APMA 0350 – FINAL EXAM – STUDY GUIDE

This is the study guide for the exam, and is just a **guide** to help you study. Think of it as a course summary rather than "this is exactly the questions I'm going to ask you on the exam."

There will be **NO** coding on the exam, but there might be proofs, like on the homework

Format: There are 10 questions on the exam, all of them free response, no multiple choice.

A Laplace transform table will be provided, see course website. No need to print it out. You're allowed **one** 2-sided 8.5×11 cheat sheet

Useful trig identities to know:

(1) $\sin^2(x) + \cos^2(x) = 1$

(2) $1 + \tan^2(x) = \sec^2(x)$

(3) $\cos(-x) = \cos(x), \sin(-x) = -\sin(x)$

(4) $\cos(2x) = \cos^2(x) - \sin^2(x)$, $\sin(2x) = 2\sin(x)\cos(x)$

(5) $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$, $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$

Useful integration techniques to know:

(1) Integrals of $\sin, \cos, \tan, \sec$

(2) $\int \frac{1}{x^2+1} dx = \tan^{-1}(x) + C$

(3) $u-$substitution

(4) Integration by parts

(5) $\int \cos^2(x) dx$, $\int \sin^2(x) dx$; check out this video in case you forgot

(6) You need to know partial fractions to do Laplace transforms

1. Introduction (Lectures 1 – 3)

- Solve the most basic differential equation

$$\begin{cases} y' = ky \\ y(0) = y_0 \end{cases}$$

  You can go directly to $y = y(0)e^{kt}$ do not use separation of variables for this

- Show that there are no other solutions to this ODE, by calculating $\left(ye^{-kt}\right)'$ Also look at the homework problem where you calculated $\left(e^{-2t}y\right)''$

- Check if a function solves an ODE. This just means to plug in the function into the ODE and see if you get an identity.

- Find the order of an ODE and whether it is linear/nonlinear and homogeneous/inhomogeneous

- Plot the direction field of a differential equation. To be honest, this is best done with a computer. Ignore the part about the dfield app

- Find the equilibrium solutions of an autonomous ODE and draw a bifurcation diagram (the table with arrows) and use that to determine the stability of equilibria.

2. Existence-Uniqueness (Lecture 3–4)

- Study the two crazy examples of non-existence/non-uniqueness
- Apply the Existence/Uniqueness Theorem to show that there is a unique solution to an ODE, like on the homework.

3. Separable ODE (Lecture 4– 5)

- Solve an ODE using separation of variables
- This works well if you can put all the y's on one side and all the x's (or t's) on the other side
- I would tell you whether to leave your solution in implicit form or write it in explicit form
- **Careful of hidden solutions** For example, when you divide by y, you have to check if $y = 0$ is a solution. Sometimes it is, sometimes it isn't. Or if you divide by $y^2 - 1$, you have to check if $y = \pm 1$ is a solution.
- Solve the logistic differential equation $y' = Py\left(1 - \frac{y}{M}\right)$ with $y(0) = y_0$.
- In order to save you some time, you do **NOT** have to show the fact that $\frac{20}{y(20-y)} = \frac{1}{y} + \frac{1}{20-y}$

4. Integrating Factors (Lecture 5–6)

- Solve an ODE using **integrating factors**, both in the case $y' + ay$, where you multiply by e^{at} and $y' + Py$ where you multiply by $e^{\int P}$
- Make sure the coefficient of y' is 1 here, people usually lose points on that
- In general, integrating factors work if you can put your equation in the form $y' + Py =$ something

5. Applications (Lecture 6–7)

- All the application problems are fair game, including
  - Falling Object
  - Savings Model
  - Chemical Reactions
  - The rabbit problem on the homework
  - Newton's Law of Cooling on the homework
  - That water/pollutant problem on the homework
  - Bunny vs Foxes (end of the lecture notes)
- You do **NOT** need to memorize any physics equations, except for Newton's Second Law ($F = ma$), any other equations would be provided.
- I could ask you to derive some of those equations, using the trick with $h =$ small change in time and calculating for example $W(t+h) = W(t) +$ Change

- For the chemical tank problem, really think in terms of "what is going in" vs "what is going out," that should help you setting up the equations. Also the units are useful, usually you want something in kg/min

6. Exact Equations (Lecture 8)

- Put a differential equation in the form $Pdx + Qdy = 0$
- Solve exact ODE
- Of course this only works when $P_y = Q_x$ (PeYam = QuiXotic)
- Don't forget to check that your ODE is exact, otherwise you lose points
- Sometimes you can multiply an inexact equation by an integrating factor to make it exact, but in that case I would explicitly give you the integrating factor

7. Euler's Method (Lectures 9–10)

- Apply Euler's method with a small number of steps, like $N = 2$ or $N = 3$ steps. I'll make the algebra as reasonable as possible, since calculators are not allowed.
- Of course, know the formula for Euler's Method
- Remember that there is no coding on the exam
- You don't need to know the sections on Error and Problems with Euler

Second-Order ODE (Lectures 11 – 18)

- Solve $ay'' + by' + cy = 0$ using the method with $Dy = y'$ There are also some nice variations of this on the homework, with repeated roots and inhomogeneous equations

- Solve $ay'' + by' + cy = 0$ using auxiliary equations

- I could ask you about higher order differential equations, but in that case would give you the roots beforehand

- Find the eigenvalues and eigenfunctions of the boundary-value problem $y'' = \lambda y$ The lecture notes and the homework problem give you more or less all possible cases. Don't forget to check what the starting value of m is.

- Use undetermined coeffs to find a particular solution to a ODE:

▶ **Rule 1:** If the right hand side is e^{rt} guess Ae^{rt}

▶ **Rule 2:** If the right hand side is a polynomial like t^2 you guess $At^2 + Bt + C$ (the most general version)

▶ **Rule 3:** cos goes with sin and vice-versa

- If the root of the right-hand-side coincides with the root of y_0, then there is resonance and you add an extra t

- The section on mechanical vibrations is just practice with the techniques. You don't need to know the physical background and you don't need to know the formulas for amplitude

- Solve an ODE using variation of parameters and **make sure that the coefficient of y'' is 1**

- Be able to rederive the var of par formula. I've done an example of that in lecture and there is also one on your homework.

Laplace Transform (Lectures 19 – 25)

- Find $\mathcal{L}\{f(t)\}$ using the definition of Laplace transform.
- I recommend the method of Tabular integration, it allows you to calculate $\mathcal{L}\{t^4\}$ very quickly. Here is an example of this method.
- Prove the formulas for $\mathcal{L}\{f'(t)\}$ and $\mathcal{L}\{f''(t)\}$
- Solve a second-order ODE using Laplace transforms. I could ask you zero or nonzero initial conditions. Notice that the auxiliary equation should appear somewhere in your calculation
- Write a given function in terms of step functions u_c. The jumps could be constants, or entire functions.
- Find the Laplace transform of u_c, and know the formula

$$\mathcal{L}\{f(t-c)u_c(t)\} = e^{-cs}\mathcal{L}\{f(t)\}$$

- This also works in reverse, for example $e^{-5s}\mathcal{L}\{t^3\} = \mathcal{L}\left\{(t-5)^3 u_5(t)\right\}$
- Find the laplace transform of $e^{2t}\sin(3t)$ where you "shift" the Laplace transform.
- Also know how to do it in reverse, like find a function whose Laplace transform is $\frac{1}{(s-2)^2+1}$
- This sometimes requires to complete the square, like $s^2+4s+5 = (s+2)^2+1$

- You may have to combine both, like the $\frac{3(s-2)e^{-3s}}{s^2-4s+5}$ example from lecture.

- Solve ODEs with jumps. This means first write $f(t)$ in terms of jump functions (if not already done) and take Laplace transforms

- Feel free to write your solutions in terms of $h(t)$, like in lecture, but you **need** to define what $h(t)$ is!

- Solve ODE involving Dirac Delta

- The definition of convolution will be provided but know that $\mathcal{L}\{f \star g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$

- Find the Laplace transform of a function that is written as a convolution, like the Laplace transform of $\int_0^t (t-\tau)^2 e^\tau d\tau$

- Find an inverse Laplace transform and express your answer as an integral, like the inverse Laplace transform of $\left(\frac{1}{s^2+1}\right)\left(\frac{1}{s^2+4}\right)$. I would explicitly tell you "Leave your solution as an integral"

- Use Laplace transforms and convolution to solve an ODE in terms of integrals. An excellent example is $y'' - 4y' + 4y = \tan(t)$ I would explicitly tell you "Leave your solution as an integral"

- Solve integral equations and integro-differential equations (see lecture)

- Solve the integral question with

$$\int_0^1 x^3(1-x)^7 dx$$

8. Systems of ODE (Lectures 28 – 34)

- Write a second-order ODE in the form $\mathbf{x}' = A\mathbf{x}$ and solve it
- Solve $\mathbf{x}' = A\mathbf{x}$

▶ **Case 1:** Two distinct eigenvalues

▶ **Case 2:** Complex eigenvalues. Here I recommend using the $e^{(2+3i)t}$ trick from lecture. Remember to pick one row and make the other one full of $0'$s

▶ **Case 3:** Repeated eigenvalues

- **Note:** I might force you to write you answers in terms of integers, so don't forget to re-scale your eigenvectors. Do **NOT** re-scale the *generalized* eigenvectors.
- Draw phase portraits of $\mathbf{x}' = A\mathbf{x}$:
  - ▶ For distinct eigenvalues, you draw the axes generated by the eigenvectors. Then you determine the arrows on each axis, and follow the arrows to draw the other curves
  - ▶ For complex eigenvalues, the solutions are ellipses, or spirals. You do **NOT** need to know how to find the axes here or whether it goes clockwise or counterclockwise.
  - ▶ For repeated eigenvalues, the eigenvector gives you the axis, and the solutions go in or out of the axis.
- Solve $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ using variation of parameters.
- I could ask you a word problem where you set up a system of ODE of the form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, like the chemical tank problem on the HW or the ones on the practice final

9. Nonlinear ODE (Lecture 34)

- Find and classify the equilibrium points of a nonlinear system
- Don't forget to look at the word problems in the HW and the practice exams.
- The Ecology model and the COVID models will **NOT** be on the final