APMA 1941G – HOMEWORK 10

Problem 1: (5 points)

This problem refers to "Example 4: Earth-Moon Spacecraft Problem"

Show that the limiting velocity V_∞ satisfies

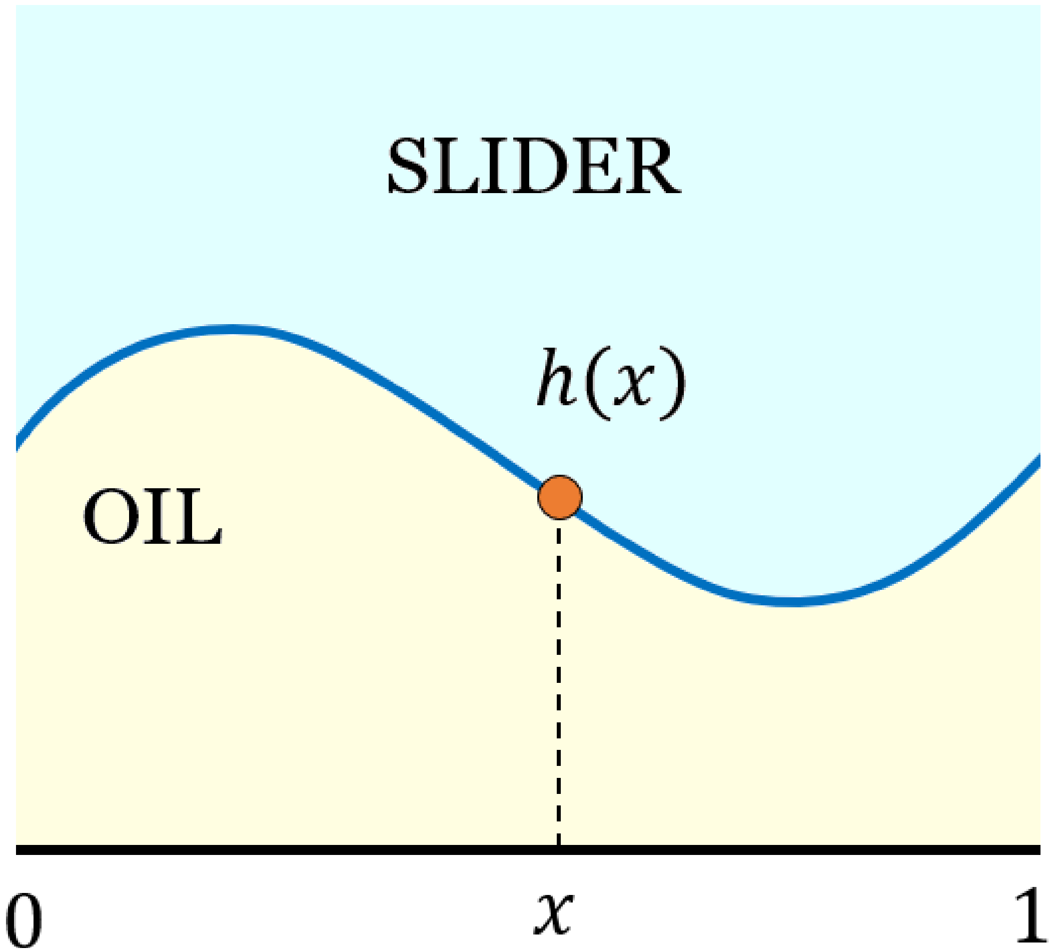
$$(V_\infty)^2 = \frac{\tan(\alpha)}{b}$$

Hint: Use the formula that relates V_∞ to V_0, as well as the definition of e (at the beginning of the section on classical orbits), the formula for e in terms of α, and finally the definition of b.

(TURN PAGE)

Problem 2: [Reynold's equation for sliders] (15 points, 5 points each)

Consider a slider that you put in oil, as in the following picture:



This slider exerts a certain pressure p^ϵ which satisfies the ODE

$$\begin{cases} -\epsilon\left(\left(h^3\right)p^\epsilon p^\epsilon_x\right)_x = \left(p^\epsilon h\right)_x \\ \qquad p^\epsilon(0) = p^\epsilon(1) = 1 \end{cases} \tag{ODE}$$

Here $p^\epsilon = p^\epsilon(x)$ and $x \in (0,1)$ and $h = h(x) : [0,1] \to (0,\infty)$ is a given height, with $h(1) = 1$ and ϵ is a 'viscosity coefficient'

We expect there to be a boundary layer near $x = 1$, and our goal is to find a good approximation $p^\star$ of p^ϵ.

(a) [Outer solution, near $x = 0$]

Apply the Ansatz:

$$p^\epsilon(x) = p^0(x) + \epsilon\, p^1(x) + \cdots$$

Compare the $O(1)$-terms and get an equation for $p^0(x)$. Impose $p^0(0) = 1$, solve for p^0 in terms of $h(0)$ and $h(x)$

(b) [Inner solution, near $x = 1$]

Let $y = \frac{x-1}{\epsilon}$ and $\bar{p}_\epsilon(y) = p_\epsilon(x)$ and $\bar{h}(y) = h(x)$

Rewrite (ODE) in terms of $\bar{p}_\epsilon$ and $\bar{h}$, and apply the Ansatz:

$$\bar{p}_\epsilon(y) = \bar{p}_0(y) + \epsilon\, \bar{p}_1(y) + \cdots ,$$

and moreover Taylor expand the function $\bar{h}(y) = h(1 + \epsilon y)$

Compare the $O(1)-$terms and recall $h(1) = 1$ to obtain

$$\bar{p}_0\, (\bar{p}_0)_y + \bar{p}_0 = -A$$

Solve for y in terms of $\bar{p}_0$ using separation of variables

Note: At some point, it may be useful to note that

$$\frac{\bar{p}_0}{A+\bar{p}_0} = 1 - \frac{A}{A+\bar{p}_0}$$

Impose the condition $\bar{p}_0(0) = 1$ (which is the same as $p^0(1) = 1$) and ultimately obtain that

$$-y = \bar{p}_0 - A\ln\left|\frac{A+\bar{p}_0}{A+1}\right| - 1 \tag{I}$$

Here A is to be determined (note that there are many equivalent ways to write your answer)

This gives us an implicit formula for $\bar{p}_0(y)$ in terms of A.

(c) [Matching]

Here, the matching condition is

$$\lim_{x\to 1} p^0(x) = \lim_{y\to -\infty} \bar{p}_0(y)$$

Since we want a finite answer in (I), the only way this works if if the term in our ln goes to 0^+, and this is achieved only if $\bar{p}_0$ goes to $-A$. In other words, the right-hand-side of our matching conditon has to be $-A$.

Use this to solve for A and rewrite (I) in terms of the answer A that you found (this should be quick); this gives us an implicit formula for $\bar{p}_0$.

Finally, calculate your composite solution $p^\star(x)$; leave the $\bar{p}_0$ term as $\bar{p}_0\left(\frac{x-1}{\epsilon}\right)$