APMA 1941G Homework 9 Solutions
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April 12, 2024

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Problem 1

Consider the following ODE on $(0,1)$, where $u^\epsilon = u^\epsilon(x)$,

$$\begin{gathered} \epsilon u^\epsilon_{xx} + u^\epsilon_x = 2x \\ u^\epsilon(0) = 1, u^\epsilon(1) = 1 \end{gathered} \tag{1}$$

We expect there to be a boundary layer at $x = 0$. Follow the method used in lecture to find a good approximation of u^* of u^ϵ that incorporates the boundary layer. You will only need to use the $O(1)$ terms and the matching part can be done with any method.

Solution. First, we construct the outer solution, which is near $x = 1$.

Outer Solution. For this, we use the usual ansatz, $u^\epsilon(x) = u^0(x) + \epsilon u^1(x) + ...$ Substituting this into (1), we obtain

$$(\epsilon u^0_{xx} + \epsilon^2 u^1_{xx}) + (u^0_x + \epsilon u^1_x) + ... = 2x. \tag{2}$$

The $O(1)$ terms yield just

$$u^0_x = 2x. \tag{3}$$

We can solve this to obtain that $u^0(x) = x^2 + C$. Imposing the boundary condition at 1, we have that $u(1) = 1 + C = 1$, so $C = 0$. Then our outer solution is just

$$\boxed{u^0(x) = x^2}.$$

Inner Solution. Now we construct the solution near $x = 0$. We use the change of variables $y = x/\epsilon^\alpha$, where α is to be determined. Let $\overline{u}^\epsilon(y) \doteq u^\epsilon(x)$. Then we can compute

$$\begin{aligned} u^\epsilon_x &= \frac{d\overline{u}^\epsilon}{dy}\frac{dy}{dx} = \frac{1}{\epsilon^\alpha}\overline{u}^\epsilon_y \\ u^\epsilon_{xx} &= \frac{1}{\epsilon^{2\alpha}}\overline{u}^\epsilon_{yy} \end{aligned}$$

from which the original ODE becomes

$$\epsilon^{1-2\alpha}\overline{u}^\epsilon_{yy} + \epsilon^{-\alpha}\overline{u}^\epsilon_y - 2y\epsilon^\alpha = 0. \tag{4}$$

In order to apply the method of dominant balance, label $A \doteq \epsilon^{1-2\alpha}\overline{u}^\epsilon_{yy}$, $B \doteq \epsilon^{-\alpha}\overline{u}^\epsilon_y$, and $C \doteq -2y\epsilon^\alpha$. If $A \sim B$ and C is smaller, then $1 - 2\alpha = -\alpha$, so $\alpha = 1$, in which case C is of the order ϵ. This is indeed smaller that ϵ^{-1} when taking the limit as $\epsilon \to 0^+$, so this works. To be sure, we verify the other options from dominant balance do not work. If $A \sim C$ and B is smaller, then $1 - 2\alpha = \alpha$, so $\alpha = 1/3$, but then B is of the order $\epsilon^{-1/3}$ which is not smaller than $\epsilon^{1/3}$ as $\epsilon \to 0^+$, violating our assumption B is smaller. If $B \sim C$ and A is smaller, then $\alpha = 0$, but then $y = x$ and we have no boundary layer. Accordingly, we do need $\alpha = 1$. Rewriting (5) with this knowledge,

$$\epsilon^{-1}\overline{u}^\epsilon_{yy} + \epsilon^{-1}\overline{u}^\epsilon_y - 2y\epsilon = 0. \tag{5}$$

Multiplying through by ϵ,

$$\overline{u}^\epsilon_{yy} + \overline{u}^\epsilon_y - 2y\epsilon^2 = 0..$$

Now we can make our usual ansatz, $\overline{u}^\epsilon(u) = \overline{u}^0(y) + \epsilon\overline{u}^1(y) + ...$ Substituting this in, we obtain

$$\overline{u}^0_{yy} + \epsilon\overline{u}^1_{yy} + \overline{u}^0_y + \epsilon\overline{u}^1_y - 2y\epsilon^2 + ... = 0.$$

The $O(1)$ terms yield

$$\overline{u}^0_{yy} + \overline{u}^0_y = 0.$$

Solving this,

$$\overline{u}^0(y) = Ae^{-y} + B.$$

Using the boundary term for the inner solution, so near zero, we have $\overline{u}^0(0) = 1$. Then $1 = A + B$, so $B = 1 - A$, and $\overline{u}^0(y) = A(e^{-y} - 1) + 1$. We need to match this with the outer solution to find A. We do so by using the first method from class. We see that

$$\lim_{x \to 0} x^2 = 0 \text{ and } \lim_{y \to \infty} A(e^{-y} - 1) = -A + 1.$$

We then need to set $A = 1$ to have matching, which means that the inner solution is

$$\boxed{\overline{u}^0(y) = e^{-y}}.$$

Overall Solution. We construct u^* by

$$u^*(x) = u^0(x) + \overline{u}^0(y) - \text{ common part } .$$

We saw the common part was zero, so assembling what we found for $u^0(x)$ and $\overline{u}^0(y)$ and using the change of variables $y = x/\epsilon$,

$$\boxed{u^*(x) = x^2 + e^{-x/\epsilon}}.$$

Problem 2

Consider the following ODE on $(0, 1)$ where $u^\epsilon = u^\epsilon(x)$

$$\begin{aligned} \epsilon u^\epsilon_{xx} + u^\epsilon_x + u^\epsilon = 0 \\ u^\epsilon(0) = 1, u^\epsilon(1) = 1 \end{aligned} \tag{6}$$

We expect there to be a boundary layer at $x = 0$. Follow the method used in lecture to find a good approximation of u^* of u^ϵ that incorporates the boundary layer. This time, go up to the $O(\epsilon)$ terms.

Solution. We follow the same overall method, first constructing the outer solution and then the inner solution, and using those to construct the overall solution.

Outer Solution. We use the ansatz $u^\epsilon(x) = u^0(x) + \epsilon u^1(x) + ...$ as before. Substituting this ansatz into (6), we obtain

$$(\epsilon u^0_{xx} + \epsilon^2 u^1_{xx}) + (u^0_x + \epsilon u^1_x) + (u^0 + \epsilon u^1) + ... = 0. \tag{7}$$

The $O(1)$ terms from (7) yield

$$u^0_x + u^0 = 0,$$

which we solve to obtain $u^0(x) = Ae^{-x}$. Since $u^0(1) = 1$ (we are in the outer region, so we use the boundary condition at 1), we obtain that $Ae^{-1} - 1 = 0$, so $A = e$. We have found the outer solution

$$\boxed{u^0(x) = e^{1-x}}.$$

Now considering the $O(\epsilon)$ terms from (7), we have

$$u^0_{xx} + u^1_x + u^1 = 0.$$

Substituting in what we found for u^0, we have that

$$u^1_x + u^1 = -e^{1-x}.$$

Solving this using the method of undetermined coefficients, we have that

$$u^1(x) = Ae^{-x} - xe^{-x+1}.$$

Since $u^1(1) = 1$ as well, $Ae^{-1} - 1 = 0$, so $A = e$. We then have found the outer solution

$$\boxed{u^1(x) = e^{1-x}(1-x)}.$$

Inner Solution. We again use the change of variables $y = x/e^\alpha$, where α is to be determined. Again denoting $\overline{u}^\epsilon(y) = u^\epsilon(x)$, we can write the ODE in terms of y:

$$\epsilon^{1-2\alpha}\overline{u}^\epsilon_{yy} + \epsilon^{-\alpha}\overline{u_y}^\epsilon + \overline{u}^\epsilon = 0. \tag{8}$$

Now label $A \doteq \epsilon^{1-2\alpha}\overline{u}^\epsilon_{yy}$, $B \doteq \epsilon^{-\alpha}\overline{u_y}^\epsilon$, and $C \doteq \overline{u}^\epsilon$. Using the method of dominant balance, we have that $A \sim B$ and C is small, so $\alpha = 1$ (indeed, then C is constant in ϵ, which is smaller than ϵ^{-1} as $\epsilon \to 0^+$). We can then rewrite (8), after multiplying through by ϵ, as

$$\overline{u}^\epsilon_{yy} + \overline{u}^\epsilon_y + \epsilon\overline{u}^\epsilon = 0.$$

Making the usual ansatz and substituting it in,

$$(\overline{u}^0_{yy} + \epsilon\overline{u}^1_{yy}) + (\overline{u}^0_y + \epsilon\overline{u}^1_y) + \epsilon(\overline{u}^0 + \epsilon\overline{u}^1) + ... = 0.$$

The $O(1)$ terms from this yield that

$$\overline{u}^0_{yy} + \overline{u}^0_y = 0,$$

which we can solve to find that $\overline{u}^0(y) = A + Be^{-y}$. Using the boundary condition $\overline{u}^0(0) = 0$, then $B = -A$, and $\overline{u}^0(y) = A(1 - e^{-y})$. In this case, we can use the first matching method to find A. Since $\lim_{x\to 0^+} e^{1-x} = e$, and $\lim_{y\to\infty} A(1 - e^{-y}) = A$, we set $A = e$ and obtain that

$$\boxed{\overline{u}^0(y) = e(1 - e^{-y})}.$$

Now we consider the $O(\epsilon)$ terms. We obtain that

$$\epsilon\overline{u}^1_{yy} + \epsilon\overline{u}^1_y + \epsilon\overline{u}^0 = 0.$$

Multiplying through by ϵ^{-1}, and substituting in what we just found for $\overline{u}^0$, we obtain that

$$\overline{u}^1_{yy} + \overline{u}_y = -e(1 - e^{-y}).$$

Solving this and using the boundary condition $\overline{u}^1(0) = 0$, we find that

$$\overline{u}^1(y) = A(1 - e^{-y}) - ye(1 + e^{-y}).$$

We need to use a matching method to find A, but we cannot use the simpler method in this case because the limit of the above as $y \to \infty$ does not converge. Instead, we use the second method. Define the intermediate variable $x = x/\epsilon^\beta$, where $x \in (\epsilon^{\alpha_1}, \epsilon^{\alpha_2})$ as in class (this interval is within $(0, 1)$, and z is defined so that it is between x and $y = x/\epsilon$). Notice that $y = \epsilon^{\beta-1}z$. Now rewriting $u^0(x), \epsilon u^1(x)$ and $\overline{u}^0(y) + \epsilon\overline{u}^1(y)$ in terms of z,

$$\begin{aligned} u^0(x) + \epsilon u^1(x) &= e^{1-\epsilon^\beta z} + \epsilon(1 - \epsilon^\beta z)e^{1-\epsilon^\beta z} \\ \overline{u}^0(y) + \epsilon\overline{u}^1(y) &= e(1 - e^{-\epsilon^{\beta-1}z}) + \epsilon(A(1 - e^{-\epsilon^{\beta-1}z})) - \epsilon \cdot \epsilon^{\beta-1}ze(1 + e^{-\epsilon^{\beta-1}z}) \end{aligned}$$

Taking the limit as $\epsilon^\beta z \to 0$ and $\epsilon^{\beta-1}z \to \infty$, we see that $u^0(x) + \epsilon u^1(x)$ goes to $e + \epsilon e$, while $\overline{u}^0(y) + \epsilon\overline{u}^1(y)$ goes to $e + \epsilon A - 0 = e + \epsilon A$. Matching these, we have that $e + \epsilon e = e + \epsilon A$, which means we want to set $A = e$. Then

$$\boxed{\overline{u}^1(y) = e(1 - e^{-y}) - ye(1 + e^{-y})}.$$

Overall Solution. We construct u^* by

$$u^*(x) = u^0(x) + \epsilon u^1(x) + \overline{u}^0(y) + \epsilon\overline{u}^1(y) - \text{ common part }.$$

From the previous step, we saw that the common part was $e + \epsilon e = e(1 + \epsilon)$. By assembling all the parts we found and changing variables $y = x/\epsilon$, we obtain the overall solution

$$\boxed{u^*(x) = e^{1-x} + \epsilon e^{1-x}(1-x) + e(1 - e^{-x/\epsilon}) + \epsilon\left(e(1 - e^{-x/\epsilon}) - \frac{x}{\epsilon}e(1 + e^{-x/\epsilon})\right) - e(1+\epsilon)}.$$