LECTURE: FINAL EXAM – REVIEW

1. Nonlinear ODE

Example 1:

Find and classify the equilibrium points of the following system

$$\begin{cases} x' = y \\ y' = -\sin(x) \end{cases}$$

STEP 1: Equilibrium Points:

$$\begin{cases} y = 0 \\ -\sin(x) = 0 \end{cases}$$

This gives $y = 0$ and $\sin(x) = 0$ so $x = \pi m$ and $y = 0$

Equilibrium points: $(\pi m, 0)$

STEP 2: Classification:

$$\nabla F(x, y) = \begin{bmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial(-\sin(x))}{\partial x} & \frac{\partial(-\sin(x))}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cos(x) & 0 \end{bmatrix}$$

$$\nabla F(\pi m, 0) = \begin{bmatrix} 0 & 1 \\ -\cos(\pi m) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(-1)^m & 0 \end{bmatrix}$$

Eigenvalues:

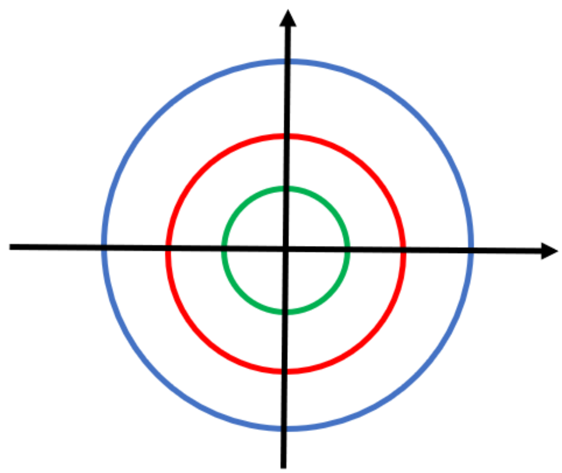
$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -(-1)^m & -\lambda \end{vmatrix} = \lambda^2 + (-1)^m = 0$$

Case 1: m even

Then $(-1)^m = 1$ and so we get $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

In that case $(\pi m, 0)$ is neither stable, unstable, or a saddle

Aside: This is called a center since the solutions are circles/ellipses

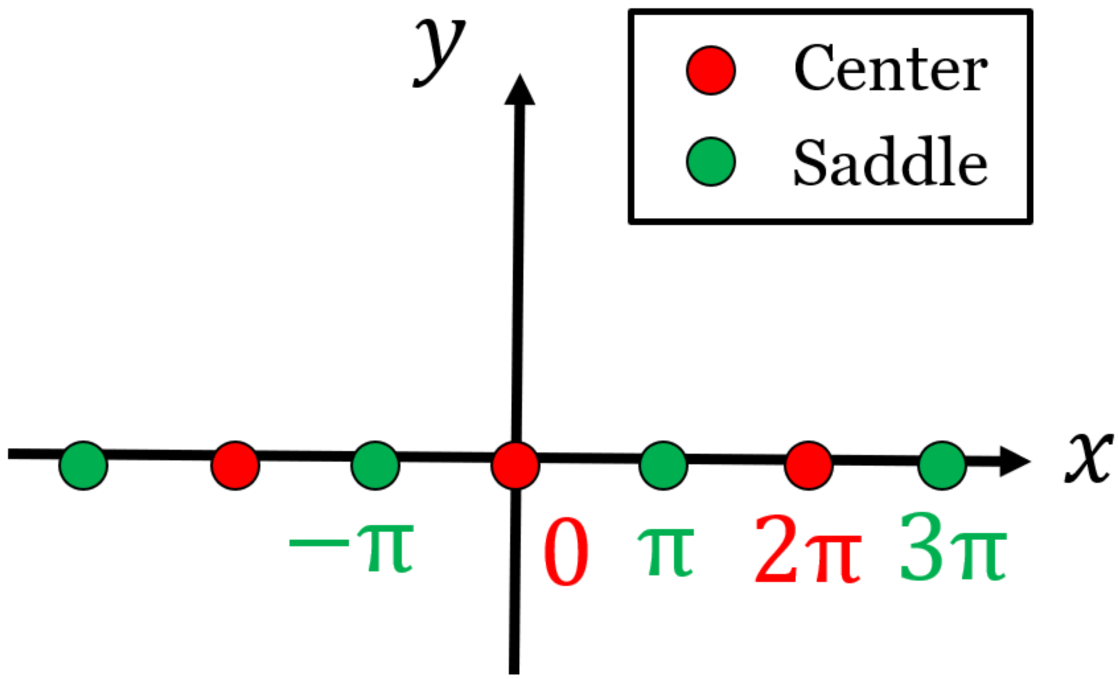


Case 2: m odd

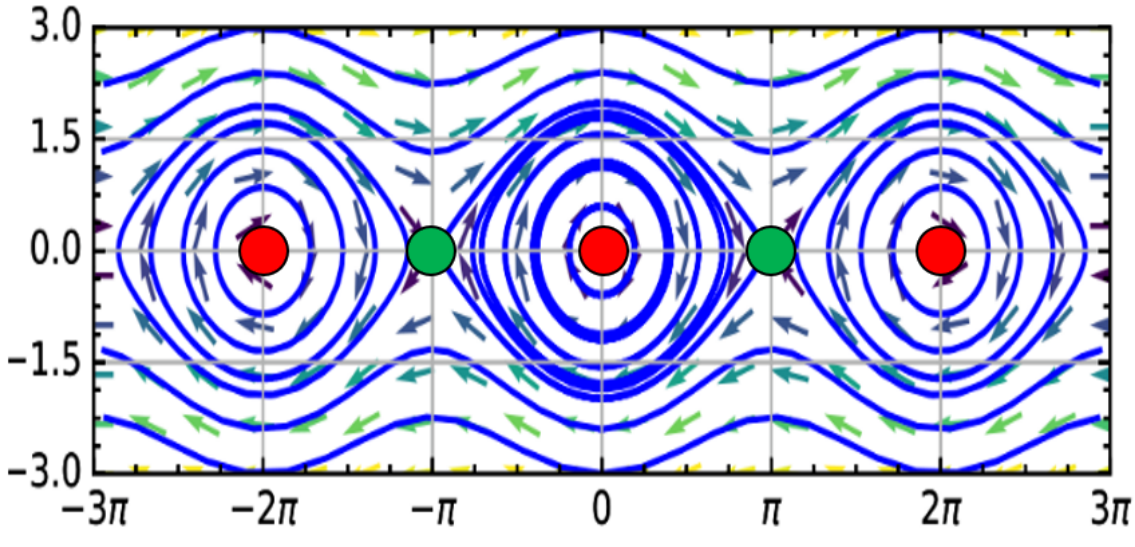
Then $(-1)^m = -1$ and so $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

In that case $(\pi m, 0)$ is a saddle

$(\pi m, 0)$ is a center if m is even and a saddle if m is odd



Application: The ODE above are in fact the equations of a pendulum, and we can see in the picture below how the equilbrium points alternate between saddles and centers



Example 2: (more practice)

Find and classify the equilibrium points of

$$\begin{cases} x' = (x-1)^2 + y \\ y' = x^2 + y \end{cases}$$

STEP 1: Equlibrium Points: Set $x' = 0$ and $y' = 0$

$$\begin{cases} (x-1)^2 + y = 0 \\ x^2 + y = 0 \end{cases}$$

Subtracting the second equation from the first, we get

$$(x-1)^2 - x^2 = 0 \Rightarrow (x-1)^2 = x^2 \Rightarrow \cancel{x - 1 = x} \text{ or } x - 1 = -x \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$$

And from the second equation, $y = -x^2 = -\left(\frac{1}{2}\right)^2 = -\frac{1}{4}$

Equlibrium point: $\left(\frac{1}{2}, -\frac{1}{4}\right)$

STEP 2: Stability

$$\nabla F(x,y) = \begin{bmatrix} 2(x-1) & 1 \\ 2x & 1 \end{bmatrix}$$

$$\nabla F\left(\frac{1}{2}, \frac{1}{4}\right) = \begin{bmatrix} 2\left(\frac{1}{2} - 1\right) & 1 \\ 2\left(\frac{1}{2}\right) & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = A$$

Eigenvalues:

$$\begin{aligned} \left|A - \lambda I\right| &= \begin{vmatrix} -1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (-1-\lambda)(1-\lambda) - 1 = -1 + \cancel{\lambda} - \cancel{\lambda} + \lambda^2 - 1 \\ &= \lambda^2 - 2 = 0 \end{aligned}$$

Which gives $\lambda = \pm\sqrt{2}$. This is both positive and negative, so

Conclusion: $\left(\frac{1}{2}, -\frac{1}{4}\right)$ is a **saddle**

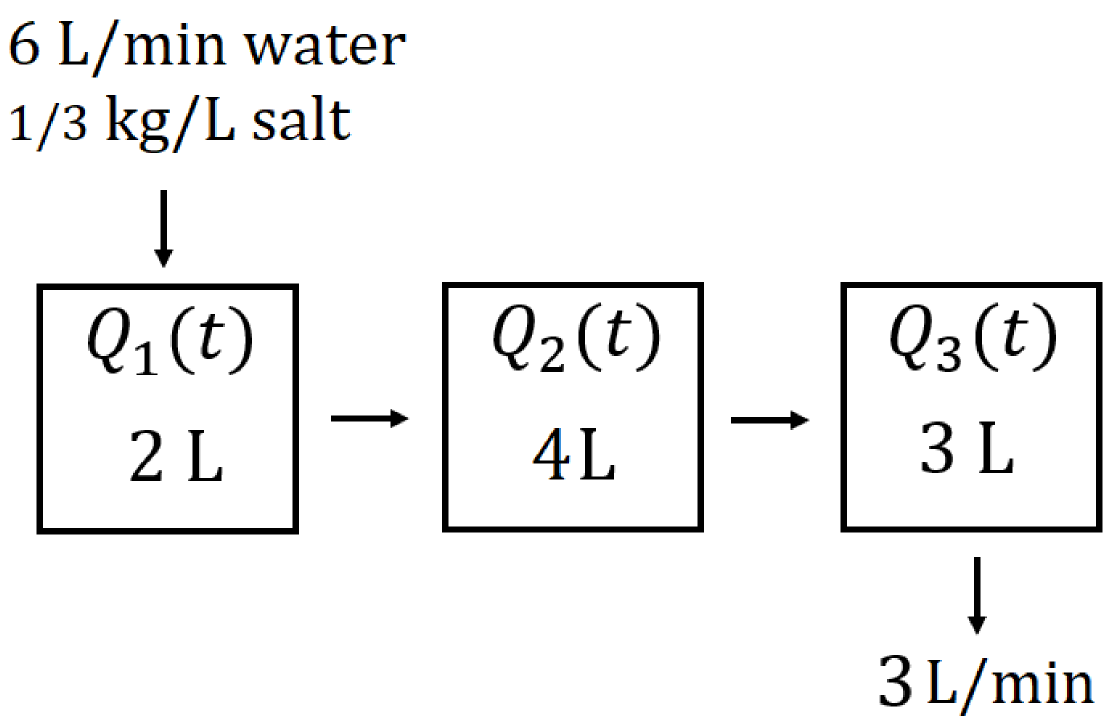
2. Chemical Tanks

Example 3:

Consider the following configuration of chemical tanks. Assume the two arrows in the middle are 4 L/min and the amount of water in each tank is constant. Set up a system

$$\mathbf{Q}'(t) = A\mathbf{Q}(t) + \mathbf{b}$$

Where $Q_i(t)$ is the amount of salt in tank i (in kg)



No matter what the configuration, always think "What is going in and what is going out?"

Tank 1:

$$Q_1'(t) = \text{ In } - \text{ Out } = 6 \times \left(\frac{1}{3}\right) - 4 \times \left(\frac{Q_1}{2}\right) = 2 - 2Q_1(t)$$

Tank 2:

$$Q_2'(t) = \text{ In } - \text{ Out } = 4 \times \left(\frac{Q_1}{2}\right) - 4 \times \left(\frac{Q_2}{4}\right) = 2Q_1(t) - Q_2(t)$$

Tank 3:

$$Q_3'(t) = \text{ In } - \text{ Out } = 4 \times \left(\frac{Q_2}{4}\right) - 3 \times \left(\frac{Q_3}{3}\right) = Q_2(t) - Q_3(t)$$

System:

$$\begin{cases} Q_1'(t) = -2Q_1(t) + 2 \\ Q_2'(t) = 2Q_1(t) - Q_2(t) \\ Q_3'(t) = Q_2(t) - Q_3(t) \end{cases}$$

This is of the form $\mathbf{Q}'(t) = A\mathbf{Q}(t) + \mathbf{b}$ where

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

3. Variation of Parameters

Example 4:

Use var of par to find a particular solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ where

$$A = \begin{bmatrix} 1/t & -1 \\ 1 & 1/t \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} t \\ -t \end{bmatrix}$$

Note: Simplify your final answer and write it as a single vector. Assume the general solution of the homogeneous equation is

$$\mathbf{x_0}(t) = C_1 \begin{bmatrix} t\sin(t) \\ -t\cos(t) \end{bmatrix} + C_2 \begin{bmatrix} t\cos(t) \\ t\sin(t) \end{bmatrix}$$

STEP 1:

$$\mathbf{x_p}(t) = u(t) \begin{bmatrix} t\sin(t) \\ -t\cos(t) \end{bmatrix} + v(t) \begin{bmatrix} t\cos(t) \\ t\sin(t) \end{bmatrix}$$

$$\begin{bmatrix} t\sin(t) & t\cos(t) \\ -t\cos(t) & t\sin(t) \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} t \\ -t \end{bmatrix}$$

STEP 2:

$$\text{Denominator: } \begin{vmatrix} t\sin(t) & t\cos(t) \\ -t\cos(t) & t\sin(t) \end{vmatrix} = t^2\sin^2(t) + t^2\cos^2(t) = t^2$$

$$u'(t) = \frac{\begin{vmatrix} t & t\cos(t) \\ -t & t\sin(t) \end{vmatrix}}{t^2} = \frac{t^2\sin(t) + t^2\cos(t)}{t^2} = \sin(t) + \cos(t)$$

$$v'(t) = \frac{\begin{vmatrix} t\sin(t) & t \\ -t\cos(t) & -t \end{vmatrix}}{t^2} = \frac{-t^2\sin(t) + t^2\cos(t)}{t^2} = -\sin(t) + \cos(t)$$

$$u(t) = \int \sin(t) + \cos(t)dt = -\cos(t) + \sin(t)$$
$$v(t) = \int -\sin(t) + \cos(t)dt = \cos(t) + \sin(t)$$

STEP 3:

$$\begin{aligned}
\mathbf{x_p}(t) =& (-\cos(t) + \sin(t)) \begin{bmatrix} t\sin(t) \\ -t\cos(t) \end{bmatrix} + (\cos(t) + \sin(t)) \begin{bmatrix} t\cos(t) \\ t\sin(t) \end{bmatrix} \\
=& \begin{bmatrix} (-\cos(t) + \sin(t))\, t\sin(t) + (\cos(t) + \sin(t))\, t\cos(t) \\ (-\cos(t) + \sin(t))\,(-t\cos(t)) + (\cos(t) + \sin(t))\, t\sin(t) \end{bmatrix} \\
=& \begin{bmatrix} -\cancel{t\cos(t)\sin(t)} + t\sin^2(t) + t\cos^2(t) + \cancel{t\cos(t)\sin(t)} \\ t\cos^2(t) - \cancel{t\sin(t)\cos(t)} + \cancel{t\cos(t)\sin(t)} + t\sin^2(t) \end{bmatrix} \\
=& \begin{bmatrix} t\left(\cos^2(t) + \sin^2(t)\right) \\ t\left(\cos^2(t) + \sin^2(t)\right) \end{bmatrix} \\
=& \begin{bmatrix} t \\ t \end{bmatrix}
\end{aligned}$$

4. Repeated Eigenvalues

Example 5:

Solve $\mathbf{x}' = A\mathbf{x}$ where
$$A = \begin{bmatrix} 6 & -1 \\ 4 & 2 \end{bmatrix}$$

STEP 1: Eigenvalues

$$\begin{aligned}|A-\lambda I| &= \begin{vmatrix} 6-\lambda & -1 \\ 4 & 2-\lambda \end{vmatrix} \\ &= (6-\lambda)(2-\lambda)-(-1)(4) \\ &= 12-6\lambda-2\lambda+\lambda^2+4 \\ &= \lambda^2-8\lambda+16 \\ &= (\lambda-4)^2\end{aligned}$$

Hence $\lambda = 4$ (repeated)

STEP 2:

$\boxed{\lambda = 4}$

$$\text{Nul }(A-4I) = \left[\begin{array}{cc|c} 6-4 & -1 & 0 \\ 4 & 2-4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 4 & -2 & 0 \end{array}\right] \xrightarrow{(\div 2)R_2} \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$2x - y = 0$ so $y = 2x$

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

STEP 3: $(A-4I)\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 4 & -2 & 2 \end{array}\right] \xrightarrow{(\div 2)R_2} \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 2 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$2x - y = 1 \Rightarrow y = 2x - 1$

$$\mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x-1 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\mathbf{x}(t) = C_1 e^{4t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2\left(te^{4t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{4t}\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

Note: Make sure that the two vectors in blue match (the one in $(A-4I)\mathbf{w} = \begin{bmatrix}1\\2\end{bmatrix}$ and the one in $te^{4t}\begin{bmatrix}1\\2\end{bmatrix}$) Do not change this to $\begin{bmatrix}2\\4\end{bmatrix}$ for example. This is because in the proof of repeated eigenvalues, we assumed our solution is of the form $te^{4t}\begin{bmatrix}1\\2\end{bmatrix} + e^{4t}\mathbf{w}$ and then we found that $\mathbf{w}$ solves $(A-4I)\mathbf{w} = \begin{bmatrix}1\\2\end{bmatrix}$

5. Phase Portraits

Example 6:

Solve $\mathbf{x}' = A\mathbf{x}$ and draw a phase portrait, where

$$A = \begin{bmatrix}12 & -14\\ 4 & -3\end{bmatrix}$$

STEP 1: Eigenvalues

$$\begin{aligned}
\left|A-\lambda I\right| &= \begin{vmatrix}12-\lambda & -14\\ 4 & -3-\lambda\end{vmatrix}\\
&=(12-\lambda)(-3-\lambda)-(-14)(4)\\
&=-36-12\lambda+3\lambda+\lambda^2+56\\
&=\lambda^2-9\lambda+20\\
&=(\lambda-4)(\lambda-5)
\end{aligned}$$

Which gives $\lambda = 4$ or $\lambda = 5$

STEP 2: Eigenvectors

$\boxed{\lambda = 4}$

$$\text{Nul }(A - 4I) = \left[\begin{array}{cc|c} 12-4 & -14 & 0 \\ 4 & -3-4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 8 & -14 & 0 \\ 4 & -7 & 0 \end{array}\right] \xrightarrow{(\div 2)R_1} \left[\begin{array}{cc|c} 4 & -7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$4x - 7y = 0$, so $x = 7$ and $y = 4$ works

$$\lambda = 4 \rightsquigarrow \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$\boxed{\lambda = 5}$

$$\text{Nul }(A - 5I) = \left[\begin{array}{cc|c} 12-5 & -14 & 0 \\ 4 & -3-5 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 7 & -14 & 0 \\ 4 & -8 & 0 \end{array}\right] \xrightarrow{(\div 7)R_1\ (\div 4)R_2} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x - 2y = 0$ so $x = 2y$ and

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ y \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

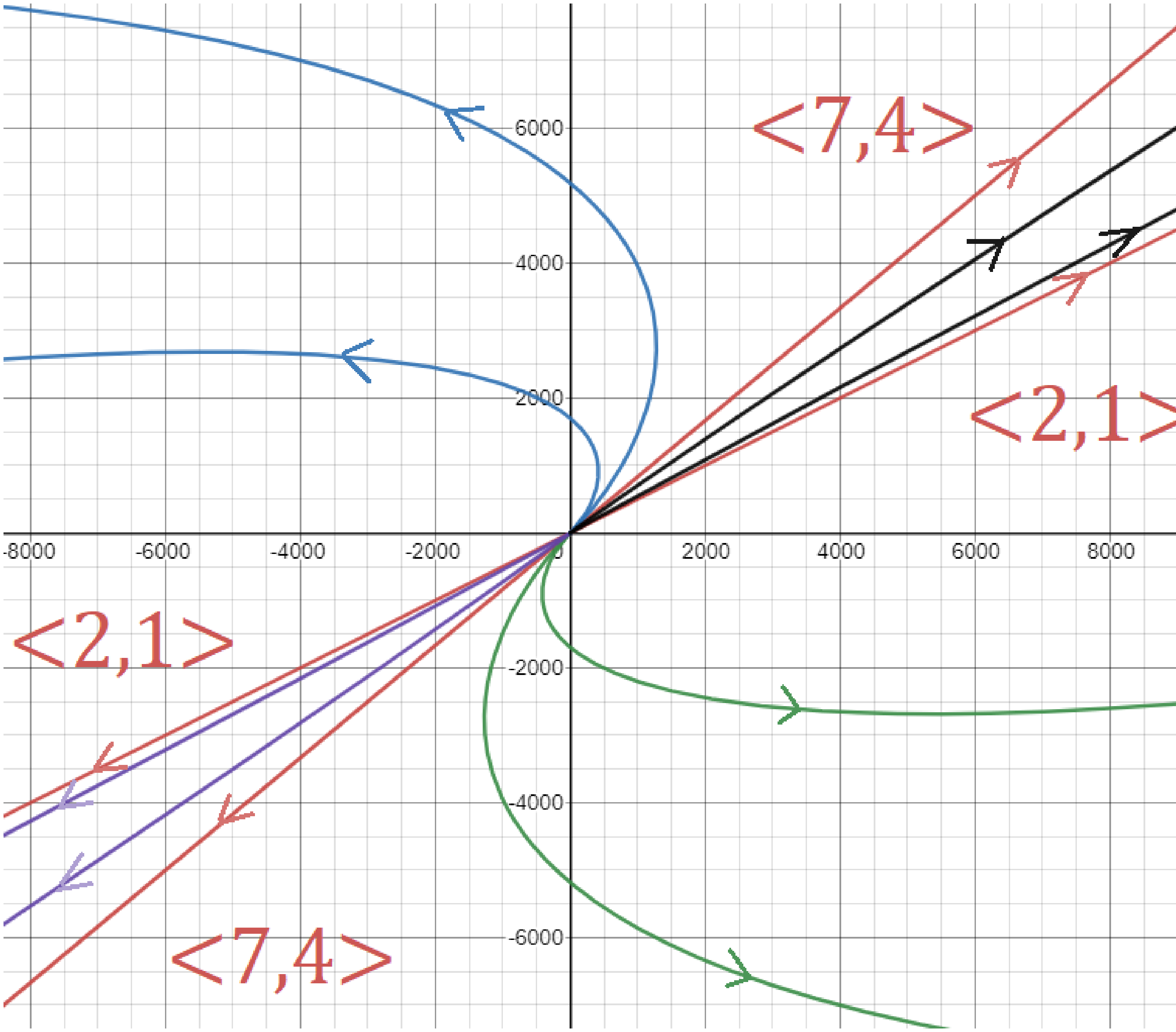
$$\lambda = 5 \rightsquigarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

STEP 3: Solution

$$\mathbf{x}(t) = C_1 e^{4t} \begin{bmatrix} 7 \\ 4 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

STEP 4: Phase Portrait

Here $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$ is steeper than $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ because $\frac{4}{7} > \frac{1}{2}$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-4000
-6000
-8000
-6000
-4000
-2000
0
2000
4000
6000
8000

Example 7:

Solve $\mathbf{x}' = A\mathbf{x}$ and draw the phase portrait, where

$$A = \begin{bmatrix} 3 & 4 \\ -1 & 3 \end{bmatrix}$$

STEP 1: Eigenvalues

$$\begin{aligned} \left|A - \lambda I\right| = & \begin{vmatrix} 3-\lambda & 4 \\ -1 & 3-\lambda \end{vmatrix} \\ = & (3-\lambda)^2 - 4(-1) \\ = & (\lambda-3)^2 + 4 = 0 \end{aligned}$$

$$(\lambda-3)^2 = -4 \Rightarrow \lambda - 3 = \pm 2i \Rightarrow \lambda = 3 \pm 2i$$

STEP 2: $\lambda = 3 + 2i$

$$\begin{aligned} \text{Nul } (A - (3+2i)I) = & \left[\begin{array}{cc|c} 3-(3+2i) & 4 & 0 \\ -1 & 3-(3+2i) & 0 \end{array}\right] \\ = & \left[\begin{array}{cc|c} -2i & 4 & 0 \\ -1 & -2i & 0 \end{array}\right] \\ \longrightarrow & \left[\begin{array}{cc|c} -1 & -2i & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2i & 0 \\ 0 & 0 & 0 \end{array}\right] \end{aligned}$$

Hence $x + (2i)y = 0$. For example $x = -2i$ and $y = 1$ satisfies this

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

STEP 3: Solution

$$e^{(3+2i)t} \begin{bmatrix} -2i \\ 1 \end{bmatrix} = e^{3t} \left(\cos(2t) + i\sin(2t)\right) \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} - i \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right)$$

$$\begin{aligned}\mathbf{x}(t) =& C_1 e^{3t}\left(\cos(2t)\begin{bmatrix}0\\1\end{bmatrix} + \sin(2t)\begin{bmatrix}2\\0\end{bmatrix}\right) + C_2 e^{3t}\left(-\cos(2t)\begin{bmatrix}2\\0\end{bmatrix} + \sin(2t)\begin{bmatrix}0\\1\end{bmatrix}\right)\\ =& C_1 e^{3t}\begin{bmatrix}2\sin(2t)\\ \cos(2t)\end{bmatrix} + C_2 e^{3t}\begin{bmatrix}-2\cos(2t)\\ \sin(2t)\end{bmatrix}\end{aligned}$$

STEP 4: Phase Portrait

Because of the e^{3t} term, the solution is spiraling outwards.