APMA 0359 - Homework 8 Solutions

November 20, 2023

1. Write the ODE in the form $x' = Ax + f$

$$y''' + 2y'' - 4y' + y = 5t.$$

Do NOT solve it!

Solution: This problem is useful for writing differential equations in the matrix form

$$x = Ax + f$$

where x and f are vectors and A is a matrix. Given the ODE

$$y''' + 2y'' - 4y' + y = 5t$$

we can assign new variables to replace the derivatives (n variables for n derivatives). For three derivatives, we write

$$\begin{aligned} x_1 = y &\implies x_1' = y' = x_2 \\ x_2 = y' &\implies x_2' = y'' = x_3. \\ x_3 = y'' &\implies x_3' = y'''. \end{aligned}$$

We can rewrite the ODE in terms of our variables, we get

$$x_3' = -x_1 + 4x_2 - 2x_3.$$

Thus, we arrive at the matrix equation

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5t \end{bmatrix}.$$

2. Solve the following systems. Do NOT draw a phase portrait

(a)

$$x' = Ax \quad \text{where } A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

Solution: We start by finding the eigenvalues of the matrix. We see that

$$\begin{aligned} \det|A - \lambda I| = \det \begin{bmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{bmatrix} = & \quad (1-\lambda)(-4-\lambda) - (-2)(3) \\ = & -4 - \lambda + 4\lambda + \lambda^2 + 6 \\ = & \lambda^2 + 3\lambda + 2 \\ = & (\lambda+1)(\lambda+2) \end{aligned}$$

Thus, we have the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

Next, we find the corresponding eigenvectors

$$\begin{aligned} \text{Nul}(A - 1I) &= \left[\begin{array}{cc|c} 1-(-1) & -2 & 0 \\ 3 & -4-(-1) & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 3 & -3 & 0 \end{array}\right] \end{aligned}$$

Then using the row operations

i. $R_2 = 1/3R_2$

ii. $R_1 = 1/2R_1$

iii. $R_2 = R_2 - R_1$

Then we see that $x - y = 0 \implies x = y$. Thus, setting $x = 1$, then we get the corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Similarly for $\lambda_2 = 2$, we find

$$\begin{aligned} \text{Nul}(A - 1I) &= \left[\begin{array}{cc|c} 1-(-2) & -2 & 0 \\ 3 & -4-(-2) & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 3 & -2 & 0 \end{array}\right] \end{aligned}$$

Then using the row operations

i. $R_2 = R_2 - R_1$,

we see that $3x - 2y = 0 \implies 3x = 2y$. Thus, setting $x = 2$, then we get the corresponding eigenvector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Finally, we obtain the solution

$$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

(b)

$$x' = Ax \quad \text{where } A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \text{ and } x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Solution: We start by finding the eigenvalues of the matrix. We see that

$$\begin{aligned} \det|A - \lambda I| = &\det \begin{bmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{bmatrix} = \qquad (5-\lambda)(1-\lambda) - (-1)(3) \\ =&5 - 5\lambda - \lambda + \lambda^2 - 3 \\ =&\lambda^2 - 6\lambda - 8 \\ =&(\lambda - 2)(\lambda - 4) \end{aligned}$$

Thus, we have the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 4$.

Next, we find the corresponding eigenvectors

$$\begin{aligned} \text{Nul}(A - 2I) &= \left[\begin{array}{cc|c} 5-2 & -1 & 0 \\ 3 & 1-2 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right] \end{aligned}$$

Then using the row operations

i. $R_2 = R_2 - R_1$

Then we see that $3x - y = 0 \implies 3x = y$. Thus, setting $x = 1$, then we get the corresponding eigenvector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Similarly for $\lambda_2 = 4$, we find

$$\begin{aligned} \text{Nul}(A - 4I) &= \left[\begin{array}{cc|c} 5-4 & -1 & 0 \\ 3 & 1-4 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right] \end{aligned}$$

Then using the row operations

i. $R_2 = 1/3R_2$

ii. $R_2 = R_2 - R_1$,

we see that $x - y = 0 \implies x = y$. Thus, setting $x = 1$, then we get the corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Finally, we obtain the solution

$$x(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We then use the initial condition $x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Thus, we see that

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Using the row operations

i. $R_2 + (-3R_1)$

ii. $R_2 = R_2/(-2)$

iii. $R_1 = R_1 + (-R_2)$

We find $c_1 = -3/2$ and $c_2 = 7/2$. Thus, the final solution can be written as

$$x(t) = -\frac{3}{2}e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{7}{2}e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

3. Solve the system and draw a phase portrait by hand for

$$x' = Ax \text{ where } A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}.$$

Solution: We start by finding the eigenvalues of the matrix. We see that

$$\begin{aligned} \det|A - \lambda I| = \det \begin{bmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{bmatrix} = & \qquad (3-\lambda)(-2-\lambda) - (-2)(2) \\ = -6 - 3\lambda + 2\lambda + \lambda^2 + 4 & \\ = \lambda^2 - \lambda - 2 & \\ = (\lambda+1)(\lambda-2) & \end{aligned}$$

Thus, we have the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$.

Next, we find the corresponding eigenvectors

$$\begin{aligned} \text{Nul}(A - 1I) &= \left[\begin{array}{cc|c} 3-(-1) & -2 & 0 \\ 2 & -2-(-1) & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 4 & -2 & 0 \\ 2 & -1 & 0 \end{array}\right] \end{aligned}$$

Then using the row operations

(a) $R_1 = 1/2R_1$

(b) $R_2 = R_2 - R_1$

Then we see that $2x - y = 0 \implies 2x = y$. Thus, setting $x = 1$, then we get the corresponding eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Similarly for $\lambda_2 = 2$, we find

$$\text{Nul}(A - 1I) = \left[\begin{array}{cc|c} 3-(2) & -2 & 0 \\ 2 & -2-(2) & 0 \end{array}\right]$$
$$= \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

Then using the row operations

(a) $R_2 = 1/2R_2$

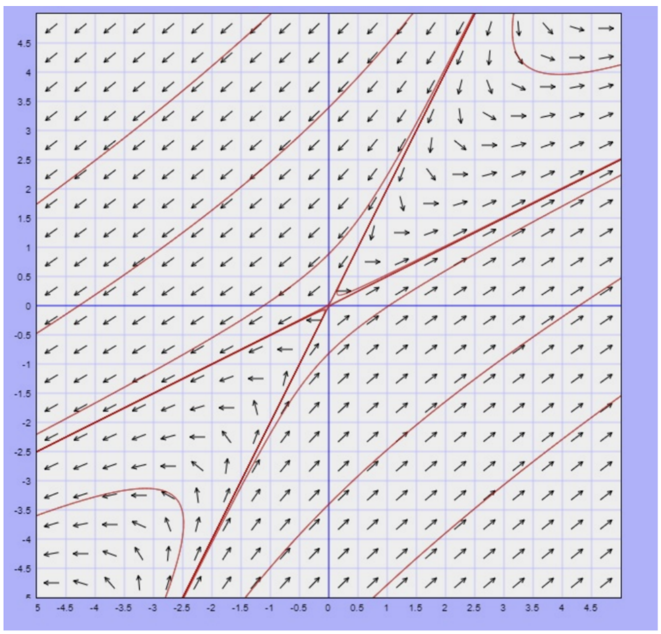
(b) $R_2 = R_2 - R_1$

Then we see that $x - 2y = 0 \implies x = 2y$. Thus, setting $x = 2$, then we get the corresponding eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Finally, we obtain the solution

$$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

and the phase portrait should resemble the following plot, but should be done by hand:



4. Solve $y'' - 5y' + 6y = 0$ by writing it as a system $x' = Ax$ and solving that system. Do NOT use another method to solve this.

Solution: Since we have two derivatices, we use two varilables and find

$$x_1 = y \implies x_1' = x_2$$
$$x_2 = y' \implies x_2' = 5x_2 - 6x_1.$$

Thus, we arrive at the system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We start by finding the eigenvalues of the matrix. We see that

$$\begin{aligned} \det |A - \lambda I| =& \det \begin{bmatrix} 0 - \lambda & 1 \\ -6 & 5 - \lambda \end{bmatrix} \\ =& -5\lambda + \lambda^2 - 6 \\ =& \lambda^2 - 5\lambda - 6 \\ =& (\lambda - 3)(\lambda - 2) \end{aligned}$$

Thus, we have the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$.

Next, we find the corresponding eigenvectors

$$\begin{aligned} \text{Nul}(A - 3I) =& \left[\begin{array}{cc|c} 3 & -1 & 0 \\ -6 & 5 - (3) & 0 \end{array} \right] \\ =& \left[\begin{array}{cc|c} 3 & -1 & 0 \\ -6 & 2 & 0 \end{array} \right] \end{aligned}$$

Then using the row operations

(a) $R_2 = -1/2R_2$

(b) $R_2 = R_2 - R_1$

Then we see that $3x - y = 0 \implies 3x = y$. Thus, setting $x = 1$, then we get the corresponding eigenvector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Similarly for $\lambda_2 = 2$, we find

$$\begin{aligned} \text{Nul}(A - 2I) =& \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -6 & 5 - (2) & 0 \end{array} \right] \\ =& \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -6 & 3 & 0 \end{array} \right] \end{aligned}$$

Then using the row operations

(a) $R_2 = -1/3R_2$

(b) $R_2 = R_2 - R_1$

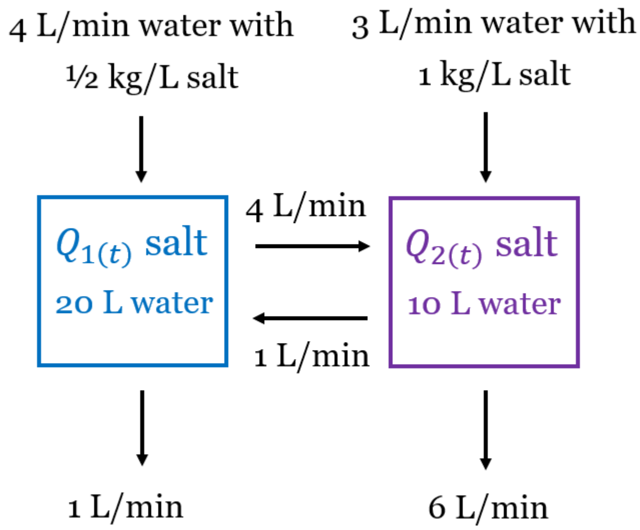
Then we see that $2x - y = 0 \implies 2x = y$. Thus, setting $x = 1$, then we get the corresponding eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Finally, we obtain the solution

$$x(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Since $x_1(t) = y(t) \implies y(t) = c_1 e^{3t} + c_2 e^{2t}$.

5. Consider the following system of interconnected tanks that have an inflow and outflow of salt-water mixture.



Set up but do NOT solve a system of ODEs of the form

$$Q'(t) = AQ(t) + b$$

where $Q(t) = \begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix}$ with $Q_1(t)$ as the amount of salt in tank 1 and $Q_2(t)$ as the amount of salt in tank 2 and b as a constant vector.

Note: For simplicity, assume that the amount of water in each tank is constant.

Hint: For each tank, carefully think about how much salt goes in/out and whether that amount depends on Q_1 or Q_2 or not. It might help to think in terms of units, you want kg/min everywhere.

Solution: We can write out

$$Q_1^{'} = 4[\text{L/min}] \cdot 1/2[\text{kg/L}] = -5[\text{L/min}] \cdot \frac{Q_1}{V_1}[\text{kg/L}] + 1[\text{L/min}] \cdot \frac{Q_2}{V_2}[\text{kg/L}]$$

Thus, since $V_1 = 20$ and $V_2 = 10$, we find

$$Q_1' = 2 - \frac{Q_1}{4} + \frac{Q_2}{10}.$$

Similarly for Q_2, we find

$$Q_2' = 3 - \frac{7Q_2}{10} + \frac{Q_1}{5}.$$

Therefore, our system can be written as

$$\begin{bmatrix} Q_1' \\ Q_2' \end{bmatrix} = \begin{bmatrix} -1/4 & 1/10 \\ 1/5 & -7/10 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$