APMA 1941G – HOMEWORK 8

Problem 1: $(10 = 4 + 4 + 2$ points)

For this problem, you may use the following integration-by-parts formula, assuming the boundary terms are 0

$$\int_W Df \cdot Dg \, d\mathbf{x} = -\int_W \operatorname{div}(Df) \, g \, d\mathbf{x}$$

(a) Suppose u is a minimizer of

$$I[u] = \int_W \frac{1}{2} |Du|^2 + f(u) + |\mathbf{x}|^2 \, d\mathbf{x}$$

Where $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $W \subseteq \mathbb{R}^n$ and $f = f(s)$ and $' = \frac{d}{ds}$

Mimic the derivation of the Euler-Lagrange equation to show

$$-\Delta u + f'(u) = 0$$

(b) This part refers to Example 7 (An Eikonal/Continuity PDE):

Suppose that a^0 and θ^0 minimize $I^0\left[a^0, \theta^0\right]$

By doing a variation in θ^0 show that θ^0 must satisfy the equation

$$-\operatorname{div}\left((a_0)^2 D\theta^0\right) = 0$$

(c) Let f and ϕ be fixed functions. Find $L = L(p, z, x)$ so that

$$-\Delta u + D\phi \cdot Du = f(x)$$

Is the Euler-Lagrange equation corresponding to the functional

$$I[u] = \int_W L(Du, u, x)dx$$

Hint: First find a Lagrangian for the simpler PDE $-\Delta u = f$. Then, to solve our original problem, multiply the Lagrangian you just found by a simple exponential term involving ϕ

Problem 2: (10 = 4 + 4 + 2 points)

Consider the following ODE on $(0, 1)$.

$$\begin{cases} \epsilon\, u_\epsilon'' + u_\epsilon' + u_\epsilon = 0 \\ u_\epsilon(0) = 0, \quad u_\epsilon(1) = 1 \end{cases}$$

Apply the Ansatz

$$u^\epsilon(x) = u^0\left(x, \frac{x}{\epsilon}\right) + \epsilon\, u^1\left(x, \frac{x}{\epsilon}\right) + \cdots$$

Where $u^k = u^k(x, \tau)$

(a) Find the $O\left(\frac{1}{\epsilon}\right)$ terms to get a formula for $u^0(x, \tau)$ in terms of constants $A(x)$ and $B(x)$

(b) Find the $O(1)-$terms to get an ODE of u^1 in terms of u^0. Kill the resonance terms to find (the general form of) $A(x)$ and $B(x)$

(c) Finally, impose the conditions $u^0(0) = 0$ and $u^0(1) = 1$ to find an explicit formula for $u^0(x) = u^0\left(x, \frac{x}{\epsilon}\right)$