# APMA 0350 − Homework 9 − Solutions

Problem 1: (a) **Finding Eigenvalues:**

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 - (-4)1 = 0$$

$$\implies \lambda^2 + 2\lambda + 5 = 0$$

By the quadratic formula, the roots of this equation are

$$\lambda = -1 \pm 2i$$

**Finding Eigenvectors:**
Consider $\lambda = -1 + 2i$.

$$\text{Nul}(A - \lambda I) = \left[\begin{array}{cc|c} -2i & -4 & 0 \\ 1 & -2i & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -2i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so

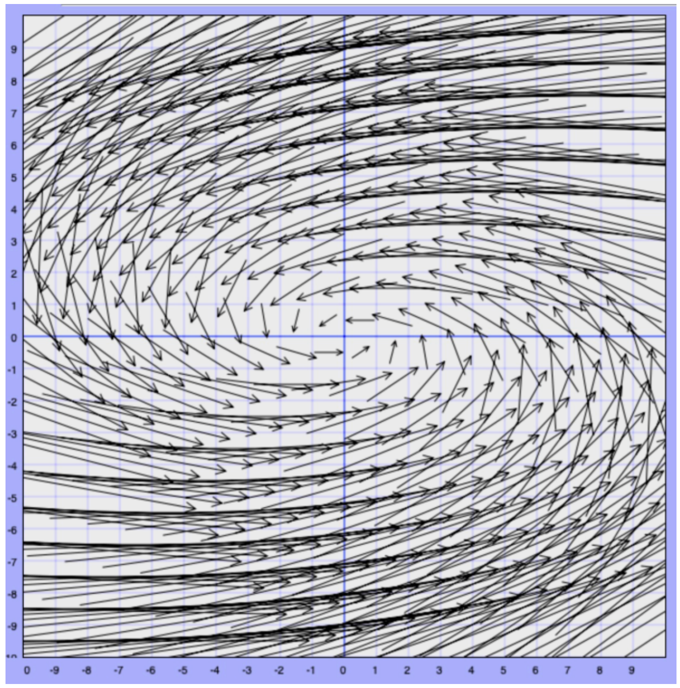
$$\mathbf{v} = \begin{bmatrix} 2 \\ -i \end{bmatrix} = \mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

is an eigenvector. The general solution is then

$$\mathbf{x}(t) = C_1 e^{-t} \left( \cos(2t) \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \sin(2t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

$$+ C_2 e^{-t} \left( \cos(2t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin(2t) \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right).$$

**Phase Portrait:**



(b) **Finding Eigenvalues:**

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 1(-1) = 0$$

$$\implies (\lambda - 2)^2 = 0,$$

so we have a repeated root at $\lambda = 2$.

**Finding Eigenvectors:**

$$\text{Nul}(A - 2I) = \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so

$$\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

is an eigenvector. To find the second term, we solve

$$(A - 2I)\mathbf{w} = \mathbf{v}$$

$$\implies \left[\begin{array}{cc|c} -1 & -1 & -1 \\ 1 & 1 & 1 \end{array}\right]$$

$$= \begin{bmatrix} -1 & -1 & \big| & -1 \\ 0 & 0 & \big| & 0 \end{bmatrix}. \qquad (R_2 = R_2 - R_1)$$

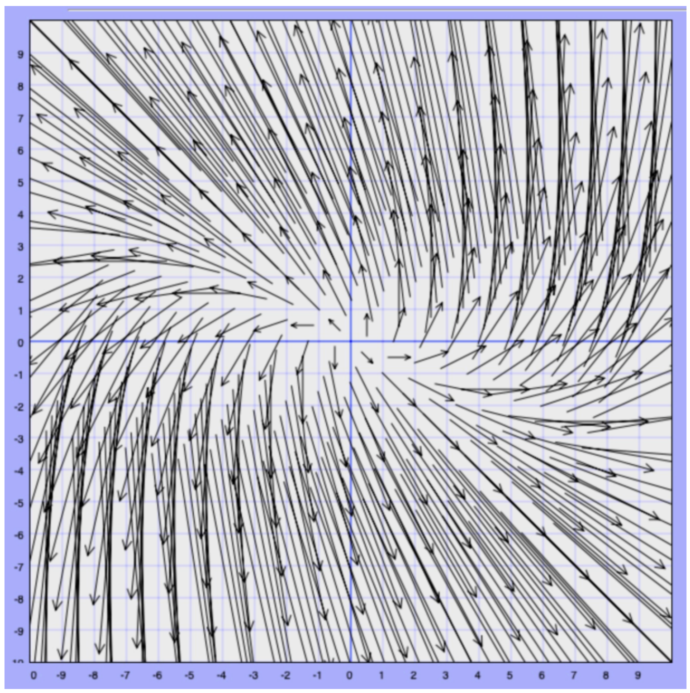
This is solved by

$$\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The general solution is then

$$\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{2t} \left( t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

**Phase Portrait:**




Problem 2: (a) **Finding Eigenvalues:**

$$|A - \lambda I| = \begin{vmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = (-3-\lambda)(-1-\lambda) - 2(-1) = 0$$

$$\implies \lambda^2 + 4\lambda + 5 = 0.$$

By the quadratic formula, the roots of this equation are

$$\lambda = -2 \pm i.$$

**Finding Eigenvectors:**

Consider $\lambda = -2 + i$.

$$\text{Nul}(A - \lambda I) = \left[\begin{array}{cc|c} -1-i & 2 & 0 \\ -1 & 1-i & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -1 & 1-i & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so

$$\mathbf{v} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

is an eigenvector. The general solution is then

$$\mathbf{x}(t) = C_1 e^{-2t} \left( \cos(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$

$$+ C_2 e^{-2t} \left( \cos(t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right).$$

**Using the Initial Condition:**

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\implies C_1 - C_2 = 1,$$

$$C_1 = -2$$

$$\implies C_2 = -3$$

Thus the solution is

$$\mathbf{x}(t) = -2e^{-2t} \left( \cos(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$

$$-3e^{-2t} \left( \cos(t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

(b) **Finding Eigenvalues:**

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -2 \\ 8 & -5-\lambda \end{vmatrix} = (3-\lambda)(-5-\lambda) - (-2)(8) = 0$$

$$\implies \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0,$$

so we have a repeated root at $\lambda = -1$.

**Finding Eigenvectors:**

$$\text{Nul}(A + I) = \left[\begin{array}{cc|c} 4 & -2 & 0 \\ 8 & -4 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is an eigenvector. To find the second term, we solve

$$(A + I)\mathbf{v} = \mathbf{w}$$

$$\Longrightarrow \left[\begin{array}{cc|c} 4 & -2 & 1 \\ 8 & -4 & 2 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cc|c} 4 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right], \qquad (R_2 = R_2 - 2R_1)$$

which is solved by

$$\mathbf{w} = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}.$$

The general solution is then

$$\mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-t} \left( t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} \right).$$

**Using the Initial Condition:**

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$$

$$\Longrightarrow C_1 = 2,$$

$$2C_1 - 1/2C_2 = 2$$

$$\Longrightarrow C_2 = 4C_1 - 4 = 4$$

Thus the solution is

$$\mathbf{x}(t) = 2e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4e^{-t} \left( t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} \right)$$

$$= e^{-t} \begin{bmatrix} 4t + 2 \\ 8t + 2 \end{bmatrix}.$$

Problem 3: Finding the eigenvalues, we find

$$\begin{aligned} \det(A - \lambda I) = & \det \begin{bmatrix} 2 - \lambda & -1 \\ 3 & -2 - \lambda \end{bmatrix} \\ = & (2 - \lambda)(-2 - \lambda) - (-1)(3) \\ = & \lambda^2 - 1 = 0 \end{aligned}$$

Then, $\lambda_1 = 1$ and $\lambda_2 = -1$.

For the eigenvectors, we find

$$\begin{bmatrix} 2-1 & -1 \\ 3 & -2-1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $d_1 - d_2 = 0 \implies d_1 = d_2$. Setting $d_1 = 1 \implies d_2 = 1$. Therefore, the eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For $\lambda_2$, we find

$$\begin{bmatrix} -2+1 & -1 \\ 3 & -2+1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, $3d_1 - d_2 = - \implies 3d_1 = d_2$ if we set $d_2 = 3 \implies d_1 = 1$. Therefore, the eigenvector is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. So, we find

$$x_0 = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

With variation of parameters, we find

$$x_p = u(t) \begin{bmatrix} e^t \\ e^t \end{bmatrix} + v(t) \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}$$

With the Var of Par equations, we find

$$\underbrace{\begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}}_{B} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$

Then $\det(B) = (e^t)(3e^{-t}) = (e^t)(e^{-t}) = 3 - 1 = 2.$

Therefore,

$$\begin{aligned}
u'(t) =& \frac{\det \begin{bmatrix} e^t & e^{-t} \\ -e^t & 3e^{-t} \end{bmatrix}}{2} \\
=& \frac{(e^t)(3e^{-t}) - (e^{-t})(-e^t)}{2} \\
=& \frac{3+2}{2} \\
=& 2
\end{aligned}$$

and

$$\begin{aligned}
u'(t) =& \frac{\det \begin{bmatrix} e^t & e^t \\ e^t & -e^t \end{bmatrix}}{2} \\
=& \frac{(-e^t)(e^t) - (e^t)(e^t)}{2} \\
=& \frac{-e^{2t} - e^{2t}}{2} \\
=& \frac{-2e^{2t}}{2} \\
=& -e^{2t}.
\end{aligned}$$

Therefore, $u(t) = 2\int dt = 25$ and $v(t) = -\int e^{2t} = \frac{-e^{2t}}{2}$. And

$$\begin{aligned}
x_p =& 2t \begin{bmatrix} e^t \\ e^t \end{bmatrix} - \frac{e^{2t}}{2} \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix} \\
=& \begin{bmatrix} 2te^t - \frac{e^t}{2} \\ 2te^t - \frac{3}{2}e^t \end{bmatrix} \\
=& e^t \begin{bmatrix} 2t - \frac{1}{2} \\ 2t - \frac{3}{2} \end{bmatrix}.
\end{aligned}$$

Therefore, the general solution is

$$x(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + e^t \begin{bmatrix} 2t - \frac{1}{2} \\ 2t - \frac{3}{2} \end{bmatrix}$$