APMA 1941G – HOMEWORK 11

Problem 1: (10 = 3 + 3 + 4 points) Consider the following PDE:

$$\epsilon^2 u^\epsilon_t - \epsilon^2 u^\epsilon_{xx} + (f(x))^2 \sin(2u^\epsilon) = 0$$

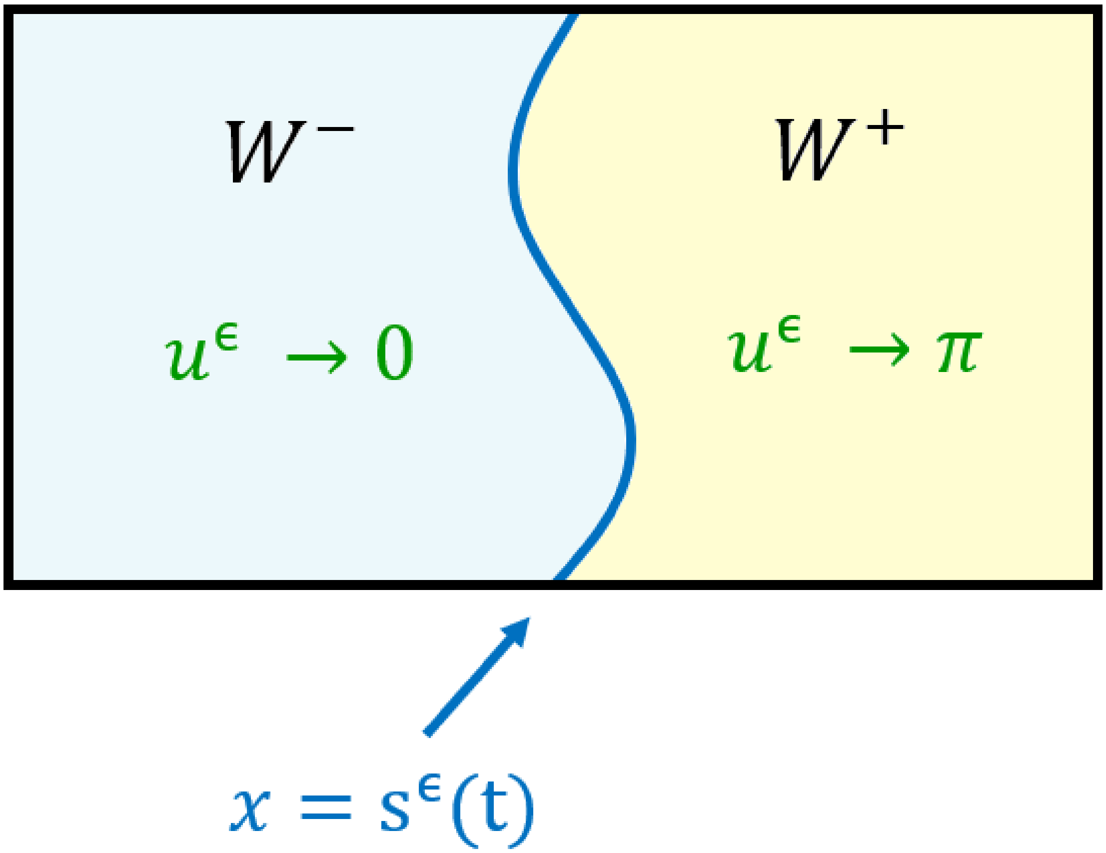
Where $u^\epsilon = u^\epsilon(t,x)$ and $x \in \mathbb{R}$ and $f(x) > 0$

Here think of $\sin(2u^\epsilon)$ as being our $\Phi'(u^\epsilon)$

Similar to the situation in "A Singular Variational Problem" (Ex 5) we expect that there are two regions $W^\pm$ separated by a curve $x = s^\epsilon(t)$ such that

$$u^\epsilon(t,x) \to \begin{cases} 0 \text{ if } x < s^\epsilon(t) \\ \pi \text{ if } x > s^\epsilon(t) \end{cases}$$

as in the following picture:



Goal: Find a differential equation for $s^0(t)$

(a) Let $u^\epsilon(t,x) = \bar{u}^\epsilon\left(t, \frac{x - s^\epsilon(t)}{\epsilon}\right)$ where $\bar{u}^\epsilon = \bar{u}^\epsilon(t,y)$

Rewrite the above PDE in terms of $\bar{u}^\epsilon$.

(b) Apply the following Ansätze to the PDE in (a)

$$\begin{cases} \bar{u}^\epsilon = & \bar{u}^0 + \epsilon \bar{u}^1 + \cdots \\ s^\epsilon = & s^0 + \epsilon s^1 + \cdots \end{cases}$$

Here $\bar{u}^k = \bar{u}^k(t, y)$ and $s^k = s^k(t)$

What are the $O(1)$ and $O(\epsilon)$ terms?

Note: You may need to Taylor expand the $\sin(2\bar{u}^\epsilon)$ and the $f(x) = f(s + \epsilon y) = f(s_0 + \epsilon(s_1 + y))$ terms

(c) Show that s_0 must satisfy the differential equation:

$$s_0'(t) = -\frac{f'(s_0)}{f(s_0)}$$

Here $s_0' = \frac{ds^0}{dt}$ and $f' = \frac{df}{ds}$

Hint: Multiply the $O(\epsilon)$ term equation by $\bar{u}_y^0$ and integrate with respect to y on $\mathbb{R}$. This gives you four terms.

For the term with $\bar{u}_{yy}^1$, integrate by parts and use the $O(1)-$term. For the term with $\cos(2\bar{u}^0)\bar{u}_y^0$ (recognize this as a derivative), integrate by parts. You should see some cancellation happening now when you add the 4 terms. For the term with $\sin(2\bar{u}^0)$, don't integrate by parts yet; first use your $O(1)$ term to get rid of the $\sin -$term and then integrate by parts.

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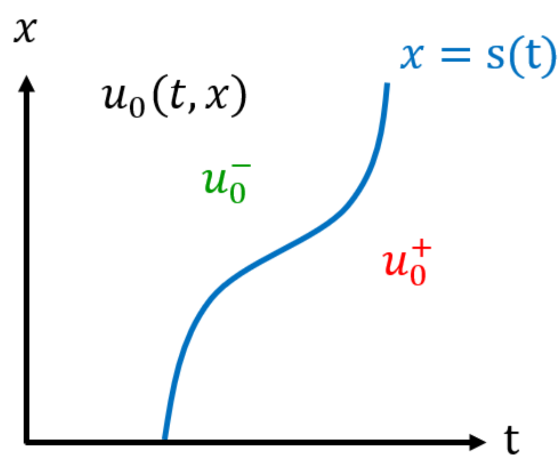
Problem 2: (10 = 3 + 4 + 3 points) Consider Burgers' equation

$$u_t^\epsilon + u^\epsilon u_x^\epsilon + \epsilon u_{xx}^\epsilon = 0 \tag{B}$$

Where $u^\epsilon = u^\epsilon(t, x)$ with $x \in \mathbb{R}$

From PDE−theory, it turns out that u^0 forms a 'shock' along a curve $x = s(t)$, that is u^0 has a jump discontinuity, where u_0 jumps from values u_0^- to u_0^+ as in the following picture:



Goal: Find a differential equation for s

(a) [Outer Solution, far from $s(t)$]

Apply the Ansatz $u^\epsilon = u^0 + \epsilon u^1 + \cdots$ and show that u^0 satisfies

$$u_t^0 + u^0 u_x^0 = 0$$

(b) [Inner Solution, on $s(t)$]

Assume that u^0 is discontinuous along the curve $x = s(t)$

Let $y = \frac{x - s(t)}{\epsilon}$ and $u^\epsilon(t,x) = \bar{u}^\epsilon(t,y)$

Rewrite (B) in terms of $\bar{u}_\epsilon$ and apply the Ansatz

$$\bar{u^\epsilon} = \bar{u}^0 + \epsilon \bar{u}^1 + \cdots$$

Compare the $O(\frac{1}{\epsilon})$ terms, and find a PDE for $\bar{u}^0$ (no need to Ansatz s here!)

(c) [Matching] Integrate the equation in (b) with respect to y from $-\infty$ to ∞. You may assume that $\lim_{y \to \pm\infty} \bar{u}^0_y(y) = 0$

The matching assumption here becomes:

$$\begin{cases} \lim_{y \to \infty} \bar{u}^0(y) = \lim_{x \to (s(t))^+} u^0 := u_0^+ \\ \lim_{y \to -\infty} \bar{u}^0(y) = \lim_{x \to (s(t))^-} u^0 := u_0^- \end{cases}$$

Use the matching assumption to find that s solves

$$s'(t) = \frac{u_0^- + u_0^+}{2}$$

Notice an averaging-phenomenon happening here, similar to Example 3: "An internal layer"