APMA 1941G Homework 8 Solutions
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Problem 1

For this problem, you may use the following integration-by-parts formula, assuming the boundary terms are zero:

$$\int_W Df \cdot Dg d\mathbf{x} = -\int_W \operatorname{div}(Df) g d\mathbf{x}, \tag{1}$$

(a)

Suppose u is a minimizer of

$$I[u] = \int_W \frac{1}{2}|Du|^2 + f(u) + |\mathbf{x}|^2 d\mathbf{x} \tag{2}$$

where $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$, $W \subseteq \mathbb{R}^n$, $f = f(s)$, and $' = \frac{d}{ds}$. Mimic the derivation of the Euler-Lagrange equation to show

$$-\Delta u + f'(u) = 0. \tag{3}$$

Proof. Let v be an arbitrary function (say smooth and compactly supported on W). Let $h \in \mathbb{R}$. Then we can evaluate

$$\Phi(\epsilon) \doteq I[u + \epsilon v] = \int_W \frac{1}{2}|D(u + \epsilon v)|^2 + f(u + \epsilon v) + |\boldsymbol{x}|^2 d\boldsymbol{x}.$$

We know since u is a minimizer by assumption, $\frac{d}{d\epsilon}\Phi(\epsilon)|_{\epsilon=0} = 0$. We now evaluate $\frac{d}{d\epsilon}\Phi$:

$$\begin{aligned}
\frac{d}{d\epsilon}\Phi(\epsilon) &= \int_W \frac{1}{2}\frac{d}{d\epsilon}|D(u + \epsilon v)|^2 + \frac{d}{d\epsilon}f(u + \epsilon v) + \frac{d}{d\epsilon}|\boldsymbol{x}|^2 d\mathbf{x} \\
&= \int_W \frac{d}{d\epsilon}\left(\frac{1}{2}|Du|^2 + \epsilon Du \cdot Dv + \epsilon^2|Dv|^2 + f(u + \epsilon v) + |\mathbf{x}|^2\right) d\mathbf{x} \\
&= \int_W Du \cdot Dv + 2\epsilon|Dv|^2 + f'(u + \epsilon v)v d\mathbf{x} \\
&= \int_W (-\Delta u + f'(u + \epsilon v))v + 2\epsilon|Dv|^2 d\mathbf{x},
\end{aligned}$$

where in the last line, we used integration by parts and the assumption that v vanishes on the boundary. We also know that when $\epsilon = 0$,

$$0 = \frac{d}{d\epsilon}\Phi(\epsilon)|_{\epsilon=0} = \int_W (-\Delta u + f'(u))v + d\mathbf{x}.$$

By the fundamental theorem of the calculus of variations though, since v was arbitrary, we may then conclude exactly (3),

$$-\Delta u + f'(u) = 0.$$

(b)

This part refers to Example 7 (An Eikonal/Continuity PDE): Suppose that a^0 and θ^0 minimize $I^0[a^0, \theta^0]$. By doing a variation in θ^0, show that θ^0 must satisfy the equation

$$-\operatorname{div}(|a_0|^2 D\theta^0) = 0. \tag{4}$$

Proof. We recall from class that

$$I^0[a^0, \theta^0] = \int_{\mathbb{R}^n} \frac{1}{2}|a^0|^2|D\theta^0|^2 + \frac{1}{2}V(x)|a^0|^2 dx. \tag{5}$$

Now let $v \in C_0^\infty(\mathbb{R}^n)$ and consider a variation in θ^0. We then evaluate

$$\Psi(\epsilon) \doteq I^0[a^0, \theta^0 + \epsilon v] = \int_{\mathbb{R}^n} \frac{1}{2}|a^0|^2(|D\theta^0|^2 + 2\epsilon Dv \cdot D\theta^0 + \epsilon^2|Dv|^2) + \frac{1}{2}V(x)|a^0|^2 d\mathbf{x}.$$

We compute as before

$$\frac{d}{d\epsilon}\Psi(\epsilon) = \int_{\mathbb{R}^n} |a^0|^2 Dv \cdot D\theta^0 + |a^0|^2\epsilon|Dv|^2 d\mathbf{x}.$$

At $\epsilon = 0$, we then have

$$0 = \int_{\mathbb{R}^n} |a^0|^2 Dv \cdot D\theta^0.$$

Using integration by parts, the assumption on v, and changing the notation from $D\cdot$ to divergence, we obtain

$$0 = -\int_W \operatorname{div}((a_0)^2 D\theta^0) = 0.$$

Again using the fundamental theorem of the calculus of variations, we have exactly the identity (4) as claimed.

(c)

Let f and ϕ be fixed functions. Find $L = L(p, z, x)$ so that

$$-\Delta u + D\phi \cdot Du = f(x)$$

is the Euler-Lagrange equation corresponding to the functional

$$I[u] = \int_W L(Du, u, x)dx.$$

Solution. Following the hint, we first find a Lagrangian corresponding to the simpler PDE $-\Delta u = f$. Inspired by an example from class, we try $L(p, z, x) = \frac{1}{2}|p|^2 - zf$. Now we check it. The Euler-Lagrange equation is given by

$$-\sum_{i=1}^{n}(L_{p_i}(p, z, x))_{x_i} + L_z(p, z, x) = 0.$$

Evaluating this for our guess, we see that we obtain

$$-\sum_{i=1}^{n}(D_i u)_{x_i} - f = 0,$$

which is exactly $-\Delta u = f$. Following the hint, we multiply by an exponential term involving ϕ. We try the simplest option:

$$L(p, z, x) \doteq e^{-\phi(x)}\left(\frac{1}{2}|p|^2 - zf\right). \tag{6}$$

Plugging this into the Euler-Lagrange equation,

$$-\sum_{i=1}^{n}(e^{-\phi(x)}D_i u)_{x_i} - e^{-\phi(x)}f = 0.$$

Using the product rule, this is

$$-\sum_{i=1}^{n} e^{-\phi(x)}D_{ii}u - D_i\phi(x)e^{-\phi(x)}D_i u = e^{-\phi(x)}f.$$

Multiplying through by $e^{\phi(x)}$,

$$-\sum_{i=1}^{n} D_{ii}u - D_i\phi(x)D_i u = f,$$

which we can rewrite as

$$-\Delta u + D\phi \cdot Du = f(x),$$

which is what we wanted. Thus, the desired Lagrangian is indeed (6).

Problem 2

Consider the following ODE on $(0,1)$

$$\begin{cases} \epsilon u_\epsilon'' + u_\epsilon' + u_\epsilon = 0 \\ u_\epsilon(0) = 0, \quad u_\epsilon(1) = 1 \end{cases}$$

and apply the ansatz

$$u^\epsilon(x) = u^0\left(x, \frac{x}{\epsilon}\right) + \epsilon u^1\left(x, \frac{x}{\epsilon}\right) + ...$$

where $u^k = u^k(x, \tau)$.

(a)

Find the $O\left(\frac{1}{\epsilon}\right)$ terms to get a formula for $u^0(x,\tau)$ in terms of constants $A(x)$ and $B(x)$.

Solution. In order to collect terms, we first need to substitute in our ansatz. To do so, we first calculate

$$\frac{d}{dx}u^\epsilon(x) = u_x^0 + \frac{1}{\epsilon}u_\tau^0 + \epsilon u_x^1 + u_\tau^1$$

and

$$\begin{aligned} \frac{d^2}{dx^2}u^\epsilon(x) &= u_{xx}^0 + \frac{1}{\epsilon u_{\tau x}^0} + \frac{1}{\epsilon}u_{\tau x}^0 + \frac{1}{\epsilon^2}u_{\tau\tau}^0 + \epsilon u_{xx}^1 + 2u_{x\tau}^1 + \frac{1}{\epsilon}u_{\tau\tau}^1 \\ &= u_{xx}^0 + \frac{2}{\epsilon}u_{x\tau}^0 + \frac{1}{\epsilon^2}u_{\tau\tau}^0 + \epsilon u_{xx}^1 + 2u_{x\tau}^1 + \frac{1}{\epsilon}u_{\tau\tau}^1. \end{aligned}$$

Substituting these into the original equation, we obtain

$$\frac{1}{\epsilon}\left(u_{\tau\tau}^0 + u_\tau^0\right) + u_{\tau\tau}^1 + 2u_{x\tau}^0 + u_x^0 + u_\tau^1 + u^0 + \epsilon(u_{xx}^0 + 2u_{x\tau}^1 + u_x^1 + u_\epsilon) + ... = 0 \tag{7}$$

Now extracting the $O(\frac{1}{\epsilon})$ terms, we have

$$u_{\tau\tau}^0 + u_\tau^0 = 0.$$

Integrating in τ once, we obtain $u_\tau^0 = A(x)e^{-\tau}$. Integrating in τ again, we obtain

$$u^0(x,\tau) = -A(x)e^{-\tau} + B(x).$$

(b)

Find the $O(1)$-terms to get an ODE of u^1 in terms of u^0. Kill the resonance terms to find (the general form of) $A(x)$ and $B(x)$.

Solution. Now considering the $O(1)$ terms from (7), we have

$$u_{\tau\tau}^1 + 2u_{x\tau}^0 + u_x^0 + u_\tau^1 + u^0 = 0. \tag{8}$$

From part (a), we find that

$$u_x^0 = -e^{-\tau}A'(x) + B'(x)$$

and

$$u_{x\tau}^0 = e^{-\tau}A'(x).$$

We then rewrite (8) as

$$u_{\tau\tau}^1 + u_\tau^1 + e^{-\tau}A'(x) + B'(x) - A(x)e^{-\tau} + B(x) = 0,$$

or equivalently,

$$u_{\tau\tau}^1 + u_\tau^1 = e^{-\tau}(A(x) - A'(x)) - B'(x) - B(x)$$

Due to the left-hand side, we see the homogeneous equation would have the same form as before, so we have to kill the resonance terms by setting

$$A'(x) = A(x) \text{ and } B'(x) = -B(x).$$

Solving these, we then get the forms $A(x) = C_1e^x$ and $B(x) = C_2e^{-x}$.

(c)

Finally, impose the conditions $u^0(0) = 0$ and $u^0(1) = 1$ to find an explicit formula for $u^0(x) = u^0(x, \frac{x}{\epsilon})$.

Solution. From (b), we have that $A(x) = C_1 e^x$ and $B(x) = C_2 e^{-x}$. Then using the result from (a), we have that

$$u^0(x, \tau) = -C_1 e^x e^{-\tau} + C_2 e^{-x}.$$

Since $u^0(0) = 0$, we have that

$$-C_1 + C_2 = 0,$$

so $C_2 = C_1$. Since $u^0(1) = 1$, we have that

$$1 = -C_1 e^{1-1/\epsilon} + C_2 e^{-1} = C_1(-e^{1-1/\epsilon} + e^{-1}).$$

We then have that

$$C_1 = \frac{1}{-e^{1-1/\epsilon} + e^{-1}} = \frac{e}{1 - e^{2-1/\epsilon}}.$$

Overall, then

$$u^0(x, \tau) = \frac{e}{1 - e^{2-1/\epsilon}}(e^{-x} - e^{x-x/\epsilon}).$$