APMA 1941G Homework 10 Solutions
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Problem 1

This problem refers to Example 4, the Earth-Moon Spacecraft Problem. Show that the limiting velocity V_∞ satsifies

$$(V_\infty)^2 = \frac{\tan(\alpha)}{b}.$$

Use the formula relating V_∞ and V_0, as well as the definition of e, the formula for e in terms of α, and finally the definition of b.

Proof. We defined $e = |r_0||V_0|^2 - 1$. We also have $e = \frac{1}{\cos(\alpha)}$ and $b = \left(\frac{\cos\alpha+1}{\sin\alpha}\right)|r_0|$. In lecture, we found that

$$(V_\infty)^2 = |V_0|^2 - \frac{2}{|r_0|}.$$

First, we can rewrite

$$|V_0|^2 = \frac{e+1}{|r_0|} = \frac{\frac{1}{\cos\alpha}+1}{|r_0|} = \frac{1+\cos\alpha}{|r_0|\cos\alpha}.$$

Then

$$\begin{aligned}(V_\infty)^2 &= |V_0|^2 - \frac{2}{|r_0|} \\ &= \frac{1+\cos\alpha}{|r_0|\cos\alpha} - \frac{2\cos\alpha}{|r_0|\cos\alpha} \\ &= \frac{1-\cos\alpha}{|r_0|\cos\alpha}.\end{aligned}$$

We can also re-arrange to find that

$$|r_0| = \frac{b}{\left(\frac{\cos\alpha+1}{\sin\alpha}\right)},$$

so

$$\begin{aligned}(V_\infty)^2 &= \frac{1-\cos\alpha}{\cos\alpha}\cdot\frac{\left(\frac{\cos\alpha+1}{\sin\alpha}\right)}{b} \\ &= \frac{\left(\frac{1-\cos^2\alpha}{\sin\alpha}\right)}{\cos\alpha b} \\ &= \frac{\left(\frac{\sin^2\alpha}{\sin\alpha}\right)}{\cos\alpha b} \\ &= \frac{\sin\alpha}{\cos\alpha b} \\ &= \frac{\tan\alpha}{b}\end{aligned}$$

as claimed.

Problem 2: Reynold's Equation for Sliders

Consider a slider that you put in oil, where x is the position between 0 and 1 and the corresponding height is $h(x)$. This slider exerts a certain pressure p^ϵ which satisfies the ODE

$$\begin{aligned}-\epsilon((h^3)p^\epsilon p^\epsilon_x)_x &= (p^\epsilon h)_x \\ p^\epsilon(0) = p^\epsilon(1) &= 1\end{aligned} \tag{1}$$

Here $p^\epsilon = p^\epsilon(x)$ and $x \in (0,1)$, and $h = h(x) : [0,1] \to (0,\infty)$ is a given height, with $h(1) = 1$ and ϵ is a "viscosity coefficient." We expect there to be a boundary layer near $x = 1$, and our goal is to find a good approximation p^* of p^ϵ.

(a) Outer solution, near $x = 0$

Apply the ansatz $p^\epsilon(x) = p^0(x) + \epsilon p^1(x) + ...$ Compare the $O(1)$ terms and get an equation for $p^0(x)$. Impose $p^0(0) = 1$, solve for p^0 in terms of $h(0)$ and $h(x)$.

Solution. Substituting the ansatz into (1), we obtain

$$-\epsilon(h^3(p^0 + \epsilon p^1 + ...)(p^0_x + \epsilon p^1_x + ...))_x = ((p^0 + \epsilon p^1 + ...)h)_x = 0.$$

We can see that the only $O(1)$ term will be $(p^0 h)_x$ on the right-hand side, which gives

$$(p^0 h)_x = 0.$$

Integrating this in x one time, we have

$$p^0(x)h(x) = C.$$

Then

$$p^0(x) = \frac{C}{h(x)}.$$

Imposing the boundary condition $p^0(0) = 1$, we have that

$$1 = \frac{C}{h(0)} \implies C = h(0).$$

Thus, we can express

$$\boxed{p^0(x) = \frac{h(0)}{h(x)}}.$$

(b) Inner solution, near $x = 1$

Let $y = \frac{x-1}{\epsilon}$ and $\overline{p}_\epsilon(y) = p_\epsilon(x)$ and $\overline{h}(y) = h(x)$. Rewrite (1) in terms of $\overline{p}_\epsilon$ and $\overline{h}$, and apply the ansatz $\overline{p}_\epsilon(y) = \overline{p}_0(y) + \epsilon\overline{p}_1(y) + ...$, and moreover Taylor expand the function $\overline{h}(y) = h(1 + \epsilon y)$. Compare the $O(1)$-terms and recall $h(1) = 1$ to obtain

$$\overline{p}_0(\overline{p_0})_y + \overline{p}_0 = -A.$$

Solve for y in terms of $\overline{p}_0$ using separation of variables.

Note. At some point, it may be useful to note that

$$\frac{\overline{p}_0}{A + \overline{p}_0} = 1 - \frac{A}{A + \overline{p}_0}.$$

Impose the condition $\overline{p}_0(0) = 1$, which is the same as $p^0(1) = 1$, and ultimately obtain that

$$-y = \overline{p}_0 - A \ln\left|\frac{A + \overline{p}_0}{A + 1}\right| - 1$$

where A is to be determined.

Proof. Rewriting,

$$-\epsilon((h^3)\overline{p}^\epsilon \overline{p}^\epsilon_x)_x = (\overline{p}^\epsilon h)_x$$

Now Taylor expanding $\overline{h}$, we have that $\overline{h}(y) \approx h(1) + \epsilon y h'(1)$. Since $h(1) = 1$, $\overline{h}(y) \approx 1 + \epsilon y h'(1)$. Considering our ansatz and using the chain rule,

$$\overline{p}^\epsilon_x = \frac{1}{\epsilon}\left(\overline{p}^0_y + \epsilon\overline{p}^1_y + ...\right)$$

so we have, after moving the $1/\epsilon$ outside the derivative, transforming the x-derivatives into y-derivatives, and multiplying through by ϵ,

$$-\left((h^3)(\overline{p}^0 + \epsilon\overline{p}^1)(\overline{p}^0_y + \epsilon\overline{p}^1_y)\right)_y = (h(\overline{p}^0 + \epsilon\overline{p}^1 + ...))_y$$

Using the Taylor expansion for h, we see that $O(1)$ terms are:

$$-(\overline{p}^0\overline{p}^0_y)_y = \overline{p}^0_y$$

Re-arranging and integrating in y, we find that there is a constant A such that

$$\overline{p}^0\overline{p}^0_y + \overline{p}^0 = -A.$$

Rewriting this,

$$\frac{d\overline{p}_0}{dy} = \frac{-A - \overline{p}_0}{\overline{p}_0}.$$

Using separation of variables, we have that

$$\begin{aligned}
-y &= \int \frac{\overline{p}_0}{A + \overline{p}_0} d\overline{p}_0 \\
&= \int 1 - \frac{A}{A + \overline{p}_0} d\overline{p}_0 \\
&= \overline{p}_0 - A \int \frac{1}{A + \overline{p}_0} d\overline{p}_0 \\
&= \overline{p}_0 - A \ln|A + \overline{p}_0| + C
\end{aligned}$$

Since $p_0(1) = 1$, $\overline{p}_0(0) = 1$, so $0 = 1 - A \ln|A + 1| + C$. Thus

$$-y = \overline{p}_0 - A \ln\left|\frac{A + \overline{p}_0}{A + 1}\right| - 1, \tag{2}$$

so we have the desired implicit formula for $\overline{p}_0(y)$ in terms of A.

(c) Matching

Here, the matching condition is

$$\lim_{x\to 1} p^0(x) = \lim_{y\to -\infty} \overline{p}^0(y).$$

Since we want a finite answer in (2), the only way this works is if the term inside of the logarithm goes to 0^+, and this is achieved only if $\overline{p}_0$ goes to $-A$. Use this to solve for A and rewrite (2) in terms of the answer A. Finally, calculate your composite solution $p^*(x)$; leave the $\overline{p}_0$ term as $\overline{p}_0\left(\frac{x-1}{\epsilon}\right)$.

Solution. Since

$$\lim_{x\to 1} p^0(x) = \frac{h(0)}{h(1)} = h(0),$$

$-A = h(0)$. Thus (2) becomes

$$-y = \overline{p}_0 + h(0) \ln\left|\frac{-h(0) + \overline{p}_0}{-h(0) + 1}\right| - 1.$$

The composite solution is given by

$$p^*(x) = p_0(x) + \overline{p}_0(y) - \text{ common part }.$$

Based on how we have been defining the common part in class, and the above, the common part is $A = -h(0)$. Thus, leaving the $\overline{p}_0$ term as $\overline{p}_0\left(\frac{x-1}{\epsilon}\right)$, we obtain

$$\boxed{p^*(x) = \frac{h(0)}{h(x)} + \overline{p}_0\left(\frac{x-1}{\epsilon}\right) + h(0).}$$