APMA 1941G – HOMEWORK 7

Problem 1: (10 points, 5 points each)

This example concerns "Ex 4: A nonlinear oscillator with damping."

(a) Let A be the area of the region

$$\left\{ (u_0, V) \in \mathbb{R}^2 \mid \frac{1}{2}\left(V^2\right) + \Phi\left(u_0\right) \leq E \right\}$$

$$\text{Show that } A = \int_0^{2\pi} \omega^0 \left(u_\eta^0\right)^2 d\eta$$

Hint: There is no Multivariable Calculus involved here, just use single-variable calculus. Also, use $\sqrt{2(E - \Phi(u_0))} = \omega_0 u_\eta^0$ as well as the change of variable $u_0 = u_0(\eta)$. Finally, remember ω_0 does not depend on η and that $(u_0)^2$ is 2π periodic and even

(b) Show that

$$\frac{dA}{dE} = \frac{2\pi}{\omega^0(E)}$$

Hint: Use the formula of A you found in (a) that involves square roots and $a(E)$ and $b(E)$, and differentiate this with respect to E. You may use the following differentiation formula:

$$\frac{d}{dE}\int_{a(E)}^{b(E)} f(x,E)dx = \int_{a(E)}^{b(E)} f_E(x,E)dx + b'(E)f(b(E),E) - a'(E)f(a(E),E)$$

Problem 2: (10 points = 4 + 4 + 2 points)

This problem is similar to Examples 4 and 5 in lecture, so if you're stuck, just look back at what we did in those two examples. Consider

$$u_\epsilon''(t) + \omega^2(\epsilon t)\sin(u_\epsilon(t)) = 0$$

Here $u^\epsilon = u^\epsilon(t)$ and $\omega = \omega(s) > 0$.

Suppose u_ϵ has the form

$$u_\epsilon(t) = u\left(\frac{\theta(\epsilon t, \epsilon)}{\epsilon}, \epsilon t, \epsilon\right)$$

Where $u = u(\eta, \tau, \epsilon), \theta = \theta(\tau, \epsilon)$ and $\eta \mapsto u(\eta, \tau, \epsilon)$ are 2π-periodic.

Apply the (usual) Ansatz

$$u = u^0 + \epsilon u^1 + \cdots$$
$$\theta = \theta^0 + \epsilon\theta^1 + \cdots$$

Now just like the WKB method, **choose** θ_0 such that

$$\theta^0_\tau = \omega \text{ and let } \omega_0 := \theta^0_\tau = \omega \text{ and } \omega_1 := \theta^1_\tau$$

(a) Let E be the energy

$$E(\tau, \eta) = \frac{1}{2}\left(u^0_\eta\right)^2 - \cos\left(u^0\right)$$

Show, using the $O(1)$-terms, that $E = E(\tau)$

(b) Differentiate the $O(1)$-terms with respect to θ and let $W = u^0_\eta$ to show that W solves a linear PDE in W. Then multiply the $O(\epsilon)$-terms by $W = u^0_\eta$ and integrate by parts and show that

$$\int_0^{2\pi} \omega^0 \left(u_\eta^0\right)^2 d\eta = C$$

(c) Finally, calculate θ_0 in terms of ω

Note: Just like in Example 5 (Nonlinear Wave equation), it turns out that we can solve the three equations in (a),(b),(c) to find u_0, ω_0 and θ_0