APMA 1941G Homework 11 Solutions
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Problem 1

Consider the following PDE:

$$\epsilon^2 u_t^\epsilon - \epsilon^2 u_{xx}^\epsilon + (f(x))^2 \sin(2u^\epsilon) = 0. \tag{1}$$

where $u^\epsilon = u^\epsilon(t,x)$, $x \in \mathbb{R}$, and $f(x) > 0$. Here think of $\sin(2u^\epsilon)$ as being our $\Psi'(u^\epsilon)$. Similar to the situation in "A Singular Variational Problem" (Ex 5), we expect that there are two regions $W^\pm$ separated by a curve $x = s^\epsilon(t)$ such that

$$u^\epsilon(t,x) \to \begin{cases} 0 \text{ if } x < s^\epsilon(t) \\ \pi \text{ if } x > s^\epsilon(t) \end{cases}. \tag{2}$$

Our goal is to find a differential equation for $s^0(t)$.

(a)

Let $u^\epsilon(t,x) = \overline{u}^\epsilon\left(t, \frac{x-s^\epsilon(t)}{\epsilon}\right)$ where $\overline{u}^\epsilon = \overline{u}^\epsilon(t,y)$. Rewrite (1) in terms of $\overline{u}^\epsilon$.

Solution. Applying the chain rule, we obtain the equivalent equation for (1),

$$\epsilon^2 \overline{u}_t^\epsilon - \epsilon \overline{u}_y^\epsilon s_t^\epsilon - \overline{u}_{yy}^\epsilon + (f(\epsilon y + s^\epsilon(t)))^2 \sin(2\overline{u}^\epsilon) = 0. \tag{3}$$

(b)

Apply the following ansatz to the PDE in (a):

$$\begin{cases} \overline{u}^\epsilon &= \overline{u}^0 + \epsilon \overline{u}^1 + ... \\ s^\epsilon &= s^0 + \epsilon s^1 + ... \end{cases}$$

Here $\overline{u}^k = \overline{u}^k(t,y)$ and $s^k = s^k(t)$. What are the $O(1)$ and $O(\epsilon)$ terms? Note that you may need to Taylor expand the $\sin(2\overline{u}^\epsilon)$ and the $f(x) = f(s + \epsilon y) = f(s_0 + \epsilon(s_1 + y))$ terms.

Solution. In order to substitute the above ansatz into (3), first note that via Taylor expansion,

$$\sin(2\overline{u}^\epsilon) = \sin(2(\overline{u}^0 + \epsilon \overline{u}^1 + ...)) = \sin(2\overline{u}^0) + 2\epsilon \overline{u}^1 \cos(2\overline{u}^0) + ...$$

and

$$f(s^\epsilon + \epsilon y) = f(s^0 + \epsilon(s^1 + y) + ...) = f(s^0) + \epsilon(s^1 + y) f'(s_0) + ...$$

Then we obtain

$$\begin{aligned} \epsilon^2(\overline{u}_t^0 - \epsilon \overline{u}_t^1 + ...) + \epsilon(\overline{u}_y^0 + \epsilon \overline{u}_y^1 + ...)(s_t^0 + \epsilon s_t^1 + ...) - (\overline{u}_{yy}^0 + \epsilon \overline{u}_{yy}^1 + ...) \\ + (f(s^0) + \epsilon(s^1 + y) f'(s_0) + ...)^2 (\sin(2\overline{u}^0) + 2\epsilon \overline{u}^1 \cos(2\overline{u}^0) + ...) = 0. \end{aligned}$$

The $O(1)$ terms then yield:

$$\overline{u}_{yy}^0 - f(s^0)^2 \sin(2\overline{u}^0) = 0. \tag{4}$$

The $O(\epsilon)$ terms yield:

$$-\overline{u}_{yy}^1 + 2f(s^0)^2 \overline{u}^1 \cos(2\overline{u}^0) + 2(s^1 + y) f'(s^0) f(s^0) \sin(2\overline{u}^0) - \overline{u}_y^0 s_t^0 = 0 \tag{5}$$

(c)

Show that s_0 must satisfy the differential equation:

$$s_0'(t) = -\frac{f'(s_0)}{f(s_0)} \tag{6}$$

here $s_0' = \frac{ds^0}{dt}$ and $f' = \frac{df}{ds}$.

Solution. First, we multiply the $O(\epsilon)$ equation (5) by $\overline{u}_y^0$ and integrate with respect to y on $\mathbb{R}$. Doing this, the first term in (5) becomes, using integration by parts,

$$\begin{aligned}-\int_{\mathbb{R}} \overline{u}_{yy}^1 \overline{u}_y^0 &= -\overline{u}_y^1 \overline{u}_y^0|_{-\infty}^{\infty} + \int_{\mathbb{R}} \overline{u}_y^1 \overline{u}_{yy}^0 \\ &= \int_{\mathbb{R}} f(s^0)^2 \sin(2\overline{u}^0)\overline{u}_y^1 dy.\end{aligned}$$

Here we saw the boundary terms were zero because $\overline{u}^\epsilon$ tends to a constant both when $y \to \infty$ and $y \to -\infty$, due to the condition (2). Next we consider the contribution from the second term in (5) after multiplying and integrating. Notice that $\cos(2\overline{u}^0)\overline{u}_y^0 = \frac{d}{dy}\frac{1}{2}\sin(2\overline{u}^0)$. Then using integration by parts yields that

$$\begin{aligned}\int_{\mathbb{R}} 2f(s^0)^2\overline{u}^1(\cos 2\overline{u}^0)\overline{u}_y^0 dy &= \int_{\mathbb{R}} f(s_0)^2\overline{u}^1 \frac{d}{dy}\sin(2\overline{u}^0)dy \\ &= f(s_0)^2\overline{u}^1\sin(2\overline{u}^0)|_{-\infty}^{\infty} - \int_{\mathbb{R}} \sin(2\overline{u}^0)f(s_0)^2\overline{u}_y^1 dy \\ &= -\int_{\mathbb{R}} f(s^0)^2\sin(2\overline{u}^0)\overline{u}_y^1 dy.\end{aligned}$$

We have now seen that the contributions from the first two terms reduce to zero. Next we consider the third term. Since the $O(1)$-terms yield that $\sin(2\overline{u}^0) = \frac{\overline{u}_{yy}^0}{f(s^0)^2}$, we substitute this in to see that, again using integration by parts,

$$\begin{aligned}\int_{\mathbb{R}} 2(s^1+y)f'(s^0)f(s^0)\sin(2\overline{u}^0)\overline{u}_y^0 dy &= \int_{\mathbb{R}} 2(s^1+y)f'(s^0)f(s^0)\sin(2\overline{u}^0)\overline{u}_y^0 dy \\ &= \frac{f'(s^0)}{f(s^0)}\int_{\mathbb{R}}(s^1+y)\frac{d}{dy}(\overline{u}_y^0)^2 dy \\ &= \frac{f'(s^0)}{f(s^0)}\left((s^1+y)(\overline{u}_y^0)^2|_{-\infty}^{\infty} - \int_{\mathbb{R}}(\overline{u}_y^0)^2 dy\right).\end{aligned}$$

We obtained this by assuming the overall boundary terms are again zero because the derivatives go to zero due to our assumption (and say, some assumption that $(\overline{u}_y^0)^2$ goes to zero faster than y blows up). Then considering this term and the contribution from the last term of (5), we are left with

$$\int_{\mathbb{R}} s_t^0(\overline{u}_y^0)^2 dy = -\frac{f'(s^0)}{f(s^0)}\int_{\mathbb{R}}(\overline{u}_y^0)^2 dy.$$

Since s^0 is not a function of y, we can factor out and divide both sides by the integral to obtain that

$$s_0'(t) = -\frac{f'(s_0)}{f(s_0)}$$

as claimed.

Problem 2

Consider Burger's equation

$$u_t^\epsilon + u^\epsilon u_x^\epsilon + \epsilon u_{xx}^\epsilon = 0 \tag{7}$$

where $u^\epsilon = u^\epsilon(t,x)$ with $x \in \mathbb{R}$. From PDE theory, it turns out that u^0 forms a 'shock' along a curve $x = s(t)$, that is u^0 has a jump discontinuity. The goal is to find a differential equation for s.

(a) Outer solution, far from $s(t)$

Apply the ansatz $u^\epsilon = u^0 + \epsilon u^1 + ...$ and show that u^0 satisfies

$$u_t^0 + u^0 u_x^0 = 0$$

Solution. Plugging in the ansatz, we obtain

$$u_0^t + \epsilon u_t^1 + ... + (u^0 + \epsilon u^1 + ...)(u_x^0 + \epsilon u_x^1 + ...) + \epsilon(u_{xx}^0 + \epsilon u_{xx}^1 + ...) = 0.$$

Comparing the $O(1)$ terms, we obtain

$$u_0^t + u^0 u_x^0 = 0$$

as claimed.

(b) Inner solution, on $s(t)$

Assume that u^0 is discontinuous along the curve $x = s(t)$. Let $y = \frac{x - s(t)}{\epsilon}$ and $u^\epsilon(t,x) = \overline{u}^\epsilon(t,y)$. Rewrite (7) in terms of $\overline{u}_\epsilon$ and apply the ansatz

$$\overline{u}^\epsilon = \overline{u}^0 + \epsilon \overline{u}^1 + ...$$

Compare the $O(\frac{1}{\epsilon})$ terms, and find a PDE for $\overline{u}^0$ (no need to ansatz s).

Solution. First rewriting in terms of $\overline{u}_\epsilon$ and applying the chain rule, we obtain

$$\overline{u}_t^\epsilon - \frac{1}{\epsilon}\overline{u}_y^0 s'(t) + \frac{1}{\epsilon}\overline{u}^\epsilon \overline{u}_y^\epsilon + \frac{1}{\epsilon}\overline{u}_{yy}^\epsilon = 0.$$

Now substituting in the ansatz, we obtain

$$\overline{u}_t^0 - \frac{1}{\epsilon}\overline{u}_y^0 s'(t) + \epsilon \overline{u}_t^1 - \overline{u}_y^1 s'(t) + \frac{1}{\epsilon}(\overline{u}^0 + \epsilon \overline{u}^1 + ...)(\overline{u}_y^0 + \epsilon \overline{u}_y^1 + ...) + \frac{1}{\epsilon}(\overline{u}_{yy}^0 + \epsilon \overline{u}_{yy}^1 + ...) = 0.$$

Comparing the $O(\frac{1}{\epsilon})$ terms, we obtain

$$\boxed{-\overline{u}_y^0 s'(t) + \overline{u}^0 \overline{u}_y^0 + \overline{u}_{yy}^0 = 0.}$$

(c) Matching

Integrate the equation in (b) with respect to y from $-\infty$ to ∞. You may assume that $\lim_{y \to \pm\infty} \overline{u}_y^0(y) = 0$. The matching assumption here becomes:

$$\begin{cases} \lim_{y\to\infty} \overline{u}^0(y) = \lim_{x \to (s(t))^+} u^0 \doteq u_0^+ \\ \lim_{y\to-\infty} \overline{u}^0(y) = \lim_{x \to (s(t))^-} u^0 \doteq u_0^- \end{cases}$$

Use the matching assumption to find that s solves

$$s'(t) = \frac{u_0^- + u_0^+}{2}.$$

Solution. Integrating the differential equation we found and using the assumption on the boundary terms, we have that

$$-\overline{u}^0 s'(t)|_{-\infty}^{\infty} + \frac{1}{2}(\overline{u}^0)^2|_{-\infty}^{\infty} + 0 = 0.$$

This is because the term $\overline{u}^0\overline{u}^0_y = \frac{d}{dy}\frac{1}{2}(\overline{u}^0)^2$, and for the last term, we had $\int_{\mathbb{R}} \overline{u}^0_{yy} = \overline{u}^0_y|^{\infty}_{-\infty}$ which we assumed is zero. Due to the matching conditions, $\frac{1}{2}(\overline{u}^0)^2|^{\infty}_{-\infty} = \frac{(u_0^+)^2-(u_0^-)^2}{2}$ and $-\overline{u}(0)s'(t)|^{\infty}_{-\infty} = -s'(t)(u_0^+ - u_0^-)$. We then have that

$$s'(t) = \frac{(u_0^+)^2 - (u_0^-)^2}{2(u_0^+ - u_0^-)} = \frac{(u_0^+ - u_0^-)(u_0^+ + u_0^-)}{2(u_0^+ - u_0^-)} = \frac{u_0^+ + u_0^-}{2}$$

as claimed.