LECTURE: REPEATED EIGENVALUES

Today: The next case to consider is repeated eigenvalues

1. Repeated Eigenvalues

Example 1:

Solve $\mathbf{x}' = A\mathbf{x}$ and draw the phase portrait, where

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

STEP 1: Eigenvalues

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} \\ &= (1-\lambda)(3-\lambda) - (1)(-1) \\ &= 3 - \lambda - 3\lambda + \lambda^2 + 1 \\ &= \lambda^2 - 4\lambda + 4 \\ &= (\lambda - 2)^2 \end{aligned}$$

Which gives $\lambda = 2$ (*repeated* eigenvalue)

STEP 2: $\lambda = 2$

$$\text{Nul }(A - 2I) = \left[\begin{array}{cc|c} 1-2 & 1 & 0 \\ -1 & 3-2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-x + y = 0 \Rightarrow y = x \text{ and therefore } \mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 2 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

OH NO!!! There is just one eigenvector, what do we do now?

First Guess: $\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ but there should be a C_2 there

Second Guess: $\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 t e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ but this is **WRONG**

STEP 3:

Trick: Instead of solving $(A - 2I)\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ solve

$$(A - 2I)\mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow \text{ Eigenvector}$$

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow -x + y = 1 \Rightarrow y = 1 + x \text{ and so}$$

$$\mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 1 + x \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

STEP 4: Correct Solution (see below why)

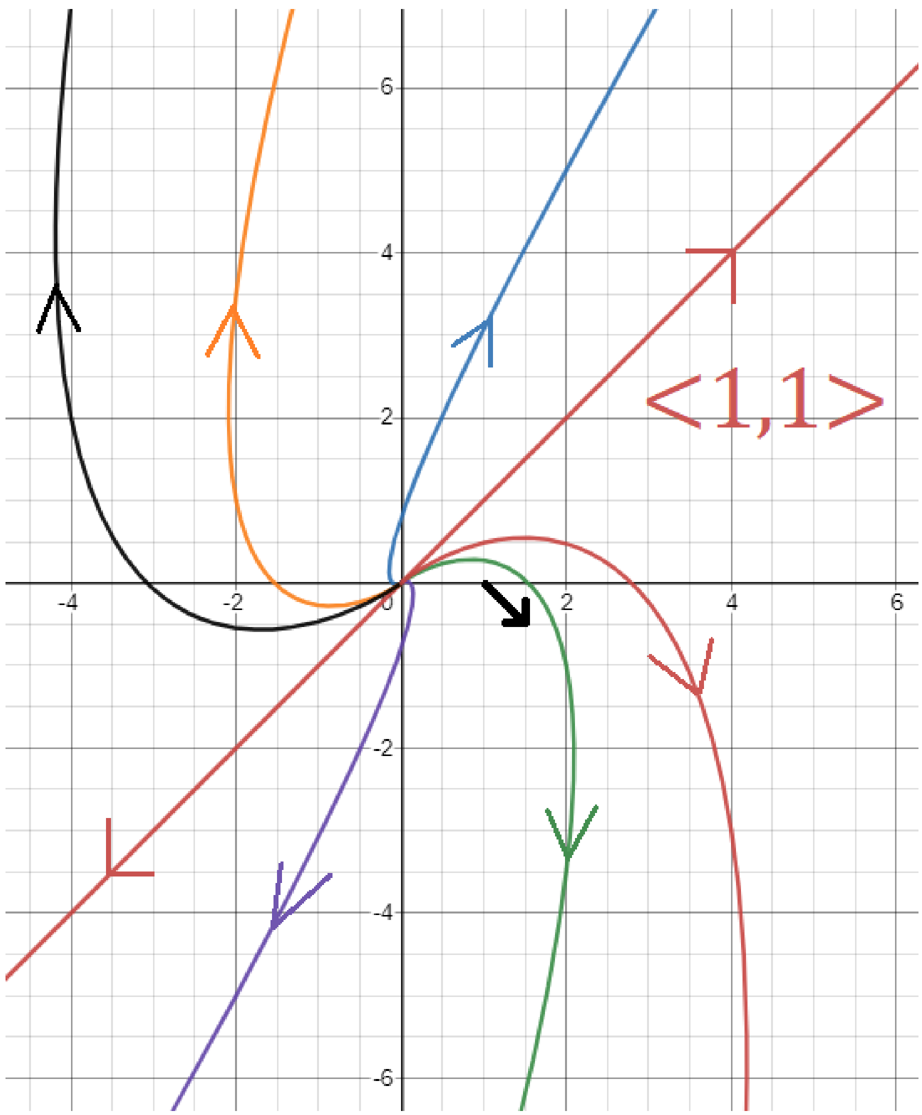
Fact:

$$\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \left(t e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Note: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is called a **generalized eigenvector** and is a great substitute when not enough eigenvectors are available.

Warning: While it is ok to rescale eigenvectors, like $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \overset{\times 2}{\rightsquigarrow} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ do **NOT** rescale **generalized** eigenvectors, don't turn $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

STEP 5: Phase portrait:



How to draw the phase portrait:

- The main axis is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- Because of the e^{2t} term, solutions on that axis *move away* from the origin.
- The other solutions curve outwards and eventually they become parallel to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ because the te^{2t} term is much bigger than the other e^{2t} terms

Note: The $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ vector plays no role in the phase portrait.

Note: One way to check whether the picture is correct is to pick any initial condition, say $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (any other non-eigenvector is fine too) and then by the ODE, we have

$$\mathbf{x}'(0) = A\mathbf{x}(0) = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \mathbf{x}'(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore at the point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ the solutions move in the $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ direction. This is illustrated in the picture with the black arrow that moves in the southeast direction.

2. Why this works

Let's see why we need to solve $(A - 2I)\mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Since it is not enough to assume that $\mathbf{x}(t) = te^{2t}\begin{bmatrix}1\\1\end{bmatrix}$ let's suppose

$$\mathbf{x}(t) = te^{2t}\begin{bmatrix}1\\1\end{bmatrix} + e^{2t}\mathbf{w} \qquad \mathbf{w} \text{ TBA}$$

To find an equation for $\mathbf{w}$, plug into the ODE

$$\begin{aligned}
\mathbf{x}' &= A\mathbf{x} \\
\left(te^{2t}\begin{bmatrix}1\\1\end{bmatrix} + e^{2t}\mathbf{w}\right)' &= A\left(te^{2t}\begin{bmatrix}1\\1\end{bmatrix} + e^{2t}\mathbf{w}\right) \\
\left(te^{2t}\right)'\begin{bmatrix}1\\1\end{bmatrix} + \left(e^{2t}\right)'\mathbf{w} &= te^{2t}A\begin{bmatrix}1\\1\end{bmatrix} + e^{2t}\mathbf{w} \\
e^{2t}\begin{bmatrix}1\\1\end{bmatrix} + \cancel{te^{2t}\,2\begin{bmatrix}1\\1\end{bmatrix}} + 2e^{2t}\mathbf{w} &= \cancel{te^{2t}\,2\begin{bmatrix}1\\1\end{bmatrix}} + e^{2t}A\mathbf{w}
\end{aligned}$$

Here we used $A\begin{bmatrix}1\\1\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix}$ since $\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = 2$

We therefore get:

$$\begin{aligned}
\cancel{e^{2t}}\begin{bmatrix}1\\1\end{bmatrix} + 2\cancel{e^{2t}}\mathbf{w} &= \cancel{e^{2t}}A\mathbf{w} \\
\begin{bmatrix}1\\1\end{bmatrix} + 2\mathbf{w} &= A\mathbf{w} \\
A\mathbf{w} - 2\mathbf{w} &= \begin{bmatrix}1\\1\end{bmatrix} \\
(A - 2I)\mathbf{w} &= \begin{bmatrix}1\\1\end{bmatrix} \checkmark
\end{aligned}$$

Therefore $\mathbf{w}$ has to be a generalized eigenvector of A corresponding to $\lambda = 2$

Note: For a more direct way of finding $\mathbf{x}(t)$ you can use the "matrix exponential" e^{At} which is the matrix analog of the exponential function e^{at}. This is beyond the scope of this lecture.

3. Initial Conditions

Example 2: (more practice)

Solve $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ where $A = \begin{bmatrix} -1 & -1 \\ 4 & -5 \end{bmatrix}$

STEP 1: Eigenvalues

$$\begin{aligned} |A - \lambda I| = & \begin{vmatrix} -1-\lambda & -1 \\ 4 & -5-\lambda \end{vmatrix} \\ = & (-1-\lambda)(-5-\lambda) - (-1)(4) \\ = & 5 + \lambda + 5\lambda + \lambda^2 + 4 \\ = & \lambda^2 + 6\lambda + 9 \\ = & (\lambda+3)^2 \end{aligned}$$

Which gives $\lambda = -3$

STEP 2: $\lambda = -3$

$$\text{Nul } (A - (-3)I) = \left[\begin{array}{cc|c} -1-(-3) & -1 & 0 \\ 4 & -5-(-3) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 4 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$2x - y = 0 \Rightarrow y = 2x \text{ and so}$$

$$\lambda = -3 \rightsquigarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

STEP 3: Generalized Eigenvector

$$(A - (-3)I)\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 4 & -2 & 2 \end{array}\right] \xrightarrow{(\div 2)R_2} \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 2 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $2x - y = 1 \Rightarrow y = 2x - 1$ and so

$$\mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x - 1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

WARNING: Do **NOT** rescale this to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$!!!

STEP 4: Solution

$$\mathbf{x}(t) = C_1 e^{-3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \left(t e^{-3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-3t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

(The phase portrait would be like the previous example, but with the arrows reversed)

STEP 5: Initial Condition

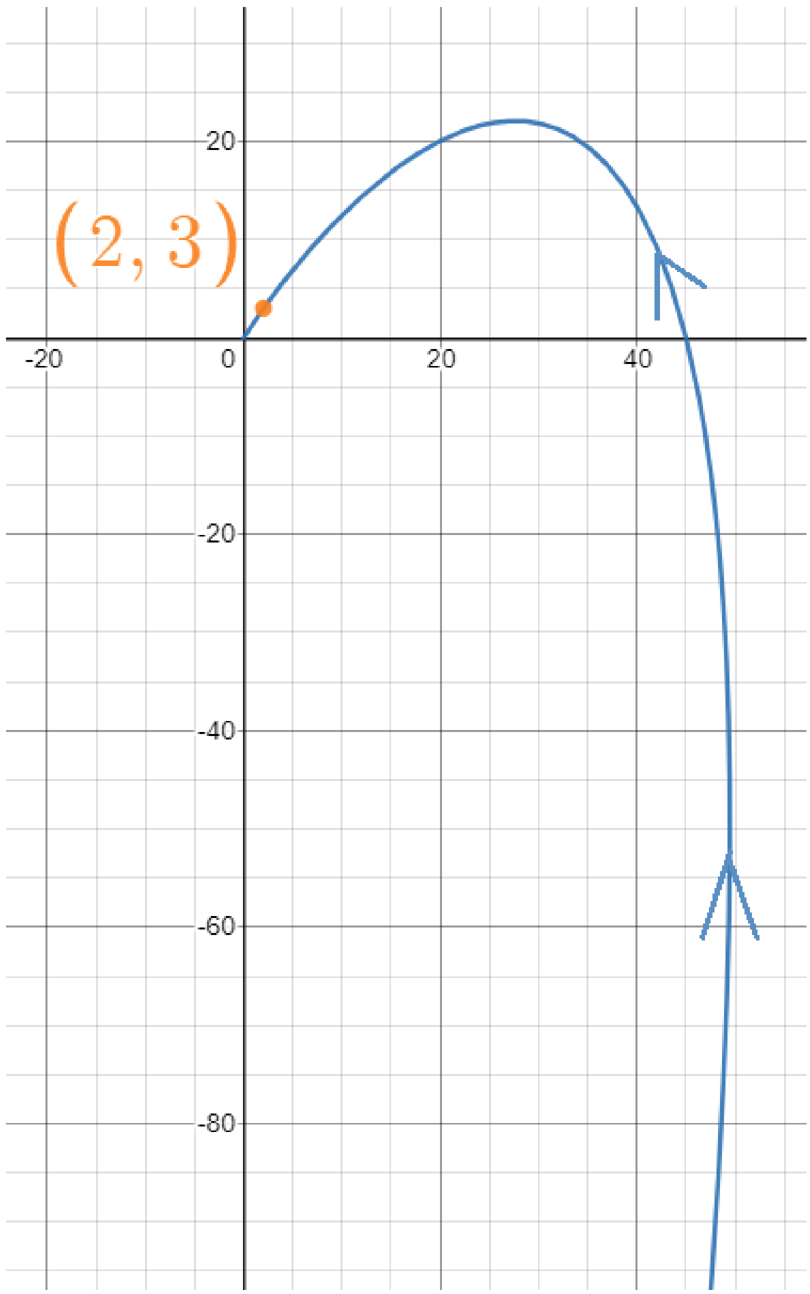
$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$C_1 e^0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \left(0 e^0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^0 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{cases} C_1 = 2 \\ 2C_1 - C_2 = 3 \end{cases} \Rightarrow \begin{cases} C_1 = 2 \\ C_2 = 2C_1 - 3 = 2(2) - 3 = 1 \end{cases} \Rightarrow \begin{cases} C_1 = 2 \\ C_2 = 1 \end{cases}$$

$$\mathbf{x}(t) = 2e^{-3t}\begin{bmatrix}1\\2\end{bmatrix} + 1\left(te^{-3t}\begin{bmatrix}1\\2\end{bmatrix} + e^{-3t}\begin{bmatrix}0\\-1\end{bmatrix}\right) = e^{-3t}\begin{bmatrix}2+t\\3+2t\end{bmatrix}$$



4. PPLANE APP

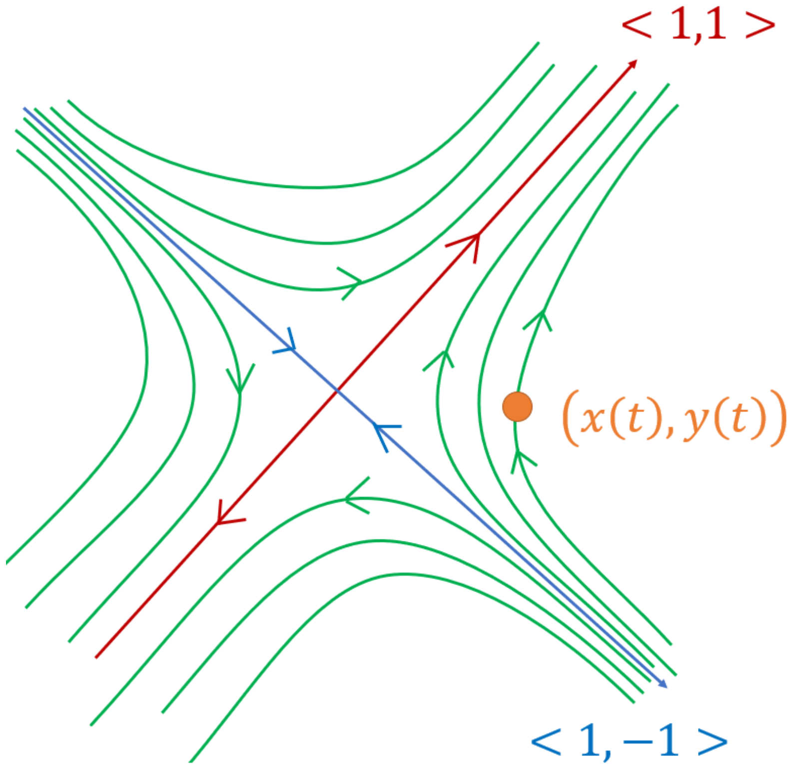
App: pplane app

Just like direction fields, in practice you draw phase portraits with the help of a computer.

Example 3:

$$\mathbf{x}' = A\mathbf{x} \text{ where } A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

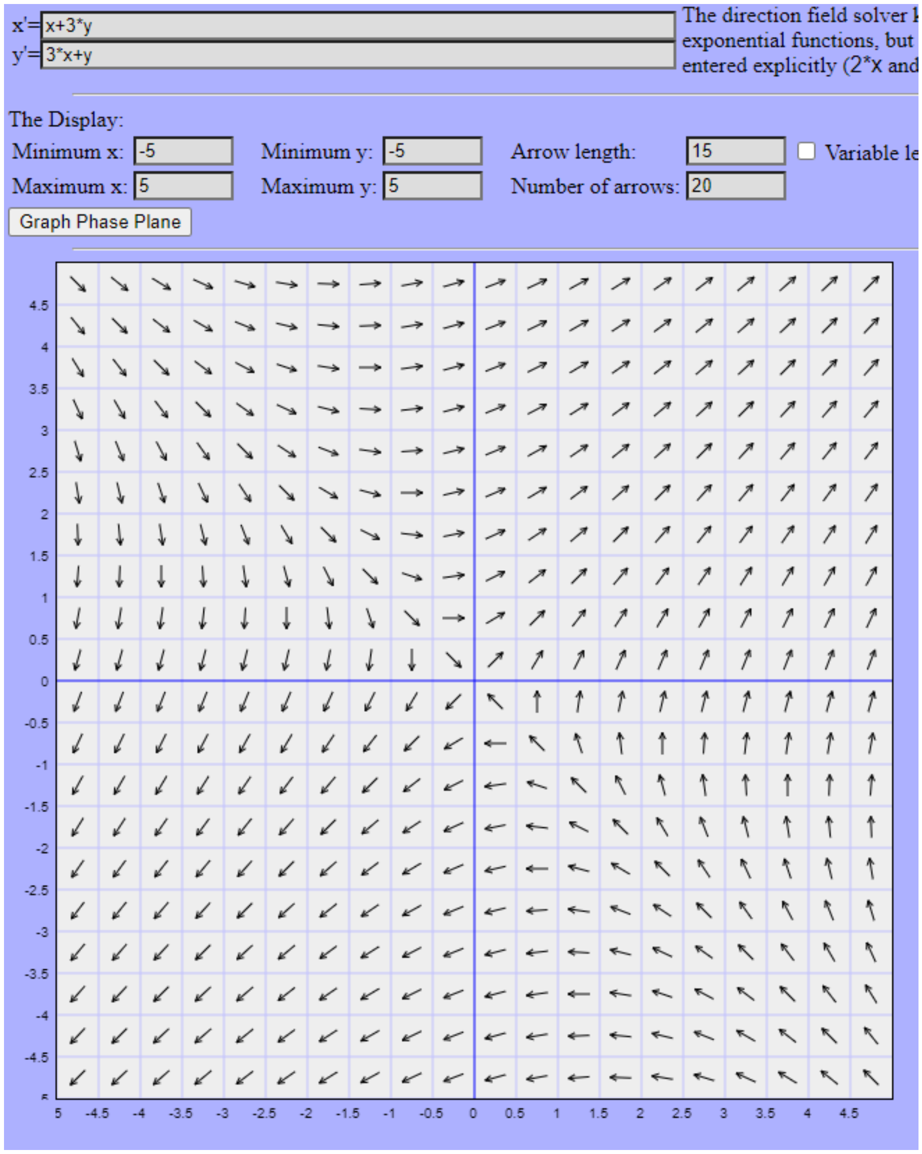
Recall that the solution was: $\mathbf{x}(t) = C_1 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



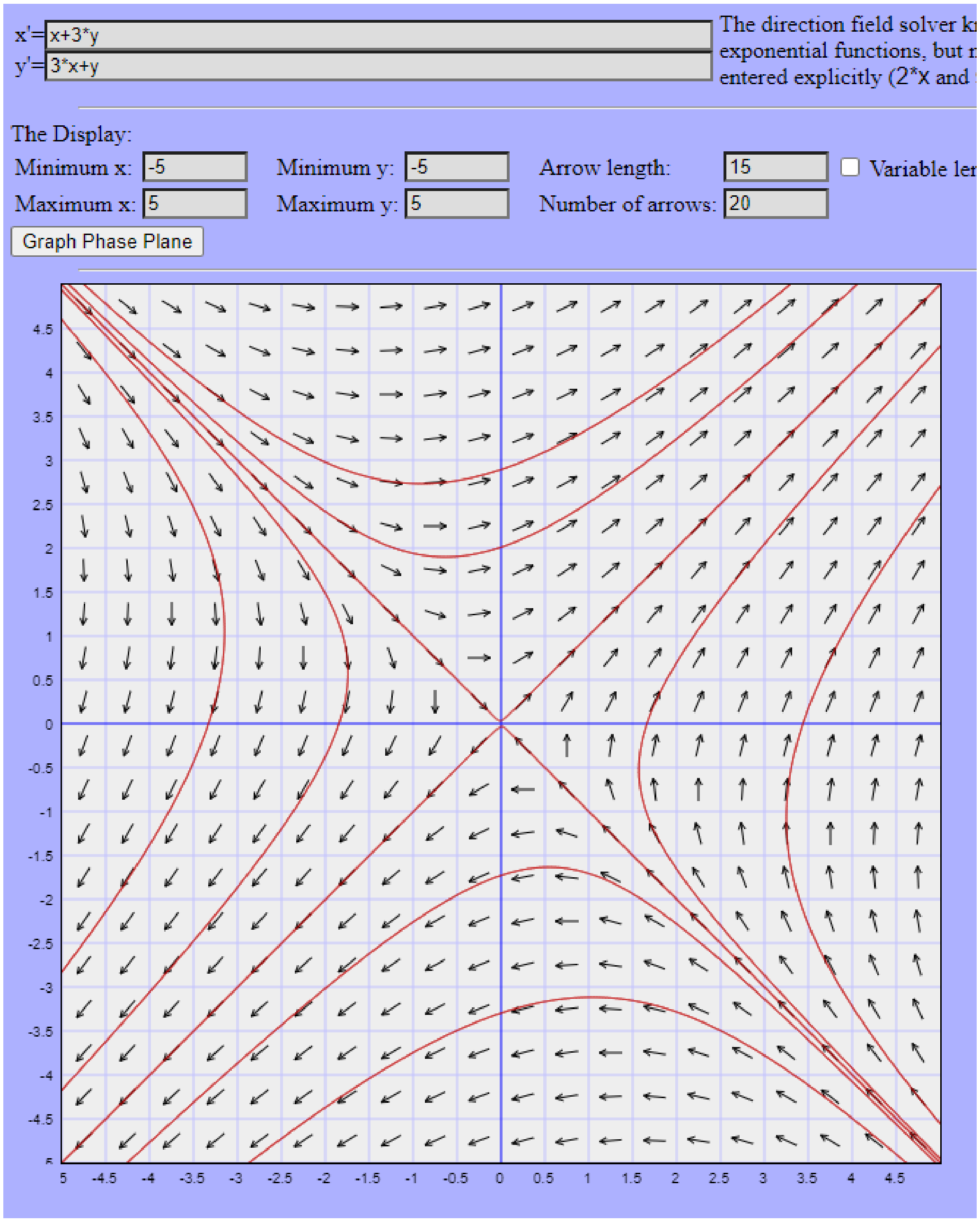
To plot it using the pplane app, you write this as

$$\begin{cases} x' = x + 3 \star y \\ y' = 3 \star x + y \end{cases}$$

The arrows tell you where the solutions are going.



And by clicking, you can plot a couple of trajectories to get a general idea of what the solutions look like. You can even click on the axes, provided that you know where they are.


x'= x+3*y
y'= 3*x+y
The direction field solver k
exponential functions, but r
entered explicitly (2^x and
The Display:
Minimum x: -5
Minimum y: -5
Arrow length: 15
Variable le
Maximum x: 5
Maximum y: 5
Number of arrows: 20
Graph Phase Plane

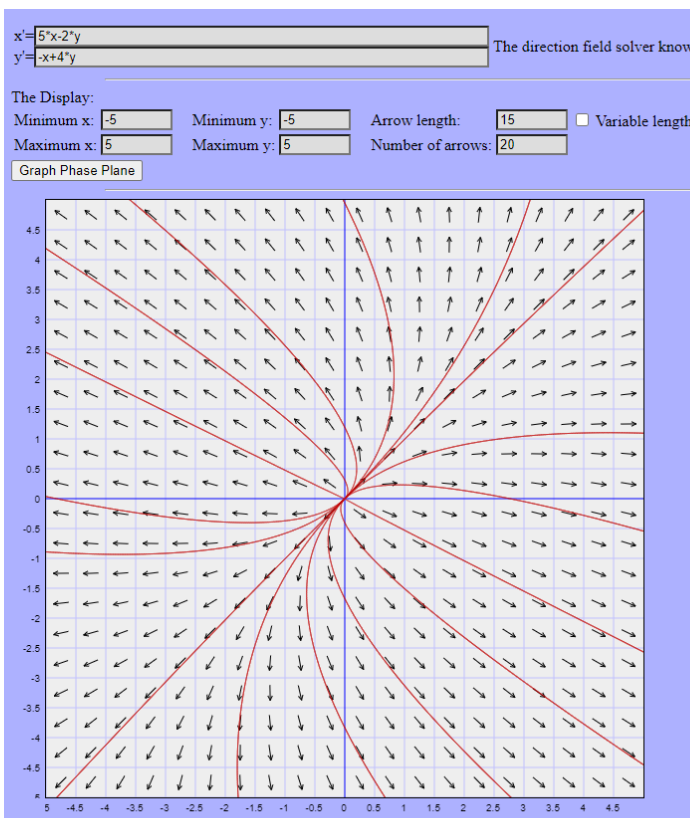
Example 4:

$$\mathbf{x}' = A\mathbf{x} \text{ where } A = \begin{bmatrix} 5 & -2 \\ -1 & 4 \end{bmatrix}$$

$$\lambda = 3 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda = 6 \rightsquigarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\mathbf{x}(t) = C_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Here the solutions move **away** from both axes.

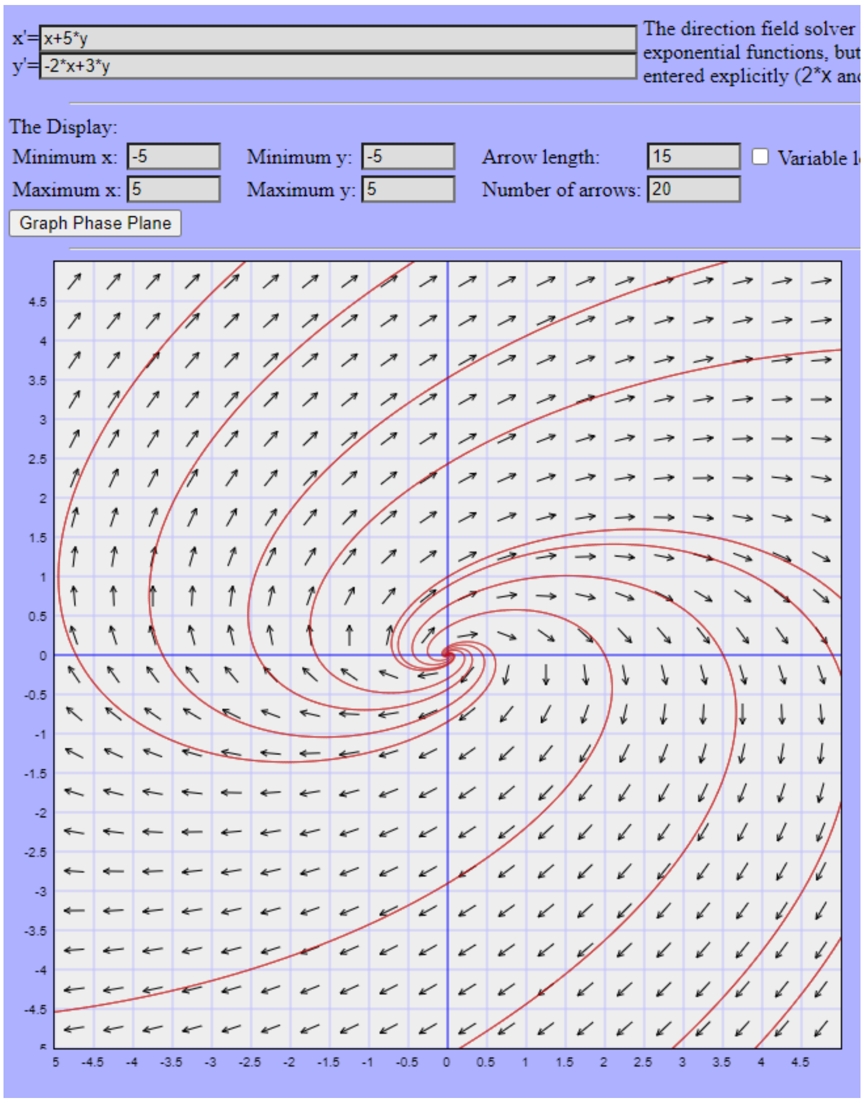


(Opposite scenario if both eigenvalues are negative)

Example 5:

$$\mathbf{x}' = A\mathbf{x} \text{ where } A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$$

As before, we saw that the solutions are spiraling away

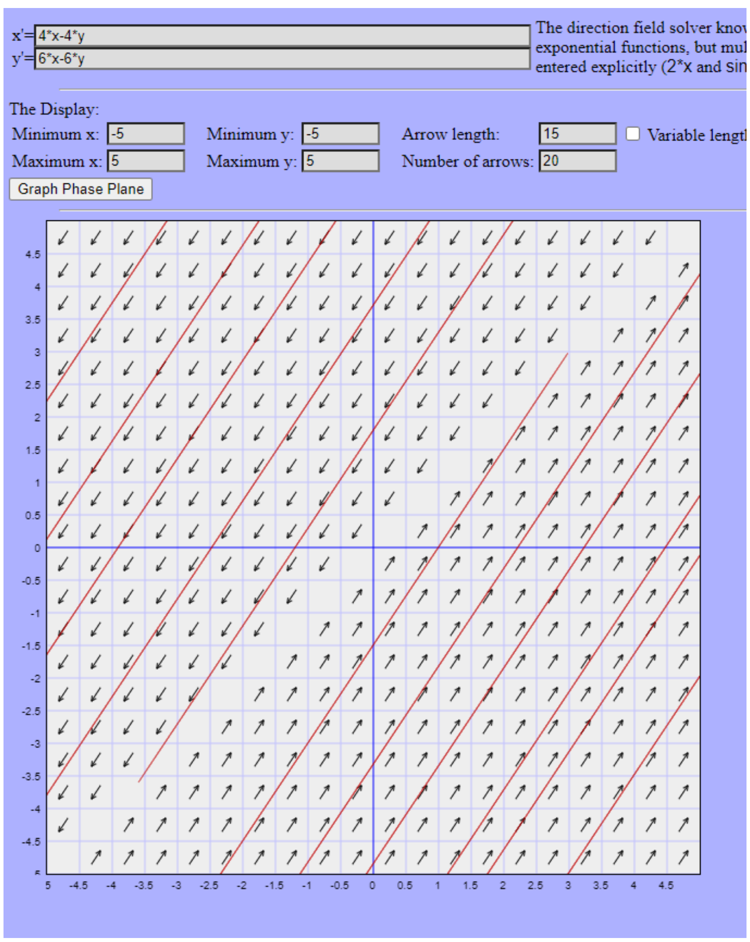


Example 6:

$$\mathbf{x}' = A\mathbf{x} \text{ where } A = \begin{bmatrix} 4 & -4 \\ 6 & -6 \end{bmatrix}$$

$$\lambda = 0 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \lambda = -2 \rightsquigarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



The solutions are lines parallel to $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Notice the change in arrows, which is when C_2 changes from negative to positive. In the region parallel to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ the solutions are just points (where $C_2 = 0$)