HOMEWORK 7 SOLUTIONS

Problem 1. Find the Laplace transform of

$$f(t) = \begin{cases} t & \text{if } 0 \leq t < 2 \\ 2 & \text{if } 2 \leq t < 5 \\ 7 - t & \text{if } 5 \leq t < 7 \\ 0 & \text{if } t \geq 7 \end{cases}$$

Start with t

At $t = 2$ jump by $2 - t = -(t-2)$

At $t = 5$ jump by $(7-t) - 2 = 5 - t = -(t-5)$

At $t = 7$ jump by $0 - (7-t) = t - 7$

$$f(t) = t - (t-2)u_2(t) - (t-5)u_5(t) + (t-7)u_7(t)$$

Therefore:

$$\begin{aligned}
\mathcal{L}\{f(t)\} =& \mathcal{L}\{t\} - \mathcal{L}\{(t-2)u_2(t)\} - \mathcal{L}\{(t-5)u_5(t)\} + \mathcal{L}\{(t-7)u_7(t)\} \\
=& \left(\frac{1}{s^2}\right) - e^{-2s}\mathcal{L}\{t\} - e^{-5s}\mathcal{L}\{t\} + e^{-7s}\mathcal{L}\{t\} \\
=& \left(\frac{1}{s^2}\right) - e^{-2s}\left(\frac{1}{s^2}\right) - e^{-5s}\left(\frac{1}{s^2}\right) + e^{-7s}\left(\frac{1}{s^2}\right) \\
=& \frac{1}{s^2}\left(1 - e^{-2s} - e^{-5s} + e^{-7s}\right)
\end{aligned}$$

Problem 2. Find a function whose Laplace transform is

$$\frac{8}{s^2 - 4s + 4}$$

$$= \frac{8}{(s-2)^2}$$

First notice that

$$\mathcal{L}^{-1}\left\{\frac{8}{s^2}\right\} = 8\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = 8t.$$

Recall the shifting fact: if

$$\mathcal{L}\{f(t)\} = F(s),$$

then

$$\mathcal{L}\left\{e^{ct}f(t)\right\} = F(s-c).$$

Therefore, because

$$\mathcal{L}\{8t\} = \frac{8}{s^2},$$

we have

$$\mathcal{L}\left\{8te^{2t}\right\} = \frac{8}{(s-2)^2.}$$

So the desired function is $f(t) = 8te^{2t}$.

Problem 3. Find a function whose Laplace transform is

$$\frac{(s-1)e^{-3s}}{s^2 - 4s + 5}$$

$$= e^{-3s}\frac{s-1}{(s-2)^2+1}.$$

Notice that $s-1-1+1=(s-2)+2$, so the above expression can be rewritten as

$$e^{-3s}\frac{(s-2)+1}{(s-2)^2+1}=e^{-3s}\left(\frac{s-2}{(s-2)^2+1}+\frac{1}{(s-2)^2+1}\right). \tag{1}$$

Notice that the expressions $\frac{s-2}{(s-2)^2+1}$ and $\frac{1}{(s-2)^2+1}$ are the expressions $\frac{s}{s^2+1}$ and $\frac{1}{s^2+1}$ shifted by 2. We know that

$$\frac{s}{s^2+1}+\frac{1}{s^2+1}=\mathcal{L}\{\cos(t)+\sin(t)\},$$

and therefore

$$\frac{s-2}{(s-2)^2+1}+\frac{1}{(s-2)^2+1}=\mathcal{L}\left\{e^{2t}\cos(t)+e^{2t}\sin(t)\right\}.$$

Therefore, (1) can be rewritten as

$$e^{-3s}\mathcal{L}\left\{e^{2t}\cos(t)+e^{2t}\sin(t)\right\},$$

and using the rule for exponentials outside of the Laplace transform, this is equal to

$$\mathcal{L}\left\{e^{2(t-3)}\cos(t-3)u_3(t)+e^{2(t-3)}\sin(t-3)u_3(t)\right\}.$$

Thus the desired function is $f(t)=e^{2(t-3)}\cos(t-3)u_3(t)+e^{2(t-3)}\sin(t-3)u_3(t)$.

Problem 4. Solve

$$\begin{cases} y''+4y=3\sin(t)-3u_{2\pi}(t)\sin(t-2\pi) \\ y(0)=0 \\ y'(0)=0. \end{cases}$$

We first take the Laplace transform of the above equation:

$$\mathcal{L}\left\{y''\right\} + 4\mathcal{L}\left\{y\right\} = 3\mathcal{L}\left\{\sin(t)\right\} - 3\mathcal{L}\left\{u_{2\pi}(t)\sin(t-2\pi)\right\}. \tag{2}$$

Recall that

$$\mathcal{L}\left\{f''(t)\right\} = s^2\mathcal{L}\left\{f(t)\right\} - sf(0) - f'(0),$$

so using the initial conditions, we have

$$\mathcal{L}\left\{y''\right\} = s^2\mathcal{L}\left\{y\right\} - sy(0) - y'(0)$$

$$= s^2\mathcal{L}\{y\}. \tag{3}$$

Also notice that

$$\mathcal{L}\left\{\sin(t)\right\} = \frac{1}{s^2+1}, \tag{4}$$

and by the shift rule,

$$\mathcal{L}\left\{u_{2\pi}(t)\sin(t-2\pi)\right\} = e^{-2\pi s}\mathcal{L}\left\{\sin(t)\right\}$$

$$= \frac{e^{-2\pi s}}{s^2+1}. \tag{5}$$

Substituting (3), (4), and (5) into (2), we have

$$(s^2+4)\mathcal{L}\{y\} = \frac{3}{s^2+1}\left(1 - e^{-2\pi s}\right)$$

$$\implies \mathcal{L}\{y\} = \frac{3}{(s^2+1)(s^2+4)}\left(1 - e^{-2\pi s}\right).$$

We can finally solve for y using partial fractions.

$$\frac{3}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$\implies 3 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$
$$= As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2 + Cs + D$$

<u>s^3 terms:</u>

$$0s^3 = (A+C)s^3$$
$$\implies A + C = 0$$
$$\implies A = -C$$

<u>s^2 terms:</u>

$$0s^2 = (B+D)s^2$$
$$\implies B + D = 0$$
$$\implies B = -D$$

<u>s terms:</u>

$$0s = (4A+C)$$
$$\implies 4A + C = 0$$
$$\implies 4A = -C$$

Because we also know that $A = -C$, this implies that $A = C = 0$. <u>Constant terms:</u>

$$3 = 4B + D$$

$$\implies D = 3 - 4B$$

Recall that $B = -D$, so

$$D = 3 + 4D$$

$$\begin{aligned}\implies -3D &= 3\\ \implies D &= -1\\ \implies B &= 1.\end{aligned}$$

We thus have that

$$\frac{3}{(s^2+1)(s^2+4)} = \frac{1}{s^2+1} - \frac{1}{s^2+4},$$

so we have

$$\begin{aligned}\mathcal{L}\{y\} &= \left(1 - e^{-2\pi s}\right)\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)\\ &= \left(1 - e^{-2\pi s}\right)\mathcal{L}\left\{-\frac{1}{2}\sin(2t) + \sin(t)\right\}.\end{aligned}$$

For the clarity of the next step, let $h(t)$ denote the argument of the Laplace transform in the above expression, i.e. $h(t) := -\frac{1}{2}\sin(2t) + \sin(t)$. We have

$$\begin{aligned}\mathcal{L}\{y\} &= \left(1 - e^{-2\pi s}\right)\mathcal{L}\{h(t)\}\\ &= \mathcal{L}\{h(t)\} - e^{-2\pi s}\mathcal{L}\{h(t)\},\end{aligned}$$

and using the shift rule on the second term in this expression, this is equal to

$$\begin{aligned}&\mathcal{L}\{h(t)\} - \mathcal{L}\left\{h(t-2\pi)u_{2\pi}(t)\right\}\\ &= \mathcal{L}\left\{h(t) - h(t-2\pi)u_{2\pi}(t)\right\}.\end{aligned}$$

Therefore, we have that

$$y = h(t) - h(t-2\pi)u_{2\pi}(t).$$

Plugging in $-\frac{1}{2}\sin(2t) + \sin(t)$ for $h(t)$, we have

$$y = -\frac{1}{2}\sin(2t) + \sin(t) + \left[\frac{1}{2}\sin(2(t-2\pi)) - \sin(t-2\pi)\right]u_{2\pi}(t).$$