APMA 0350 – MIDTERM 2 – SOLUTIONS

1. **STEP 1: Auxiliary Equation**

$$r^2 + 9 = 0 \Rightarrow r = \pm 3i$$

STEP 2: Particular Solution

$3\sin(3t)$ corresponds to $r = \pm 3i$ which coincides with one of the roots of the auxiliary equation.

Hence there is resonance and we must guess

$$y_p = At\cos(3t) + Bt\sin(3t)$$

$$(y_p)' = A\cos(3t) - 3At\sin(3t) + B\sin(3t) + 3Bt\cos(3t)$$

$$\begin{aligned}(y_p)'' &= -3A\sin(3t) - 3A\sin(3t) - 9At\cos(3t) + 3B\cos(3t) + 3B\cos(3t) - 9Bt\sin(3t) \\ &= -6A\sin(3t) - 9At\cos(3t) + 6B\cos(3t) - 9Bt\sin(3t)\end{aligned}$$

$$(y_p)'' + 9(y_p) = 3\sin(3t)$$

$$-6A\sin(3t) - 9At\cos(3t) + 6B\cos(3t) - 9Bt\sin(3t) + 9(At\cos(3t) + Bt\sin(3t)) = 3\sin(2t)$$

$$-6A\sin(3t) - \cancel{9At\cos(3t)} + 6B\cos(3t) - \cancel{9Bt\sin(3t)} + \cancel{9At\cos(3t)} + \cancel{9Bt\sin(3t)} = 3\sin(3t)$$

$$-6A\sin(3t) + 6B\cos(3t) = 3\sin(3t) + 0\cos(3t)$$

$$\begin{cases} -6A = 3 \\ 6B = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{2} \\ B = 0 \end{cases}$$

$$y_p = -\frac{t}{2}\cos(3t)$$

2. **STEP 0: Standard Form** ✓

STEP 1: Homogeneous Solution

$$\textbf{Aux } r^2 + 9 = 0 \Rightarrow r = \pm 3i$$

$$y_0 = A\cos(3t) + B\sin(3t)$$

STEP 2: Var of Par

$$y_p(t) = u(t)\cos(3t) + v(t)\sin(3t)$$

$$\begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\cos(3t) & 3\sin(3t) \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 3\sin(3t) \end{bmatrix}$$

Denominator:

$$\begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = 3\cos^2(3t) + 3\sin^2(3t) = 3$$

Using Cramer's rule, we get

$$u'(t) = \frac{\begin{vmatrix} 0 & \sin(3t) \\ 3\sin(3t) & 3\cos(3t) \end{vmatrix}}{3} = \frac{0 - 3\sin^2(3t)}{3} = -\sin^2(3t)$$

$$v'(t) = \frac{\begin{vmatrix} \cos(3t) & 0 \\ -3\sin(3t) & 3\sin(3t) \end{vmatrix}}{3} = \frac{3\cos(3t)\sin(3t) - 0}{3} = \cos(3t)\sin(3t)$$

$$u(t) = \int -\sin^2(3t)dt = \int -\frac{1}{2} + \frac{1}{2}\cos(6t)dt = -\frac{t}{2} + \frac{1}{12}\sin(6t)$$

$$v(t) = \int \cos(3t)\sin(3t)dt = \int \frac{1}{2}\sin(6t)dt = -\frac{1}{12}\cos(6t)$$

$$y_p(t) = \left(-\frac{t}{2} + \frac{1}{12}\sin(6t)\right)\cos(3t) - \frac{1}{12}\cos(6t)\sin(3t)$$

Note: You could simplify this to $-\frac{t}{2}\cos(3t) + \frac{1}{12}\sin(3t)$ using $\sin(a)\cos(b) - \cos(a)\sin(b) = \sin(a-b)$ but that wasn't required

3. **STEP 1: Write f in terms of u_c**

Start at 3

At $t = 4$ jump by $1 - 3 = -2$

At $t = 8$ jump by $5 - 1 = 4$

$$f(t) = 3 - 2u_4(t) + 4u_8(t)$$

$$\mathcal{L}\{f(t)\} = \left(\frac{3}{s}\right) - 2\left(\frac{e^{-4s}}{s}\right) + 4\left(\frac{e^{-8s}}{s}\right)$$

Laplace Transforms

$$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} = 10\,\mathcal{L}\{f(t)\}$$

$$\left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) + 6(s\mathcal{L}\{y\} - y(0)) + 10\mathcal{L}\{y\} = 10\left(\left(\frac{3}{s}\right) - 2\left(\frac{e^{-4s}}{s}\right) + 4\left(\frac{e^{-8s}}{s}\right)\right)$$

$$\left(s^2 + 6s + 10\right)\mathcal{L}\{y\} = \left(\frac{10}{s}\right)\left(3 - 2e^{-4s} + 4e^{-8s}\right)$$

$$\mathcal{L}\{y\} = \frac{10}{s\left(s^2 + 6s + 10\right)}\left(3 - 2e^{-4s} + 4e^{-8s}\right)$$

STEP 2: Partial Fractions: From the note, we get

$$\frac{10}{s\left(s^2 + 6s + 10\right)} = \left(\frac{1}{s}\right) - \left(\frac{s+6}{s^2 + 6s + 10}\right)$$

STEP 3:

We know $\frac{1}{s} = \mathcal{L}\{1\}$ and also

$$\frac{s+6}{s^2+6s+10} = \frac{s+6}{(s+3)^2 - 9 + 10} = \frac{s+6}{(s+3)^2 + 1} = \frac{(s+3)+3}{(s+3)^2+1} = \frac{s+3}{(s+3)^2+1} + \frac{3}{(s+3)^2+1}$$

This is a shifted version by -3 units of

$$\frac{s}{s^2+1}+\frac{3}{s^2+1}=\cos(t)+3\sin(t)$$

$$\text{Hence } \frac{s+6}{s^2+6s+10}=\mathcal{L}\left\{e^{-3t}\left(\cos(t)+3\sin(t)\right)\right\}$$

$$\frac{10}{s\left(s^2+6s+10\right)}=\left(\frac{1}{s}\right)-\left(\frac{s+6}{s^2+6s+10}\right)=\mathcal{L}\left\{1-e^{-3t}\left(\cos(t)-3\sin(t)\right)\right\}$$

STEP 4:

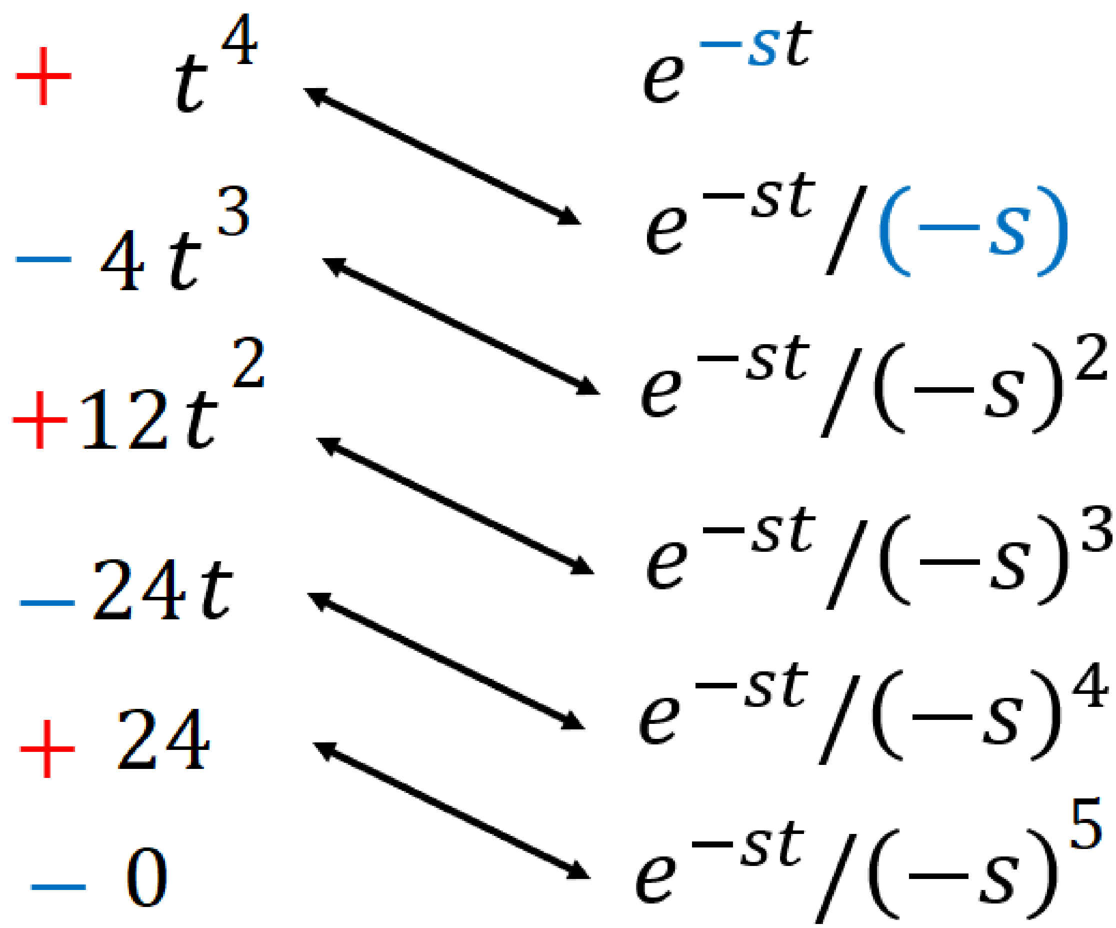
$$\begin{aligned}\mathcal{L}\{y\}=&\frac{10}{s\left(s^2+6s+10\right)}\left(3-2e^{-4s}+4e^{-8s}\right)\\ =&\mathcal{L}\left\{1-e^{-3t}\cos(t)+3e^{-3t}\sin(t)\right\}\left(3-2e^{-4s}+4e^{-8s}\right)\\ =&\mathcal{L}\{h(t)\}\left(3-2e^{-4s}+4e^{-8s}\right)\\ =&\mathcal{L}\{3h(t)-2h(t-4)u_4(t)+4h(t-8)u_8(t)\}\end{aligned}$$

STEP 5: Answer:

$$\begin{aligned}y(t)=&3h(t)-2h(t-4)u_4(t)+4h(t-8)u_8(t) \text{ where}\\ h(t)=&1-e^{-3t}\cos(t)-3e^{-3t}\sin(t)\end{aligned}$$

4.

$$\mathcal{L}\left\{t^4\right\} = \int_0^\infty t^4 e^{-st} dt$$



Using tabular integration, this integral equals

$$= \left[+t^4\left(\frac{e^{-st}}{(-s)}\right) - 4t^3\left(\frac{e^{-st}}{(-s)^2}\right) + 12t^2\left(\frac{e^{-st}}{(-s)^3}\right) - 24t\left(\frac{e^{-st}}{(-s)^4}\right) + 24\left(\frac{e^{-st}}{(-s)^5}\right)\right]_{t=0}^{t=\infty}$$

$$= (0 - 0 + 0 - 0 + 0)$$

$$- \left(0\left(\frac{1}{(-s)}\right) - 0\left(\frac{1}{(-s)^2}\right) + 0\left(\frac{1}{(-s)^3}\right) - 0\left(\frac{1}{(-s)^4}\right) + 24\left(\frac{1}{(-s)^5}\right)\right)$$

$$= -\frac{24}{(-s)^5}$$

$$= -\left(\frac{24}{-\left(s^5\right)}\right)$$

$$= \frac{24}{s^5}$$