APMA 1941G – MIDTERM – SOLUTIONS

Problem 1:

(a) Our Ansatz is

$$u^{\epsilon} = u^{0}\left(t, \epsilon t\right) + \epsilon u^{1}\left(t, \epsilon t\right) + \cdots$$

Therefore for all k we have

$$\begin{aligned} \left(u_k\right)' &= u_t^k + \epsilon u_\tau^k \\ \left(u_k\right)'' &= u_{tt}^k + 2\epsilon u_{t\tau}^k + \epsilon^2 u_{\tau\tau}^k \end{aligned}$$

Plugging into the ODE, we get

$$\begin{aligned} u_\epsilon'' - \epsilon \cos(t) \sin\left(u^\epsilon\right) &= 0 \\ u_0'' + \epsilon u_1'' - \epsilon \cos(t) \sin\left(u_0 + \epsilon u_1\right) &= 0 \\ \left(u_{tt}^0 + 2\epsilon u_{t\tau}^0 + \epsilon^2 u_{\tau\tau}^0\right) + \left(\epsilon u_{tt}^1 + 2\epsilon^2 u_{t\tau}^1\right) + \epsilon^3 u_{\tau\tau}^1 - \epsilon \cos(t) \sin\left(u_0\right) - \epsilon^2 \cos(t) u_1 \cos\left(u_0\right) &= 0 \end{aligned}$$

We used a Taylor expansion: $\sin(x+h) = \sin(x) + h\cos(x) + o(h)$

$O(1)-$**terms:** $u_{tt}^0 = 0$

Multiply by u^0 and integrate by parts over $[0, 2\pi]$ (no boundary terms since u_t^0 is 2π-periodic)

$$\int_0^{2\pi} u_{tt}^0 u^0 dt = \int_0^{2\pi} 0 dt \Rightarrow -\int_0^{2\pi} \underbrace{\left(u_t^0\right)^2}_{\geq 0} dt = 0$$

Therefore $u_t^0 = 0$ and so $u^0 = u^0(\tau)$ as desired

(b) $O(\epsilon)-$**terms:** Since u^0 doesn't depend on t, we get

$$
\begin{aligned}
\cancelto{0}{2u^0_{t\tau}} + u^1_{tt} - \cos(t)\sin(u_0) &= 0\\
u^1_{tt} &= \cos(t)\sin(u_0)\\
u^1_t &= \int \cos(t)\sin(u_0)\,dt = \sin(t)\sin(u_0) + A\\
u^1_t &= \sin(t)\sin(u_0) \quad (\text{Setting } A = 0)\\
u^1 &= \int \sin(t)\sin(u_0)\,dt = -\cos(t)\sin(u_0) + B
\end{aligned}
$$

Setting $B = 0$, we get $u^1 = -\cos(t)\sin(u_0)$ as desired

(c) $O\left(\epsilon^2\right)-$**terms:** Since $u^1 = -\cos(t)\sin(u_0)$ we get

$$
\begin{aligned}
u^0_{\tau\tau} + 2u^1_{t\tau} - u^1\cos\left(u^0\right)\cos(t) &= 0\\
u^0_{\tau\tau} + 2\left(-\cos(t)\sin\left(u^0(\tau)\right)\right)_{t\tau} - \left(-\cos(t)\sin\left(u^0\right)\right)\cos(t)\sin\left(u^0\right) &= 0\\
u^0_{\tau\tau} + 2\sin(t)\cos\left(u^0\right)u^0_\tau + \cos^2(t)\sin\left(u^0\right)\cos\left(u^0\right) &= 0
\end{aligned}
$$

Integrate with respect to t on $[0, 2\pi]$:

$$
\int_0^{2\pi} u^0_{\tau\tau} + 2\sin(t)\cos\left(u^0\right)u^0_\tau + \cos^2(t)\sin\left(u^0\right)\cos\left(u^0\right)dt = \int_0^{2\pi} 0\,dt
$$

$$
\begin{aligned}
2\pi u^0_{\tau\tau} + 2\cos(u_0)\,u^0_\tau \cancelto{0}{\int_0^{2\pi}\sin(t)dt} + \sin(u_0)\cos(u_0)\underbrace{\int_0^{2\pi}\cos^2(t)dt}_{\pi} &= 0\\
2\pi u^0_{\tau\tau} + \pi\sin(u_0)\cos(u_0) &= 0
\end{aligned}
$$

Hence we get $u^0_{\tau\tau} + \frac{1}{2}\sin(u_0)\cos(u_0) = 0$ as desired

Problem 2:

STEP 1: Start with

$$\begin{aligned}(n-1)! &= \int_0^\infty e^{-t} t^{n-1} dt \\ &= \int_0^\infty e^{-ns} (ns)^{n-1} n ds \qquad \left(s = \frac{t}{n} \Rightarrow dt = nds\right) \\ &= n^n \int_0^\infty e^{-ns} s^n \left(\frac{1}{s}\right) ds \\ &= n^n \int_0^\infty e^{-ns} e^{n\ln(s)} \left(\frac{1}{s}\right) ds \\ &= n^n \int_0^\infty e^{\frac{-s+\ln(s)}{\epsilon}} \left(\frac{1}{s}\right) ds \qquad \left(n = \frac{1}{\epsilon}\right)\end{aligned}$$

STEP 2: Now apply the general Laplace's method with $a(s) = \frac{1}{s}$ and $\phi(s) = -s + \ln(s)$

Notice in particular that $\phi'(s) = 0 \Rightarrow -1 + \frac{1}{s} = 0 \Rightarrow s = 1$ and $\phi''(s) = -\frac{1}{s^2}$ so $\phi''(1) = -1 < 0$

So if you let $x_0 = 1$ then ϕ has a global max at 1 with $\phi(1) = -1$ and so by general Laplace, we get:

$$\int_0^\infty e^{\frac{-s+\ln(s)}{\epsilon}} \left(\frac{1}{s}\right) ds = \sqrt{\frac{2\pi\epsilon}{|\phi''(1)|}} e^{\frac{\phi(1)}{\epsilon}} (1 + o(1)) = \sqrt{\frac{2\pi}{n}} e^{-n} (1 + o(1))$$

STEP 3: Therefore, we get

$$(n-1)! = n^n \int_0^\infty e^{\frac{-s+\ln(s)}{\epsilon}} \left(\frac{1}{s}\right) ds = n^n \sqrt{\frac{2\pi}{n}} e^{-n} (1 + o(1)) = \sqrt{2\pi} n^{n-\frac{1}{2}} e^{-n} (1 + o(1))$$

Multiplying both sides by n we get:

$$n\,(n-1)! = n\sqrt{2\pi}n^{n-\frac{1}{2}}e^{-n}\,(1+o(1))$$

$$\text{Therefore } n! = \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}\,(1+o(1))$$

This gives us the result because then (Remember that since $n = \frac{1}{\epsilon}$, as $\epsilon \to 0$ we have $n \to \infty$)

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}} = \lim_{n\to\infty} 1 + o(1) = 1$$

$$\text{Hence } n! \sim \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n} \text{ as desired } \checkmark$$

Problem 3: Let $n > 0$ be given, then (see next page for explanations)

$$
\begin{aligned}
&\left| f(\epsilon)g(\epsilon) - \sum_{k=0}^{n} c_k \epsilon^k \right| \\
&\overset{\text{DEF}}{=} \left| f(\epsilon)g(\epsilon) - \sum_{k=0}^{n} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) \epsilon^k \right| \\
&= \left| f(\epsilon)g(\epsilon) - \sum_{k=0}^{n} \sum_{i=0}^{k} a_i b_{k-i} \epsilon^k \right| \\
&\overset{\text{HINT 1}}{=} \left| f(\epsilon)g(\epsilon) - \sum_{k=0}^{n} \sum_{i=0}^{n-k} a_i b_k \epsilon^{i+k} \right| \\
&\overset{(\star)}{=} \left| f(\epsilon)g(\epsilon) - \sum_{k=0}^{n} \sum_{i=0}^{n} a_i b_k \epsilon^{i+k} + o(\epsilon^n) \right| \\
&= \left| f(\epsilon)g(\epsilon) - \sum_{k=0}^{n} \sum_{i=0}^{n} a_i \epsilon^i b_k \epsilon^k + o(\epsilon^n) \right| \\
&\overset{\text{HINT 2}}{=} \left| f(\epsilon)g(\epsilon) - \left(\sum_{i=0}^{n} a_i \epsilon^i \right) \left(\sum_{k=0}^{n} b_k \epsilon^k \right) + o(\epsilon^n) \right| \\
&= \left| f(\epsilon)g(\epsilon) - f(\epsilon) \sum_{k=0}^{n} b_k \epsilon^k + f(\epsilon) \sum_{k=0}^{n} b_k \epsilon^k - \left(\sum_{i=0}^{n} a_i \epsilon^i \right) \left(\sum_{k=0}^{n} b_k \epsilon^k \right) + o(\epsilon^n) \right| \\
&= \left| f(\epsilon) \left(g(\epsilon) - \sum_{k=0}^{n} b_k \epsilon^k \right) + \left(\sum_{k=0}^{n} b_k \epsilon^k \right) \left(f(\epsilon) - \sum_{i=0}^{n} a_i \epsilon^i \right) + o(\epsilon^n) \right| \\
&\leq |f(\epsilon)| \left| g(\epsilon) - \sum_{k=0}^{n} b_k \epsilon^k \right| + \left| \sum_{k=0}^{n} b_k \epsilon^k \right| \left| f(\epsilon) - \sum_{i=0}^{n} a_i \epsilon^i \right| + o(\epsilon^n) \\
&\overset{(\star\star)}{\leq} C o(\epsilon^n) + C o(\epsilon^n) + o(\epsilon^n) = o(\epsilon^n) \checkmark
\end{aligned}
$$

Explanations:

For the $(\star)$ step, notice that if $i > n - k$ then $i + k > n$ and so the $a_i b_k \epsilon^{i+k}$ terms are of order higher than ϵ^n and therefore $o(\epsilon^n)$

For the $(\star\star)$ step, we use that: f is bounded, $g \sim \sum_{n=0}^{\infty} b_n \epsilon^n$, the asymptotic expansion of g is bounded, and $f \sim \sum_{n=0}^{\infty} a_n \epsilon^n$ respectively.