APMA 1941G – FINAL EXAM

Problem 1: *(5 = 1 + 4 points)*

(a)

$$\text{Define: } f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n$$

(b) Use induction to show that asymptotic expansions are unique, in the sense that if

$$f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n \text{ and } f(\epsilon) \sim \sum_{n=0}^{\infty} b_n \epsilon^n$$

Then $a_i = b_i$ for all $i = 0, 1, 2, \cdots$

Problem 2: *(5 points)* Laplace's method states that if φ is a smooth function that has a global max at 0 with $\varphi(0) = 0$, $\varphi'(0) = 0$ and $\varphi''(0) < 0$, and a is any smooth function (not necessarily with compact support), then, as $\epsilon \to 0$

$$I[\epsilon] = \int_a^b a(x) e^{\frac{\varphi(x)}{\epsilon}} dx \sim \sqrt{\frac{2\pi\epsilon}{|\varphi''(0)|}} a(0) \left(1 + o(\epsilon)\right)$$

Use Laplace's method to find an asymptotic expansion of

$$\int_0^{\pi} x^2 \, e^{\frac{3+\sin^2(x)}{\epsilon}} \, dx$$

Problem 3: *(5 points)* Consider the following ODE, where $u^{\epsilon} = u^{\epsilon}(t)$

$$u_{\epsilon}'' + e^{2\epsilon t}\, u_{\epsilon} = 0$$

Apply the following WKB-Ansatz:

$$u^{\epsilon}(t) = u^{0}(\sigma^{\epsilon}(t), \epsilon t) + \epsilon\, u^{1}(\sigma^{\epsilon}(t), \epsilon t) + \cdots$$

Where $u^{k} = u^{k}(s, \tau)$ and $\sigma^{\epsilon} = \sigma^{\epsilon}(t)$ is to be selected and use this to find an *explicit* formula for $u^{0}(t) = u^{0}(\sigma^{\epsilon}(t), \epsilon t)$

Requirements: (don't forget to check them)

$$\sigma^{\epsilon}(0) = \frac{1}{\epsilon} \text{ and } \sigma_{\epsilon}'(t) = O(1) \text{ and } \sigma_{\epsilon}''(t) = O(\epsilon)$$

Problem 4: *(5 = 1 + 2 + 2 points)* Suppose that u is a minimizer of

$$I[v] = \int_{W} L(Dv, v, x)dx$$

where W is an open subset of $\mathbb{R}^n$ and $L = L(p, z, x)$

(a) What PDE must u satisfy? (just state the PDE)

(b) In the case $n = 1$, show how to derive the PDE in (a)

(c) Still in the case $n = 1$, notice that in theory maximizers of $I[u]$ also satisfy the same PDE! Show if u is a minimizer then u satisfies the following additional condition called **convexity** of L: For all v, we have

$$\int_{W} L_{pp}(u', u, x)(v')^2 + 2L_{pz}(u', u, x)v'v + L_{zz}(u', u, x)L_{zz}v^2 dx \geq 0$$

Hint: Use the second derivative test from calculus!

Problem 5: *(5 points)* Consider the following ODE on $(0,\pi)$ where $u^\epsilon = u^\epsilon(x)$ and $\alpha, \beta \in \mathbb{R}$

$$\begin{cases} \epsilon u^\epsilon_{xx} + u^\epsilon_x = \cos(x) \\ u^\epsilon(0) = \alpha \text{ and } u^\epsilon(\pi) = \beta \end{cases}$$

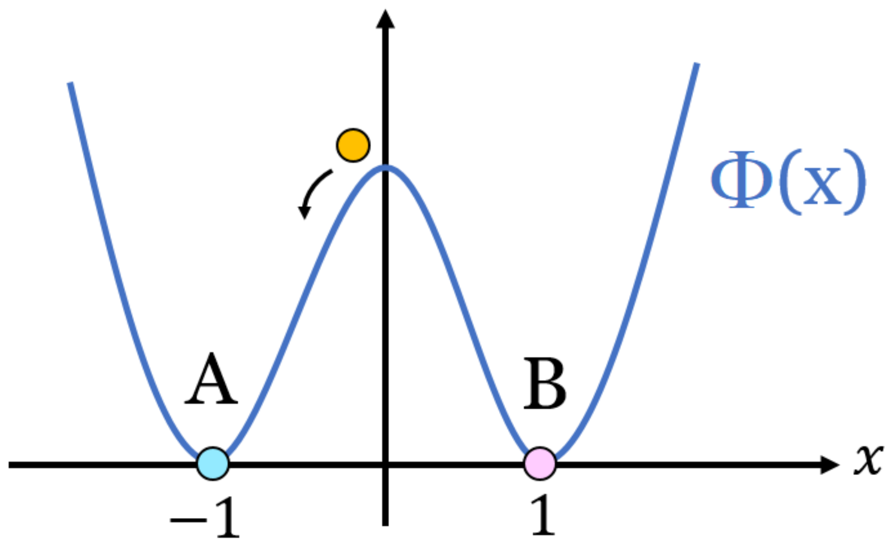
We expect there to be a boundary layer at $x = 0$

Find a good approximation $u^\star$ of u^ϵ that incorporates the fact that u^ϵ has a boundary layer. You only need to limit yourself to the $O(1)$ terms. For the matching-part, please show both methods.

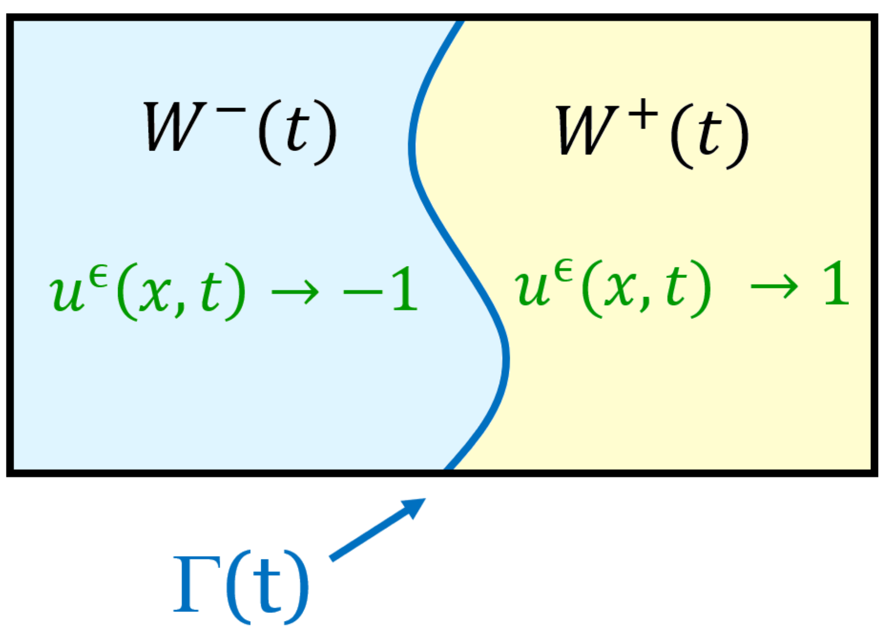
Problem 6: (10 points, 2 points each) Suppose $u^\epsilon = u^\epsilon(x,t)$ is a solution of the following reaction-diffusion PDE, where $g = g(x)$ is a fixed function

$$\begin{cases} \epsilon^2 (u^\epsilon_t - \Delta u^\epsilon) + \Phi'(u^\epsilon) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u^\epsilon(x,0) = g \end{cases}$$

Here $\Phi = \Phi(s)$ is a double-well potential function, that is a function with a local min at $s = \pm 1$ with $\Phi(\pm 1) = 0$, and a local max at $s = 0$ with $\Phi(0) = 1$, as in the following picture:



It turns out that for every t, there are two regions $W^{\pm}(t)$ where $u^{\epsilon}(x,t) \to \pm 1$ on $W^{\pm}(t)$, separated by a boundary layer $\Gamma(t)$, as in the following picture:



(a) Outer Solution, near $W^{\pm}(t)$

Apply the Ansatz:

$$u^{\epsilon}(x,t) = u^{0}(x,t) + \epsilon u^{1}(x,t) + \cdots$$

Find the $O(1)$ terms and conclude that $u^{0}(x,t) \in \{-1, 0, 1\}$

Since 0 is not a minimizer, we can ignore it, and hence we get

$$u^{0}(x,t) \in \{\pm 1\}$$

(b) [Inner Solution, near $\Gamma(t)$]

From now on, for convenience, assume $n = 2$. Suppose that the boundary layer is locally the graph of a function $x_2 = s^{\epsilon}(x_1, t)$

Assume that $s^{\epsilon}(0,0) = 0$ and $s^{\epsilon}_{x_1}(0,0) = 0$

Define $y = (y_1, y_2)$ by

$$\begin{cases} y_1 = x_1 \\ y_2 = \frac{x_2 - s^{\epsilon}(x_1,t)}{\epsilon} \end{cases}$$

Define $\overline{u}_{\epsilon} = \overline{u}_{\epsilon}(y_1, y_2)$ by

$$u^{\epsilon}(x_1, x_2, t) = \overline{u}^{\epsilon}(y_1, y_2, t) = \overline{u}^{\epsilon}\left(x_1, \frac{x_2 - s^{\epsilon}(x_1,t)}{\epsilon}, t\right)$$

Find a PDE that $\bar{u}_{\epsilon}$ satisfies (no need to simplify it)

(c) Apply the Ansätze

$$\begin{aligned}\overline{u}_\epsilon(y,t) =& \bar{u}_0(y,t) + \epsilon \overline{u}_1(y,t) + \cdots \\ s^\epsilon(x_1,t) =& s^0(x_1,t) + \epsilon s^1(x_1,t) + \cdots\end{aligned}$$

Find the $O(1)$ terms, and show $f(y_2) =: \overline{u}_0(0,y_2,0)$ solves

$$-f'' + \Phi'(f) = 0$$

(d) Find the $O(\epsilon)$ terms and show $h(y_2) =: \overline{u}_1(0,y_2,0)$ solves

$$-h'' + \Phi''(f)h - (f')\left(s_t^0(0,0) - s^0_{x_1x_1}(0,0)\right) = 0$$

(e) Multiply the equation in (d) by f' and integrate with respect to y_2 on $\mathbb{R}$ (ignore the boundary terms) to get that, at $(0,0)$

$$s_t^0(0,0) = s^0_{x_1x_1}(0,0)$$

Problem 7: *(5 points)*

Suppose $u = u(x)$ and $v = v(x)$ solve following system of PDE in $\mathbb{R}^n$:

$$\begin{cases} \Delta u = v \\ -\Delta v = u \end{cases}$$

Assume there exists a constant $C > 0$ such that for all $x \in \mathbb{R}^n$ we have

$$|u(x)| \leq \frac{C}{|x|^n} \text{ and } |v(x)| \leq \frac{C}{|x|^n} \text{ and } |Du(x)| \leq C \text{ and } |Dv(x)| \leq C$$

Show that $u = v = 0$ in $\mathbb{R}^n$

Hint: Fix $r > 0$, multiply the first equation by v and the second equation by u, add them up, integrate over $B(0, r)$, and let $r \to \infty$

You may use that the area of the sphere $\partial B(0, r)$ is $n\alpha(n)r^{n-1}$ where $\alpha(n)$ is the volume of the unit ball $B(0, 1)$ in $\mathbb{R}^n$ You don't need to figure out $\alpha(n)$ to solve this problem

Note: To show that the boundary terms go to 0, you may need to use the triangle inequality $\left|\int f\right| \leq \int |f|$ and the Cauchy-Schwarz inequality $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}| |\mathbf{b}|$