APMA 1941G – MIDTERM

Problem 1: (10 points)

Consider the following ODE, where $u_\epsilon = u_\epsilon(t)$

$$u_\epsilon''(t) - \epsilon \cos(t) \sin(u_\epsilon(t)) = 0$$

Let our Ansatz be:

$$u_\epsilon = u_0(t, \epsilon t) + \epsilon\, u_1(t, \epsilon t) + \cdots$$

Where $u_k = u_k(t, \tau)$ and $t \mapsto u_k(t, \tau)$ is 2π periodic

(a) Use the $O(1)$ terms to show that $u_0 = u_0(\tau)$
That is u_0 does not depend on t

(b) Use the $O(\epsilon)$ terms to show that $u_1 = -\sin(u_0(\tau))\cos(t)$
You may set any constants you may find equal to 0

(c) Use the $O(\epsilon^2)$ terms to conclude $u^0_{\tau\tau} + \frac{1}{2}\sin(u_0)\cos(u_0) = 0$
You may use without proof that $\int_0^{2\pi} \cos^2(t)dt = \pi$

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Problem 2: (6 points)

Note: Try your best to do this problem without looking at the HW

The general Laplace method says that:

If ϕ is a smooth function that has a max at x_0 with $\phi'(x_0) = 0$ and $\phi''(x_0) < 0$, and $a(x)$ is any smooth function (not necessarily with compact support), then, as $\epsilon \to 0$

$$\int_0^\infty a(x) e^{\frac{\phi(x)}{\epsilon}} dx = \sqrt{\frac{2\pi\epsilon}{|\phi''(x_0)|}}\, e^{\frac{\phi(x_0)}{\epsilon}}\, a(x_0)\,(1 + o(1))$$

Use Laplace's method to show that, as $n \to \infty$

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$$

Here $f \sim g$ means $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$

Make sure to specify which ϕ, x_0, a and ϵ you're using

Hint: Start with the identity (you don't have to prove this)

$$(n-1)! = \int_0^\infty e^{-t} t^{n-1} dt$$

And use the substitution $s = \frac{t}{n}$

Make sure to keep a $\left(\frac{1}{s}\right) ds$ in your integral

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Problem 3: (4 points)

Using **the definition** of asymptotic expansions, show that if

$$f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n \quad \text{and} \quad g(\epsilon) \sim \sum_{n=0}^{\infty} b_n \epsilon^n$$

And both f, g and their asymptotic expansions are bounded, then

$$f(\epsilon)g(\epsilon) \sim \sum_{n=0}^{\infty} c_n \epsilon^n \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

You may use (without proof) the following hints:

Hint 1:

$$\sum_{k=0}^{n} \sum_{i=0}^{k} a_i b_{k-i} \epsilon^k = \sum_{k=0}^{n} \sum_{i=0}^{n-k} a_i b_k \epsilon^{i+k}$$

Note: This is a discrete version of the change of variables
$u = i$ and $v = k - i$

Hint 2: In general (this is just foiling out)

$$\left(\sum_{m=0}^{n} x_m \right) \left(\sum_{k=0}^{n} y_n \right) = \sum_{m=0}^{n} \sum_{k=0}^{n} x_m y_n$$

Also remember that if $m > n$ then $\epsilon^m = o\left(\epsilon^n\right)$ For example $\epsilon^3 = o\left(\epsilon^2\right)$