APMA 1941G – MIDTERM – STUDY GUIDE

This is the study guide for the final, and is just meant to be a guide to help you study, just so that we're on the same page in terms of expectations. The exam covers the entire course.

Main Concepts

- Asymptotic expansions (Chapters 1 and 3)
- Laplace's method (Chapter 2)
- Boundary Layers (Chapter 4)

Chapter 1: Introduction

Example 1: Acoustic Approximation in Fluid Mechanics

- Make sure you are comfortable with notation like Du, ∇u, Δu, $D^2 u$, and the notation for vector-valued functions, like $D\mathbf{u}$, $\text{div}(\mathbf{u})$
- The most important thing you need to know is how to do is to do an Ansatz and plug it into an equation, and compare term by term. You are not expected to figure out exactly what equations our ρ^1 satisfies
- Know the notation $f = o(\epsilon)$ and $f = O(\epsilon)$ and its variants like $f = o(\epsilon^2)$ etc.

- Know the fact that says that if $a_0 + a_1\epsilon + \cdots = b_0 + b_1\epsilon + \cdots$, then for all i we have $a_i = b_i$

Example 2: Perturbation of eigenvalues

- Ignore this section, because it involves a bunch of linear algebra you're not responsible for. But again, know at least how to do the Ansatz and what you get when you compare the $O(1)$ and $O(\epsilon)$ terms.

Example 3: Derivation of the KdV equation

- You really don't need to understand the details of it, especially since there's this one very weird trick at the end
- For the theory of KdV, know how to plug in $u(x,t) = \phi(x - ct)$ into the PDE and know the trick of multiplying the ODE by ϕ'

Theoretical Aspects

- **Know the definition of an asymptotic expansion, that is know what $f(\epsilon) \sim \sum_{k=0}^{\infty} a_k \epsilon^k$ means**
- Prove the lemma about uniqueness of asymptotic expansions
- Show that $e^{-\frac{1}{\epsilon}}$ has asymptotic expansion 0
- In lecture, I have proven another lemma about constructing a function with a given asymptotic expansion. You don't need to memorize the proof, but make sure to understand at least the main steps. In particular, make sure to know what a support function is.

Chapter 2: Asymptotic Evaluation of Integrals

Laplace's Method

- Know the assumptions on a and φ of Laplace's method
- You do **NOT** need to know the (crazy) proof of Laplace's method in the special case, you can skip it if you like.
- Know the statement of Morse Lemma
- You do not need to memorize the proof of the general Laplace's method, but at least know the general ideas, in particular starting with the special case, and then using Morse Lemma to reduce the general case to the special case.
- Calculate the first couple of terms in the general Laplace method, like on the homework
- Know what to do in case $\varphi(0) \neq 0$ or the maximum is at x_0 instead of 0

Stationary Phase

- For stationary phase, you're not responsible for the formulas above, and you do not need to know the proof about rapid decay or how to prove the special case.
- You are **NOT** responsible for knowing things about the Fourier transform
- You do not need to memorize the formulas for stationary phase, but know how to use them

Application: Group vs. Phase Velocity

- What's interesting to know in this section is how stationary phase magically appears here! I could ask you to write $u(ct, t)$ in terms of $I[\epsilon]$

Chapter 3: Multiple Scales

Example 1: Rapidly Oscillating Coefficients

- Apply your usual Ansatz to this equation; you don't need to know why the Ansatz doesn't work
- Apply your better Ansatz to this equation, and find the $O(\frac{1}{\epsilon^2})$ and $O(\frac{1}{\epsilon})$ and $O(1)$ terms. I will always give you the Ansatz that you need to plug in.
- Know the trick of multiplying by u^0 and showing that $u_y^0 = 0$
- For the $O(\frac{1}{\epsilon})$-term know how to rewrite your equation in divergence form
- You **don't** need to know the trick of introducing w
- For the $O(1)$-term, know the trick of just integrating the equation and pulling out the terms that don't depend on y.

Example 2: An oscillator with damping

- Given conserved quantity $g(t)$, show that $g'(t) = 0$ and deduce that u^ϵ is bounded
- Again, know how to apply the given Ansatz

- Solve the ODEs in the $O(\epsilon)-$term (like on your homework). You are responsible for undetermined coefficients, but not for variation of parameters.

- Plug in the better Ansatz into your equation. Again, I will tell you which Ansatz to plug in, and know how to obtain the given equations for A and B.

- Know how we obtained the equations for A' and B'. You don't need to know how to write $\cos^3$ in terms of cos and sin, but understand that we selected A and B to kill the resonance terms.

- No need to memorize Hamilton's equations; I will give them to you if necessary

- Prove the lemma that $H(x(t), p(t))$ is a conserved quantity, and put an ODE in Hamiltonian form (all you need to do is to antidifferentiate H_x and H_p). Also, show that $|A|$ and $|B|$ are bounded from the fact that the Hamiltonian is constant.

Example 3: WKB Method

- No need to understand why our first Ansatz didn't work

- Know how to plug in the better Ansatz. I *will* give you the requirements on σ^ϵ but you do need to figure out why we chose σ^ϵ to have the form that we want.

- Everything else in this example is fair game, especially how solve for A and B and how to figure out what u^ϵ looks like. Don't worry about the change-of-variables into θ-part

Example 4: Nonlinear Oscillator with Damping

- Know how to plug in the guess into your ODE and know how to plug in the Ansatz, and know how to show that the energy is constant. Here I would give you the guess and the energy.
- Don't worry about the change of variables to obtain $\omega^0(E)$.
- Understand why w solves a linear ODE, and know how to handle the $O(\epsilon)$ terms. I will tell you what to multiply your ODE with, but I won't tell you how to simplify the B-term
- You don't need to understand that A is a function of E and a function of τ.

Example 5: Nonlinear Wave Equation

- Same rules as Example 4

Example 6: A diffusion-transport PDE

- Know how plug in your Ansatz in your PDE and show that v^0 solves a linear PDE. Also show that $v^0(\theta(t), t)$ is constant. I would provide you with v^0 and the ODE for θ

Interlude: The Calculus of Variation

- This is a **very** important section (and my favorite topic) so make sure to thoroughly study it!
- Know the formula the Euler-Lagrange equations and **know** how to derive it in the 1−dimensional case (which I did in lecture).

- Given a functional, derive its Euler-Lagrange equation
- Given a PDE, write it as an Euler-Lagrange equation for a certain functional $I[u]$. This won't be always possible. Try it for instance with $-\Delta u = f(u)$

Example 7: An eikonal-continuity equation

- I would directly give you the funtional $I^{\epsilon}[a^{\epsilon}, \theta^{\epsilon}]$
- Know how to find the $O(1)$-terms and know how to do the variation in both a^0 and θ^0 (see HW 8)

Example 8: Homogenization

- I would tell you what $I[u^{\epsilon}]$ is and I would tell you the averaging trick.

Chapter 4: Boundary Layers

Example 1: Introductory example

- For the exam, you're responsible of knowing how to do both methods
- Don't forget to show me the dominant balance, and in particular show why two of the three cases give you a contradiction
- In this example, you'd have to figure out what the matching condition is. I'll only tell you where the boundary layer is.

Example 2: Higher Orders

- Remember to find the inner solution recursively: First find $\overline{u}_0$, do matching to find the constant, and *then* find $\overline{u}_1$.
- On the exam, I would tell you whether to go to the $O(1)$ terms or to the $O(\epsilon)$ terms, so there won't be any confusion.
- Again, you'd have to figure out what the matching condition is

Example 3: An internal layer

- Don't forget that u^0 is now piece-wise defined.
- I would tell you what the matching condition is

Example 4: Earth-Moon Spacecraft problem

- You can ignore most of this section. The only thing I could ask you is how to solve for $x_0(t)$. Of course I would provide you what $t^\star$ is.

Example 5: A singular variational problem

- Derive the Euler-Lagrange equation and know how to find the outer solution. I would tell you that you can ignore the $0-$solution
- Don't worry about the derivation that $\alpha = 1$ which is non-rigorous anyway
- I would provide you with the change-of-variable, but you should know how to find the new PDE and the $O(1)$ and $O(\epsilon)$ terms

- I would tell you the assumptions of s^ϵ at 0, and I would tell you to evaluate your terms at $(0, \dots, 0, y_n)$, and I would tell you that the ODE has a unique solution
- Know how to analyze the ODE for the $O(\epsilon)$terms and to derive that s^0 satisfies Laplace's equation at 0.
- **Given** the formula for mean-curvature in terms of s which I would provide, show that this implies that our curve has mean curvature 0 at 0.

Example 6: A singular reaction-diffusion PDE

- Know how to derive all the claims that I made in this example. The same rules as Example 5 apply.
- Also **given** the formula for normal velocity, show that the normal velocity at 0 equals to the mean curvature at 0.

Example 7: Singular Perturbations of Eigenfunctions

- Know how to apply the boundary layer method to this example. I would guide you a little bit since some of the steps are not obvious. For instance, I would give you the fact that $\bar{u}_0(y) = A + \frac{B}{|y|}$. I wouldn't tell you what the matching assumption is, but I would tell you how to obtain our "guess."
- Know how to do the analysis of the $O(\epsilon)$ step. I would tell you to multiply your $O(\epsilon)$ terms by u^0 and integrate on W_δ.

- Know the integration by parts formula **with** boundary terms:

$$\int_W (\Delta u)v = \int_{\partial W} \left(\frac{\partial u}{\partial \nu}\right) v - \int_W Du \cdot Dv$$

- Know the definition of normal derivative

$$\frac{\partial u}{\partial \nu} = Du \cdot \nu$$

- Know that the normal vector to $B(0,r)$ at $\mathbf{x}$ is $\nu = \frac{\mathbf{x}}{r}$
- Know that the derivative/gradient of $|\mathbf{x}|$ is $D\,|\mathbf{x}| = \frac{\mathbf{x}}{|\mathbf{x}|}$

Example 8: The Crushed Ice Problem

- I won't ask you for the proof of the derivation of the PDE for u_0, but understand the calculations in A and B.
- Know the formula

$$\int_W \Delta u = \int_{\partial W} \frac{\partial u}{\partial \nu}$$

But this is just a particular case of the above formula with $v = 1$.

Homework Problems

- **Homework 1:** Don't worry about Problem 1, but in Problem 2, do know how to do (a) (without the hint!), (b) (again, without any help, except that I will give you what u_k is), and (c) and (d) without the hint.

- **Homework 2:** Don't worry about Problem 1, **definitely** know how to do Problem 2 without the hints! (except in (c) I would tell you what the equation is)

- **Homework 3:** Problem 1 is also an excellent problem. In that problem, I would give you the equation and the form of u, but I wouldn't give you any hints, except I would tell you the substitution. Problem 2 is also a great Laplace method question. A good exam problem would be to do (b)-(d) without the hints, except that I would give you the statement of Laplace's method, as well as the formula in (a). Know the definition of $f(n) \sim g(n)$ as $n \to \infty$.

- **Homework 4:** Obviously I won't give you anything as ridiculous as Problem 1, except that I could ask you how to do it in a really special case, and I could ask you about how to calculate $L_0 a$ (but I'd give you all the formulas that you need, except I won't define η or ψ). And I would provide you with formulas for C_2 if necessary. Problem 2 is also a great exam problem. Try to do it without the hint of second-order Taylor expansion and without the hint of the integral of $e^{-\frac{x^2}{2}}$.

- **Homework 5:** Also good problems, although Problem 2 is a bit inappropriate for the exam (this is not a calculus-course), but know how to do undetermined coefficients and how to solve second-order constant coefficient differential equations.

- **Homework 6:** Both problems are **excellent** exam problems. In Problem 1, know how to do the Ansatz and know why we chose A and B to kill the resonance terms, and know how to find the exact solution of the differential equation. For Problem 2, know how to plug in the Ansatz, know the trick of Taylor-expanding out $\sin(\theta^0 + \epsilon\theta^1)$ (I might not tell you about it!) Also know the trick of multiplying by a function and integrating by parts, and know how to do part (b). You don't need to know the formula for the integral of $\cos^2$.

- **Homework 7:** Ignore Problem 1 (it's too inappropriate), but Problem 2 is an excellent exam question (I would give you all the hints, except maybe I wouldn't tell you that you should integrate by parts)

- **Homework 8:** All the problems are **excellent** exam problems (including Problem 1c, for which I'd give you the hint). Know the integration-by-parts formula with 0-boundary terms:

$$\int_W Df \cdot Dg \, d\mathbf{x} = -\int_W \mathrm{div}(Df) g \, d\mathbf{x}$$

- **Homework 9:** Again, great exam problems (especially Problem 1), but the same remarks above from Examples 1 and 2 from Chapter 4 apply.

- **Homework 10:** Ignore Problem 1, but definitely look at Problem 2. Try to do it without the hints if possible (except I would tell you what y is and what $\overline{p}_\epsilon$ and $\overline{h}(y)$ are.

- **Homework 11:** Also two great problems. In Problem 1, I would tell you what $\bar{u}_\epsilon$ is, but everything else is fair game. Try to do all of this without the hints, except I would tell you what to multiply your $O(\epsilon)$ term with and I would tell you what to do with the $\sin(2\bar{u}_0)$ terms. For Problem 2, I would give you all the hints.