

$$-(aw') = a + C$$

Evaluating this at zero,

$$-(a(0)w'(0)) = a(0) + C.$$

Notice that left-hand side is zero by our assumption $w'(0) = 0$. Then $C = -a(0)$. We then obtain the identity

$$aw' = -a + a(0). \tag{4}$$

Splitting the integral in (1) and integrating that involving wa' by parts, we obtain

$$\overline{a} = \int_0^1 a + 2w'ady + \left(wa|_0^1 - \int_0^1 w'ady \right).$$

Notice that the boundary terms are

$$wa|_0^1 = w(1)a(1) - w(0)a(0) = w(1)(a(1) - a(0)) = 0.$$

Here, we first used the fact that $w(0) = w(1) = 0$, and secondly the fact that a is 1-periodic. We then have that

$$\overline{a} = \int_0^1 a + w'ady.$$

Substituting in (4), then

$$\overline{a} = \int_0^1 a(0)dy.$$

Now we will find what $a(0)$ is in order to get the desired form. Due to (4),

$$a(0) = a\frac{dw}{dy} + a.$$

Using separation of variables, we then obtain

$$\int_0^1 \frac{a(0)}{a} - 1dy = w(1) - w(0) = 0.$$

Re-arranging,

$$a(0)\int_0^1 \frac{1}{a(y)}dy = 1.$$

We may then conclude that

$$a(0) = \left(\int_0^1 \frac{1}{a(y)}dy\right)^{-1}.$$

Then since

$$\overline{a} = \int_0^1 a(0)dy = 1 \cdot a(0) = \left(\int_0^1 \frac{1}{a(y)}dy\right)^{-1},$$

we have shown the desired result.

(1b)

Using the formula for $\overline{a}$ from (a), find the general solution of

$$-\overline{a}u^0_{xx} = f(x).$$

Solution. Since $\overline{a}$ is a (nonzero) constant, we can integrate with respect to x to find that

$$u^0_x = -\frac{1}{\overline{a}}\int f(x)dx.$$

Integrating in x again,

$$u^0 = -\frac{1}{\overline{a}}\int\left(\int f(x)dx\right)dx.$$

This shows how we can obtain an explicit solution for u^0 for any given f. We can also write this in a different form. If we let F be such that $\frac{dF}{dx} = f$, then we could instead write

$$u^0_x = -\frac{1}{\overline{a}}(F + C)$$

where $C \in \mathbb{R}$. Then we could write the explicit solution for u in the possibly better form

$$u^0 = -\frac{1}{\overline{a}}(G(x) + Cx + B)$$

where $\frac{dG}{dx} = F(x)$ and $B \in \mathbb{R}$.

Problem 2

(a)

Using undetermined coefficients, find a particular solution of

$$w''(t) + w(t) = -\frac{1}{4}\cos(3t).$$

Solution. First note that a solution to the homogeneous problem will take the form $A\cos(t)+B\sin(t)$, whereas the non-homogeneous term is of the form $C\cos(3t)$, hence the frequencies are not the same and we do not have to worry about the issue of resonance ($A, B, C \in \mathbb{R}$). To justify our claim for the form of the solution to the homogeneous problem, we can make the ansatz $w = e^{\lambda t}$, then see that upon substituting into the homegeneous equation we obtain

$$\lambda^2 e^{\lambda t} + e^{\lambda t} = 0.$$

We can see that the characteristic polynomial $\lambda^2+1 = 0$ has roots $\pm i$, so we obtain the solutions $w_1 = c_1 e^{-it}$ and $w_2 = c_2 e^{it}$. Using Euler's identity and the superposition principle yields a homogeneous solution of

$$w_h = c_1\cos(t) - ic_1\sin(t) + c_2\cos(t) + ic_2\sin(t)$$

so

$$w_h(t) = (c_1 + c_2)\cos(t) + (ic_2 - ic_1)\sin(t)$$

Letting c_1 and c_2 be complex conjugates of each other, we can get rid of the i's so that we can re-express this as

$$w_h(t) = A\cos(t) + B\sin(t)$$

for some $A, B \in \mathbb{R}$.

With that justified, we find the particular solution. Using the method of undetermined coefficients, we make the ansatz

$$w_p(t) = A\cos(3t) + B\sin(3t).$$

In order to substitute this into the homogeneous equation, we first compute

$$w_p'(t) = -3A\sin(3t) + 3B\cos(3t)$$

and

$$w_p''(t) = -9A\cos(3t) - 9B\sin(3t).$$

Substituting, we then have

$$-9A\cos(3t) - 9B\sin(3t) + A\cos(3t) + B\sin(3t) = -\frac{1}{4}\cos(3t).$$

Hence

$$-9A + A = -\frac{1}{4}$$

so $A = 1/32$ and $B = 0$. Our particular solution is overall

$$\boxed{w_p(t) = \frac{1}{32}\cos(3t).}$$

(b)

Using undetermined coefficients, find a particular solution of

$$w''(t) + w(t) = -\frac{3}{4}\cos(t).$$

Solution. Since in this case the frequencies in the homogeneous solution and the non-homogeneous term are the same, we need to multiply the form of the solution to the homogeneous problem by t. We then have the ansatz

$$w_p = t(A\cos(t) + B\sin(t)).$$

Then

$$w_p' = -At\sin(t) + A\cos(t) + Bt\cos(t) + B\sin(t)$$

and

$$\begin{aligned} w_p'' &= -At\cos(t) - A\sin(t) - A\sin(t) - Bt\sin(t) - B\cos(t) + B\cos(t) \\ &= -At\cos(t) - Bt\sin(t) - 2A\sin(t) + 2B\cos(t) \end{aligned}$$

Substituting our ansatz in,

$$-At\cos(t) - Bt\sin(t) - 2A\sin(t) + 2B\cos(t) + At\cos(t) + Bt\sin(t) = -\frac{3}{4}\cos(t)$$

Matching the coefficients,

$$-At + 2B + At = -\frac{3}{4}$$

and

$$-Bt - 2A + Bt = 0.$$

Then $A = 0$ and $B = -\frac{3}{8}$. We thus obtain

$$\boxed{w_p = -\frac{3t}{8}\sin(t).}$$

(c)

Use $-\cos^3(t) = -\frac{3}{4}\cos(t) - \frac{1}{4}\cos(3t)$ to find the general solution

$$u_1''(t) + u_1(t) = -\cos^3(t).$$

Solution. From the superposition principle and our answers to the previous parts, the homogeneous solution is

$$u_1^h(t) = C_1 \cos(t) + C_2 \sin(t)$$

and our particular solution is

$$u_1^p(t) = \frac{1}{32}\cos(3t) - \frac{3t}{8}\sin(t),$$

by making use of the given identity. We overall have the general solution

$$\boxed{u_1(t) = C_1\cos(t) + C_2\sin(t) + \frac{1}{32}\cos(3t) - \frac{3t}{8}\sin(t)}.$$

(d)

Find the solution of (c) that satisfies $u_1(0) = 0$ and $u_1'(0) = 0$.

Solution. First,

$$\begin{aligned} u_1(0) &= C_1\cos(0) + C_2\sin(0) + \frac{1}{32}\cos(3\cdot 0) - \frac{3\cdot 0}{8}\sin(0) \\ &= C_1 + \frac{1}{32} \end{aligned}$$

Thus $C_1 = -1/32$. Next, we take the derivative of u_1 and find

$$u_1'(t) = \frac{1}{32}\sin(t) + C_2\cos(t) - \frac{3}{32}\sin(3t) - \frac{3t}{8}\cos(t) - \frac{3}{8}\sin(t)$$

Now we substitute in zero:

$$\begin{aligned} u_1'(0) &= -\frac{11}{32}\sin(0) + C_2\cos(0) - \frac{3}{32}\sin(3\cdot 0) - \frac{3\cdot 0}{8}\cos(0) \\ &= C_2 \end{aligned}$$

Thus $C_2 = 0$. Overall, we then have

$$\boxed{u_1(t) = -\frac{1}{32}\cos(t) + \frac{1}{32}\cos(3t) - \frac{3t}{8}\sin(t)}$$

as the solution to the problem with these initial conditions.