APMA 1941G Homework 7 Solutions

Lulabel Ruiz Seitz
(with collaboration from Will Kwon)
March 18, 2024

Problem 1

This relates to example four from lecture, which was about a nonlinear oscillator with damping.

(a)

Let A be the area of the region $\{(u_0, V) \in \mathbb{R}^2 \mid \frac{1}{2}V^2 + \Phi(u_0) \leq E\}$. Show that $A = \int_0^{2\pi} \omega^0 (u_\eta^0)^2 d\eta$.

Proof. First, consider the boundary of the region, which we can rewrite as $V = \pm\sqrt{2(E - \Phi(u_0))}$. By symmetry, the area is given by

$$A = 2\int_{a(E)}^{b(E)} \sqrt{2(E - \Phi(u_0))} du_0, \tag{1}$$

where $a(E)$ and $b(E)$ are the endpoints. This is because we used $a(E)$ and $b(E)$ to denote the points where $\Phi(u_0) = E$ in class; at these values, $V = 0$, so these are indeed where the function crosses the u_0-axis. We make the change of variables $u_0 = u_0(\eta)$ and assume u_0 is increasing in η, $u_0(0) = a(E)$ and $u_0(\pi) = b(E)$ as we did in class. Using the given identity, $\sqrt{2(E - \Phi(u_0))} = \omega_0 u_\eta^0$, we may then rewrite

$$A = 2\int_0^\pi \omega^0 u_\eta^0 u_\eta^0 d\eta = 2\int_0^\pi \omega^0 (u_\eta^0)^2 d\eta.$$

Using the evenness and 2π-periodicity of $(u_0)^2$, as in class, we obtain the desired relationship,

$$A = \int_0^{2\pi} \omega^0 (u_\eta^0)^2 d\eta.$$

(b)

Show that

$$\frac{dA}{dE} = \frac{2\pi}{\omega^0(E)}.$$

Proof. Following the hint, we differentiate (1) with respect to E. Using the given formula,

$$\begin{aligned}
\frac{dA}{dE} &= 2\frac{d}{dE}\int_{a(E)}^{b(E)} \sqrt{2(E - \Phi(u_0))} du_0 \\
&= 2\left(\int_{a(E)}^{b(E)} \frac{1}{\sqrt{2(E - \Phi(u_0))}} du_0 + b'(E)\sqrt{2(E - \Phi(b(E)))} - a'(E)\sqrt{2(E - \Phi(a(E)))}\right) \\
&= 2\int_{a(E)}^{b(E)} \frac{1}{\sqrt{2(E - \Phi(u_0))}} du_0
\end{aligned}$$

where we used the fact that $\Phi(a(E)) = \Phi(b(E)) = E$ to see that all of the terms added to the integral are zero. Using the same change of variables as in part (a), we then obtain

$$\frac{dA}{dE} = 2\int_0^\pi \frac{1}{\omega^0(E) u_\eta^0} u_\eta^0 d\eta = 2\frac{1}{\omega^0(E)}\int_0^\pi d\eta = \frac{2\pi}{\omega^0(E)}$$

as claimed.

Problem 2

Consider

$$u_\epsilon''(t) + \omega^2(\epsilon t)\sin(u_\epsilon(t)) = 0 \tag{2}$$

where $u^\epsilon = u^\epsilon(t)$ and $\omega = \omega(s) > 0$. Suppose u_ϵ has the form

$$u_\epsilon(t) = u\left(\frac{\theta(\epsilon t, \epsilon)}{\epsilon}, \epsilon t, \epsilon\right) \tag{3}$$

where $u = u(\eta, \tau, \epsilon)$, $\theta = \theta(\tau, \epsilon)$, and $\eta \mapsto u(\eta, \tau, \epsilon)$ are 2π-periodic. Apply the usual ansatz

$$u = u^0 + \epsilon u^1 + ...$$
$$\theta = \theta^0 + \epsilon\theta^1 + ...$$

and choose θ_0 such that $\theta_\tau^0 = \omega$. Let $\omega_0 \doteq \theta_\tau^0 = \omega$ and $\omega_1 \doteq \theta_\tau^1$.

(a)

Let E be the energy

$$E(\tau, \eta) = \frac{1}{2}(u_\eta^0)^2 - \cos(u^0).$$

Showing, using the $O(1)$-terms, that $E = E(\tau)$.

Proof. First, substitute the ansatz into (2). To do so, we compute, using (3),

$$\begin{aligned} u_\epsilon' &= (u^0 + \epsilon u^1 + ...)' \\ &= u_\eta^0 \cdot \frac{1}{\epsilon}\theta_\tau \cdot \epsilon + \epsilon u_\tau^0 + \epsilon(u_\eta^1 \cdot \frac{1}{\epsilon}\theta_\tau \cdot \epsilon + \epsilon u_\tau^1) + ... \\ &= u_\eta^0\theta_\tau^0 + \epsilon(u_\eta^0\theta_\tau^1 + u_\tau^0 + u_\eta^1\theta_t a u^0) + \epsilon^2(u_\eta^1\theta_\tau^1 + u_\tau^1) + ... \end{aligned}$$

and

$$\begin{aligned} u_\epsilon'' = {} & u_\eta^0\theta_{\tau\tau}^0\epsilon + \theta_\tau^0(u_{\eta\eta}^0(\theta_\tau^0 + \epsilon\theta_\tau^1 + ...) + u_{\eta\tau}^0\epsilon) + \epsilon(u_\eta^0\theta_{\tau\tau}^1\epsilon + \theta_\tau^1(u_{\eta\eta}^0(\theta_\tau^0 + \epsilon\theta_\tau^1 + ...) + u_{\eta\tau}^0\epsilon)) \\ & + u_{\tau\eta}^0(\theta_\tau^0 + \epsilon\theta_\tau^1 + ...) + \epsilon u_{\tau\tau}^0 + \epsilon u_\eta^1\theta_{\tau\tau}^0 + \theta_\tau^0(u_{\eta\eta}^1(\theta_\tau^0 + \epsilon\theta_\tau^1 + ...) + \epsilon u_{\eta\tau}^1) \end{aligned}$$

Just from u_ϵ'', we have:

$$\begin{aligned} O(1): &\quad u_{\eta\eta}^0(\theta_\tau^0)^2 \\ O(\epsilon): &\quad \epsilon(u_\eta^0\theta_{\tau\tau}^0 + 2\theta_\tau^0\theta_\tau^1 u_{\eta\eta}^0 + 2u_{\tau\eta}^0\theta_\tau^0 + (\theta_\tau^0)^2 u_{\eta\eta}^1) \end{aligned}$$

Notice that the term $\omega^2(\tau)\sin(u_\epsilon(t))$, substituting in our ansatz and using a Taylor expansion, becomes

$$\omega^2(\sin(u_0) + \epsilon u^1\cos(u_0) + ...)$$

Overall, the $O(1)$ terms then yield the equation

$$u_{\eta\eta}^0(\theta_\tau^0)^2 + \omega^2\sin(u^0) = 0.$$

Rewriting this to reflect that $\omega = \theta_\tau^0$, we have

$$\omega^2 u_{\eta\eta}^0 + \omega^2\sin(u^0) = 0.$$

We then have

$$u_{\eta\eta}^0 + \sin(u^0) = 0. \tag{4}$$

We want to use this to show $E = E(\tau)$. To show $E = E(\tau)$, we need to show that $\frac{\partial E}{\partial \eta} = 0$. Evaluating this using the definition of $E(\tau, \eta)$, what we want to show is that

$$u_\eta^0 u_{\eta\eta}^0 + \sin(u^0)u_\eta^0 = 0.$$

Equivalently, we want to show that

$$u_{\eta\eta}^0 + \sin(u^0) = 0.$$

Yet, this is exactly (4), the relationship we showed to be true due to the $O(1)$-terms. Thus, we have shown $E = E(\tau)$ as claimed.

(b)

Differentiate the $O(1)$-terms with respect to θ and let $W \doteq u^0_\eta$ to show that W solves a linear PDE in W. Then multiply the $O(\epsilon)-$terms by W to show that

$$\int_0^{2\pi} \omega^0 (u^0_\eta)^2 d\eta = C.$$

Proof. Differentiating (4) with respect to η and using the definition of W, we have that

$$W_{\eta\eta} + \cos(u^0)W = 0, \tag{5}$$

which is the desired PDE that W satisfies. Next, using the $O(\epsilon)$ terms we found in part (a), we obtain (using also the definition of ω^0 and ω^1)

$$u^0_\eta \omega^0_\tau + 2\omega^0\omega^1 W_\eta + 2W_\tau \omega^0 + (\omega^0)^2 u^1_{\eta\eta} + (\omega^0)^2 u^1 \cos(u^0) = 0. \tag{6}$$

Multiplying (6) by W, considering only the u^1 term, and integrating from $\eta = 0$ to $\eta = 2\pi$, we evaluate

$$\begin{aligned}\int_0^{2\pi} (\omega^0)^2 u^1_{\eta\eta} + (\omega^0)^2 u^1 \cos(u^0) W d\eta &= \int_0^{2\pi} (\omega^0)^2 u^1 W_{\eta\eta} + (\omega^0)^2 u^1 \cos(u^0) W d\eta \\ &= \int_0^{2\pi} (\omega^0)^2 u^1 (W_{\eta\eta} + \cos(u_0)W) d\eta \\ &= 0.\end{aligned}$$

In the first line we used integration by parts to get the first term in the integral, noting that the boundary terms are zero because of the 2π-periodicity assumption. In the third line, we used the PDE we just derived for W, (5). We found that multiplying this part of the overall $O(\epsilon)$ terms and integrating is zero, so we are left with (after factoring out a negative)

$$\int_0^{2\pi} W^2 \omega^0_\tau + 2\omega^0\omega^1 W_\eta W + 2\omega^0 W_\tau W d\eta = 0.$$

Evaluating this further,

$$\begin{aligned} 0 &= \omega^0\omega^1 \int_0^{2\pi} (W^2)_\eta d\eta + \int_0^{2\pi} (W^2\omega^0)_\tau d\eta \\ &= 0 + \int_0^{2\pi} (W^2\omega^0)_\tau d\eta \\ &= \frac{d}{d\tau}\int_0^{2\pi} W^2\omega^0 d\eta \end{aligned}$$

In the first line, we used that ω^0 and ω^1 do not depend on η. In the second line, we used FTC and the 2π-periodicity of W^2. From the last line, we can integrate with respect to τ and use the definition of W to find that

$$\int_0^{2\pi} \omega^0 (u^0_\eta)^2 = C,$$

as claimed.

(c)

Finally, calculate θ_0 in terms of ω.

Solution. Due to the note in the problem set stating that we can solve the three equations (from part (a), (b), and this part) to find u_0, ω^0, and θ_0, we interpret the problem statement to not be asking for a full, explicit calculation. Instead, as in class, we can integrate using the definition of ω_0 to find that

$$\theta_0(\tau) = \theta_0(0) + \int_0^\tau \omega^0(s) ds$$

is the desired equation for θ_0 in terms of $\omega = \omega^0$.