APMA 1941G Homework 6 Solutions
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Problem 1

Consider the equation

$$\begin{cases} u_\epsilon'' + \epsilon u_\epsilon' + u_\epsilon &= 0 \\ u_\epsilon(0) = 1,\ u_\epsilon'(0) &= 1 \end{cases} \tag{1}$$

Apply the ansatz

$$u_\epsilon(t) = u_0(t, \epsilon t) + \epsilon u_1(t, \epsilon t) + ...$$

Here $u_\epsilon = u_\epsilon(t)$ and $u_k = u_k(t, \tau)$ (where $\tau = \epsilon t$).

(a)

Show that $u^0(t, \tau) = A(\tau)\cos(t) + B(\tau)\sin(t)$.

Proof. First, we evaluate

$$u_\epsilon'(t) = u_t^0 + \epsilon u_\tau^0 + \epsilon u_t^1 + \epsilon^2 u_\tau^1$$
$$u_\epsilon''(t) = u_{tt}^0 + 2\epsilon u_{t\tau}^0 + \epsilon^2 u_{\tau\tau}^0 + \epsilon u_{tt}^1 + 2\epsilon^2 u_{t\tau}^2 + \epsilon^3 u_{\tau\tau}^1$$

Plugging our ansatz into (1), we obtain

$$u_{tt}^0 + 2\epsilon u_{t\tau}^0 + \epsilon^2 u_{\tau\tau}^0 + \epsilon u_{tt}^1 + 2\epsilon^2 u_{t\tau}^2 + \epsilon^3 u_{\tau\tau}^1 + \epsilon u_t^0 \epsilon^2 u_\tau^0 + \epsilon^2 u_t^1 + \epsilon^3 u_\tau^1 + ... = 0 \tag{2}$$

Comparing the $O(1)$ terms, we obtain

$$u_{tt}^0(t, \epsilon t) + u^0(t, \epsilon t) = 0$$

Solving this differential equation in t, exactly as in the previous homework, we obtain

$$u^0(t, \tau) = A(\tau)\cos(t) + B(\tau)\sin(t). \tag{3}$$

(b)

Find an ODE for A and B and solve it, and use that to solve for $u_0(t) = u_0(t, \epsilon t)$. Impose the conditions $u_0(0) = 1$ and $u_0'(0) = 1$. *Hint.* Select A and B to kill the resonance terms $\cos(t)$ and $\sin(t)$, just like we did for Duffing's equation.

Solution. To find the ODE for A and B, we make use of the $O(\epsilon)$ terms. We obtain

$$2\epsilon u_{t\tau}^0 + \epsilon u_{tt}^1 + \epsilon u_t^0 + \epsilon u^1 = 0. \tag{4}$$

Recalling (3), we evaluate

$$u_t^0 = -A(\tau)\sin t + B(\tau)\cos t$$
$$u_{t\tau}^0 = -A'(\tau)\sin t + B'(\tau)\sin t$$

Plugging this into (4), we obtain

$$u_{tt}^1 + u^1 = A(\tau)\sin t - B(\tau)\cos t + 2A'(\tau)\sin t - 2B'(\tau)\cos t.$$

Notice that we will obtain resonance if we do not choose A and B such that the right-hand side becomes zero, since the frequency for the homogeneous equation will also be 1. We then require

$$2A' + A = 0 \text{ and } -2B' - B = 0.$$

Solving this system of ODEs, we obtain $A(\tau) = C_1 e^{-\tau/2}$ and $B(\tau) = C_2 e^{-\tau/2}$. Now we need to make use of the given initial conditions. Since $u_0(0) = 1$, $A(0)\cos(0) + B(0)\sin(0) = A(0) = 1$. Then $C_1 e^0 = 1$, so $C_1 = 1$. Similarly, using the condition $u_0'(0) = 1$, we obtain that $B(0) = 1$. Then $A(\tau) = B(\tau) = e^{-\tau/2}$. We thus obtain

$$u_0(t) = e^{-\epsilon t/2}(\cos t + \sin t). \tag{5}$$

(c)

Find the exact solution of the original ODE and compare it with the solution u_0 from (b). Do we have $\lim_{\epsilon\to 0} u^\epsilon(t) = \lim_{\epsilon\to 0} u_0(t)$?

Solution. We are interested in solving the ODE $u'' + \epsilon u' + u = 0$ subject to the initial conditions $u(0) = 1$ and $u'(0) = 1$. Fix any $\epsilon > 0$. The characteristic equation is then $\lambda^2 + \epsilon\lambda + 1 = 0$. This has solutions $\lambda = -\frac{\epsilon}{2} \pm \frac{\sqrt{\epsilon^2-4}}{2}$. The general solution is then

$$u(t) = C_1 e^{-\left(\frac{\epsilon-\sqrt{\epsilon^2-4}}{2}\right)t} + C_2 e^{-\left(\frac{\epsilon+\sqrt{\epsilon^2-4}}{2}\right)t}.$$

Since $u(0) = 1, C_1 + C_2 = 1$. Since $u'(0) = 1$, $-\left(\frac{\epsilon-\sqrt{\epsilon^2-4}}{2}\right)C_1 - \left(\frac{\epsilon+\sqrt{\epsilon^2-4}}{2}\right)C_2 = 1$. Solving for C_1 and C_2 algebraically, we find that

$$C_1 = \frac{1}{\sqrt{\epsilon^2-4}}\left(1+\frac{\epsilon}{2}\right) + \frac{1}{2}$$
$$C_2 = -\frac{1}{\sqrt{\epsilon^2-4}}\left(1+\frac{\epsilon}{2}\right) + \frac{1}{2}$$

so that our solution becomes

$$u(t) = \left(\frac{1}{\sqrt{\epsilon^2-4}}\left(1+\frac{\epsilon}{2}\right) + \frac{1}{2}\right) e^{-\left(\frac{\epsilon-\sqrt{\epsilon^2-4}}{2}\right)t} + \left(-\frac{1}{\sqrt{\epsilon^2-4}}\left(1+\frac{\epsilon}{2}\right) + \frac{1}{2}\right) e^{-\left(\frac{\epsilon+\sqrt{\epsilon^2-4}}{2}\right)t}. \tag{6}$$

Now considering the limit of (5),

$$\lim_{\epsilon\to 0} u_0(t) = \lim_{\epsilon\to 0}(e^{-\epsilon t/2}(\cos t + \sin t)) = \cos t + \sin t.$$

Now we can consider the limit of (6) for comparison:

$$\begin{aligned}
\lim_{\epsilon\to 0} u(t) &= \lim_{\epsilon to 0}\left(\left(\frac{1}{\sqrt{\epsilon^2-4}}\left(1+\frac{\epsilon}{2}\right) + \frac{1}{2}\right) e^{-\left(\frac{\epsilon-\sqrt{\epsilon^2-4}}{2}\right)t} + \left(-\frac{1}{\sqrt{\epsilon^2-4}}\left(1+\frac{\epsilon}{2}\right) + \frac{1}{2}\right) e^{-\left(\frac{\epsilon+\sqrt{\epsilon^2-4}}{2}\right)t}\right) \\
&= \left(\frac{1}{2i} + \frac{1}{2}\right) e^{it} + \left(-\frac{1}{2i} + \frac{1}{2}\right) e^{-it} \\
&= \left(\frac{1}{2i} + \frac{1}{2}\right)(\cos t + i\sin t) + \left(-\frac{1}{2i} + \frac{1}{2}\right)(\cos t - i\sin t) \\
&= \frac{1}{2i} i\sin t + \frac{1}{2}\cos t + \frac{1}{2i} i\sin t + \frac{1}{2}\cos t \\
&= \cos t + \sin t.
\end{aligned}$$

The limits are indeed the same!

Problem 2: The Inverted Pendulum Problem

Consider the equation

$$\theta_\epsilon'' - \left(a + \frac{b}{\epsilon}\cos\left(\frac{t}{\epsilon}\right)\right)\sin(\theta_\epsilon) = 0 \tag{7}$$

where $a > 0$ and $b > 0$ are constants, and $\theta = \theta(t)$. Apply the ansatz

$$\theta_\epsilon(t) = \theta^0\left(t, \frac{t}{\epsilon}\right) + \epsilon\theta^1\left(t, \frac{t}{\epsilon}\right) + ...$$

where $\theta^k = \theta^k(t,\tau)$ and $\tau \mapsto \theta^k(t,\tau)$ is 2π-periodic.

(a)

Show that θ does not depend on τ, that is $\theta^0 = \theta^0(t)$.

Proof. First, we evaluate

$$\begin{aligned}\theta_\epsilon' &= \theta_t^0 + \frac{1}{\epsilon}\theta_\tau^0 + \epsilon\theta_t^1 + \theta_\tau^1 \\ \theta_\epsilon'' &= \theta_{tt}^0 + \frac{2}{\epsilon}\theta_{t\tau}^0 + \frac{1}{\epsilon^2}\theta_{\tau\tau}^0 + \epsilon\theta_{tt}^1 + 2\theta_{t\tau}^1 + \frac{1}{\epsilon}\theta_{\tau\tau}^1.\end{aligned}$$

Now, substituting the ansatz into (7), we obtain

$$\theta_{tt}^0 + \frac{2}{\epsilon}\theta_{t\tau}^0 + \frac{1}{\epsilon^2}\theta_{\tau\tau}^0 + \epsilon\theta_{tt}^1 + 2\theta_{t\tau}^1 + \frac{1}{\epsilon}\theta_{\tau\tau}^1 - \left(a + \frac{b}{\epsilon}\cos\left(\frac{t}{\epsilon}\right)\right)\sin(\theta^0 + \epsilon\theta^1 + ...) + ... = 0. \tag{8}$$

Using a Taylor expansion for the sine term, $\sin(\theta^0 + \epsilon\theta^1 + ...) = \sin(\theta^0) + \epsilon\theta^1\cos(\theta^0) + O(\epsilon^2)$. This will be useful in the next part of the problem. For now, we just consider $O\left(\frac{1}{\epsilon^2}\right)$ terms, from which we obtain

$$\frac{1}{\epsilon^2}\theta_{\tau\tau}^0 = 0 \implies \theta_{\tau\tau}^0 = 0.$$

Following the hint to multiply by θ^0 and integrate by parts, we obtain that

$$\int_0^{2\pi} \theta^0\theta_{\tau\tau}^0 d\tau = 0,$$

which implies

$$\theta^0\theta_\tau^0|_0^{2\pi} - \int_0^{2\pi}(\theta_\tau^0)^2 d\tau = 0.$$

The boundary terms are zero by the assumption that θ^0 is 2π-periodic in τ (and so is θ_τ^0). Since $(\theta_\tau^0)^2$ is a non-negative function that integrates to zero over this domain, then it must be zero on this domain. This implies that θ^0 is constant in τ on this domain (as it is also continuous), but due to periodicity, it is everywhere constant in τ. We may thus conclude that $\theta^0 = \theta^0(t)$.

(b)

Show that θ^0 satisfies the ODE

$$\theta_{tt}^0 + \frac{b^2}{4}\sin(2\theta^0) - a\sin(\theta^0) = 0.$$

Proof. Now we compare the $O\left(\frac{1}{\epsilon}\right)$ terms. From (8) with the Taylor expansion replacement for the sine term, we have

$$\frac{2}{\epsilon}\theta_{t\tau}^0 + \frac{1}{\epsilon}\theta_{\tau\tau}^1 - \frac{b}{\epsilon}\cos(\tau)\sin(\theta^0) = 0. \tag{9}$$

Since θ^0 is constant in τ, $2\theta_{t\tau}^0 = 0$, and we can re-arrange to obtain

$$\theta_{\tau\tau}^1 = b\cos(\tau)\sin(\theta^0). \tag{10}$$

Since θ^0 does not depend on τ, we may integrate (10) in τ. We then obtain

$$\theta^1_\tau = b\sin(\tau)\sin(\theta^0) + A(t).$$

Integrating again, we have

$$\theta^1 = -b\cos(\tau)\sin(\theta^0) + A(t)\tau + B(t).$$

Following the hint, without loss of generality, we set $A(t)$ and $B(t)$ to be zero. Then we have

$$\theta^1 = -b\cos(\tau)\sin(\theta^0). \tag{11}$$

Substituting (11) into (8) (with the Taylor expansion for the sine term) and just considering the $O(1)$ terms, we have

$$\begin{aligned}
0 &= \theta^0_{tt} + 2\theta^1_{t\tau} - a\sin(\theta^0) + b\theta^1\cos(\tau)\cos(\theta^0) \\
&= \theta^0_{tt} + 2(b\sin(\tau)\sin(\theta^0))_t - a\sin(\theta^0) + b(b\cos(\tau)\sin(\theta^0))\cos(\tau)\cos(\theta^0) \\
&= \theta^0_{tt} + 2b\sin(\tau)\cos(\theta^0)\theta^0_t - a\sin(\theta^0) + b^2\cos(\theta^0)\sin(\theta^0)\cos^2(\tau) \\
&= \theta^0_{tt} + 2b\sin(\tau)\cos(\theta^0)\theta^0_t - a\sin(\theta^0) + \frac{b^2}{2}\sin(2\theta^0)\cos^2(\tau).
\end{aligned}$$

In the last line, we used the double angle identity to replace $\cos(\theta^0)\sin(\theta^0) = \frac{1}{2}\sin(2\theta^0)$. Now integrating by parts over the range $[0, 2\pi]$, we obtain

$$\begin{aligned}
0 &= \int_0^{2\pi} \theta^0_{tt} + 2b\sin(\tau)\cos(\theta^0)\theta^0_t - a\sin(\theta^0) + \frac{b^2}{2}\sin(2\theta^0) + cos^2(\tau)d\tau \\
&= 2\pi(\theta^0_{tt} - a\sin(\theta^0)) + (2b\cos(\tau)\cos(\theta^0)\theta^0_t)|_0^{2\pi} + \frac{b^2}{2}\pi\sin(2\theta^0) \\
&= 2\pi(\theta^0_{tt} - a\sin(\theta^0))) + \frac{b^2}{2}\pi\sin(2\theta^0)
\end{aligned}$$

In the second line, we used the hint that $\int_0^{2\pi}\cos^2 x dx = \pi$ and the fact that θ^0 does not depend on τ. In the third line, we used the periodicity of cosine. Dividing through by 2π, we obtain

$$\theta^0_{tt} + \frac{b^2}{4}\sin(2\theta^0) - a\sin(\theta_0) = 0$$

as claimed.