Asymptotic Methods in Differential Equations

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Contents

1 Introduction

We derived the following KdV equations of general form

$$u_t + auu_x + bu_{xxx} = 0.$$

We look for solutions of the form $u(t,x) = \phi(x - ct)$, which is called solitons or traveling waves.

If we put $u(t,x) = \phi(x - ct)$ into KdV, then

$$\phi'' + f(\phi) = 0,$$

where

$$f(\phi) = \frac{a}{2b}(\phi^2) - \frac{c}{b}\phi.$$

In fact, this is explicitly solvable. Multiplying ϕ' to the equation, then we can get

$$\frac{1}{2}(\phi')^2 + F(\phi) = C,$$

where F is an antiderivative of f and C is a constant. In other words, we have

$$\frac{d\phi}{dt} = \sqrt{2(C - F(\phi))}.$$

By using separable variable, we get implicit formula for ϕ.

1.1 Theoretical aspects

What does

$$f \sim a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$$

means?

Let $\{a_k\}_{k=0}^{\infty}$ be a sequence in $\mathbb{R}$ and $f = f(\varepsilon) : (0,\infty) \to \mathbb{R}$. We write f has an *asymptotic expansion* if $f \sim \sum_{k=0}^{\infty} a_k\varepsilon^k$ as $\varepsilon \to 0$ provided that for all N

$$f(\varepsilon) - \sum_{k=0}^{N} a_k\varepsilon^k = o(\varepsilon^N)$$

as $\varepsilon \to 0$. We say that $\sum_{k=0}^{\infty} a_k\varepsilon^k$ is an asymptotic expansion for f at $\varepsilon = 0$.

Lemma 1.1. *If $f \sim \sum_{k=0}^{\infty} a_k\varepsilon^k$ and $f \sim \sum_{k=0}^{\infty} b_k\varepsilon^k$, then $a_k = b_k$ for all k.*

Proof. Note that

$$\varepsilon^{-N}\left|\sum_{k=0}^{N} a_k\varepsilon^k - \sum_{k=0}^{N} b_k\varepsilon^k\right| \le \varepsilon^{-N}\left|\sum_{k=0}^{N} a_k\varepsilon^k - f(\varepsilon)\right| + \varepsilon^{-N}\left|\sum_{k=0}^{N} b_k\varepsilon^k - f(\varepsilon)\right|.$$

Hence by letting $\varepsilon \to 0+$, we have

$$\lim_{\varepsilon\to 0+} \varepsilon^{-N}\left|\sum_{k=0}^{N} a_k\varepsilon^k - \sum_{k=0}^{N} b_k\varepsilon^k\right| = 0$$

for each N. Then the result is followed by induction. □

Remark. (a) This is not a power series expansion. If $f \sim \sum_{k=0}^{\infty} a_k \varepsilon^k$ and $g \sim \sum_{k=0}^{\infty} a_k \varepsilon^k$, then f and g may not be equal. Take $f(\varepsilon) = e^{-1/\varepsilon}$ and $g(\varepsilon) = 0$. Then 0 is the asymptotic expansion of f and g.

(b) We do not claim that the series $\sum_{k=0}^{\infty} a_k \varepsilon^k$ converges for any ε.

Lemma 1.2 (Borel's lemma)**.** *Given any sequence* $\{a_k\}_{k=0}^{\infty}$*, there exists a function* f *such that* $f \sim \sum_{k=0}^{\infty} a_k \varepsilon^k$*.*

Proof. Start with $\delta_0 = 1$ and choose $\delta_1 > 0$ such that $0 < \delta_1 < 1/2$ and $|a_1 \varepsilon| < \frac{1}{2}|a_0 \varepsilon^0|$ for $\varepsilon \in (0, \delta_1)$. Continue this procedure, i.e., choose δ_k such that $0 < \delta_k < \delta_{k-1}/2$ and $|a_{k+1}\varepsilon^{k+1}| < 2^{-1}|a_k \varepsilon^k|$ for all $\varepsilon \in (0, \delta_k)$.

For each k, choose a cut-off function ψ_k so that $\psi_k = 1$ on $[0, \delta_{k+1}]$ and $\psi_k = 0$ outside $[0, \delta_k]$. Fix k and let $l \in \mathbb{N}$. We claim that

$$|a_{k+l}\psi_{k+l}(\varepsilon)\varepsilon^{k+l}| \leq \frac{1}{2^l}|a_k \varepsilon^k|$$

for all l. Since $\psi_{k+l}(\varepsilon) = 0$ for $\varepsilon > \delta_{k+l}$, then the estimate is obvious. If $0 < \varepsilon < \delta_{k+l}$, then since $0 \leq \psi_{k+l} \leq 1$, it follows that

$$|a_{k+l}\psi_{k+l}(\varepsilon)\varepsilon^{k+l}| \leq |a_{k+l}\varepsilon^{k+l}| < \frac{1}{2}|a_{k+l-1}\varepsilon^{k+l-1}|.$$

Then by induction, we get

$$|a_{k+l}\psi_{k+l}(\varepsilon)\varepsilon^{k+l}| \leq |a_{k+l}\varepsilon^{k+l}| < \frac{1}{2^l}|a_k \varepsilon^k|$$

since $(0, \delta_{k+l}) \subset (0, \delta_k)$.

Now define

$$f(\varepsilon) = \sum_{k=0}^{\infty} (a_k \psi_k(\varepsilon))\varepsilon^k.$$

Then the function f is well-defined since

$$\sum_{k=0}^{\infty} |a_k \psi_k(\varepsilon)|\varepsilon^k \leq \sum_{k=0}^{\infty} |a_k \varepsilon^k| \leq \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k+1} |a_0| < \infty.$$

It remains for us to show that $f \sim \sum_{k=0}^{\infty} a_k \varepsilon^k$.

$$\lim_{\varepsilon \to 0+} \frac{\left| f(\varepsilon) - \sum_{k=0}^{N} a_k \varepsilon^k \right|}{\varepsilon^N}$$

□

2 Asymptotic evaluation of integrals

2.1 Laplace's method

From now on, we assume that $\phi : \mathbb{R} \to \mathbb{R}$ is smooth, has a unique global max at 0, and $\phi'(0) = 0$, $\phi''(0) < 0$. We also assume that $a \in C_c^\infty(\mathbb{R})$ and $0 \in \operatorname{supp} a$. The goal is to understand

$$I[\varepsilon] = \int_{-\infty}^{\infty} a(x) e^{\phi(x)/\varepsilon}\, dx \quad \text{as } \varepsilon \to 0.$$

We first study a special case to look at the asymptotic behavior of $I[\varepsilon]$.

Theorem 2.1. *Consider*

$$I[\varepsilon] = \int_{-\infty}^{\infty} a(x) e^{-bx^2/(2\varepsilon)}\, dx$$

for some $\varepsilon > 0$. Then

$$I[\varepsilon] \sim \sum_{k=0}^{\infty} \frac{a^{(k)}(0)}{k!} C_k \varepsilon^{\frac{\varepsilon+1}{2}},$$

where

$$C_k = \int_{-\infty}^{\infty} y^k e^{-\frac{b}{2}y^2}\, dy.$$

Observe that $C_k = 0$ if k is odd.

Proof. Fix $0 < \varepsilon < 1$ and let $r = r(\varepsilon) > 0$ which will be determined. Then

$$\begin{aligned} I[\varepsilon] &= \int_{-\infty}^{\infty} a(x) e^{-bx^2/(2\varepsilon)} dx = \int_{-r}^{r} a(x) e^{-bx^2/(2\varepsilon)}\, dx + \int_{\mathbb{R}\setminus(-r,r)} a(x) e^{-\frac{bx^2}{2\varepsilon}}\, dx \\ &= A + B. \end{aligned}$$

We first estimate B. Since a is bounded, it follows that

$$\begin{aligned} |B| &\lesssim \int_{\mathbb{R}\setminus(-r,r)} e^{-\frac{bx^2}{2\varepsilon}}\, dx \\ &\lesssim \int_{\mathbb{R}\setminus(-r,r)} e^{-\frac{bx^2}{4\varepsilon}} e^{-\frac{bx^2}{4\varepsilon}}\, dx \\ &\lesssim e^{-\frac{br^2}{4\varepsilon}} \int_{\mathbb{R}\setminus(-r,r)} e^{-\frac{b}{4\varepsilon}x^2}\, dx. \end{aligned}$$

A change of variable gives

$$\begin{aligned} &= C e^{-\frac{br^2}{4\varepsilon^2}} \sqrt{\varepsilon} \int_{\mathbb{R}\setminus(-r/\sqrt{\varepsilon}, r/\sqrt{\varepsilon})} e^{-br^2/4} dr \\ &\lesssim e^{-\frac{br^2}{4\varepsilon}} \sqrt{\varepsilon} \end{aligned}$$

Next we estimate A. By a change of variable, we have

$$A = \sqrt{\varepsilon} \int_{-r/\sqrt{\varepsilon}}^{r/\sqrt{\varepsilon}} a(\sqrt{\varepsilon} y) e^{-by^2/2} dy.$$

Now expand

$$a(\sqrt{\varepsilon} y) = \sum_{k=0}^{\infty} \frac{a^{(k)}(0)}{k!} (\sqrt{\varepsilon} y)^k = \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} x^k + O(|x|^{N+1}).$$

Then

$$A = \sqrt{\varepsilon} \int_{-r/\sqrt{\varepsilon}}^{r/\sqrt{\varepsilon}} \left(\sum_{k=0}^{N} \frac{a^{(k)}(0)}{k!} (\sqrt{\varepsilon} y)^k + O(|\sqrt{\varepsilon} y|^{N+1}) \right) e^{-\frac{b}{2} y^2} dy.$$

We further decompose this integral into the following way:

$$\begin{aligned} A = & \sqrt{\varepsilon} \int_{-\infty}^{\infty} \sum_{k=0}^{N} \frac{a^{(k)}(0)}{k!} (\sqrt{\varepsilon} y)^k e^{-\frac{b}{2} y^2} dy \\ & - \sqrt{\varepsilon} \int_{\mathbb{R} \setminus (-r/\sqrt{\varepsilon}, r/\sqrt{\varepsilon})} \sum_{k=0}^{N} \frac{a^{(k)}(0)}{k!} (\sqrt{\varepsilon} y)^k e^{-\frac{b}{2} y^2} dy \\ & + \sqrt{\varepsilon} \int_{-r/\sqrt{\varepsilon}}^{r/\sqrt{\varepsilon}} O(|\sqrt{\varepsilon} y|^{N+1}) e^{-\frac{b}{2} y^2} dy. \end{aligned}$$

To estimate the first integral, one can easily check that it is equal to

$$\sum_{k=0}^{N} \frac{a^{(k)}(0)}{k!} C_k \varepsilon^{\frac{k+1}{2}}.$$

The second integral can be estimated in a similar way as in B, which is bounded by $Ce^{-br^2/4\varepsilon}$.

To estimate the last integral, by the definition, it is bounded by

$$\begin{aligned} & \sqrt{\varepsilon} \int_{-r/\sqrt{\varepsilon}}^{r/\sqrt{\varepsilon}} (\sqrt{\varepsilon})^{N+1} |y|^{N+1} e^{-b\frac{y^2}{2}} \, dy \\ & \leq C \sqrt{\varepsilon} r^{N+1} \int_{-r/\sqrt{\varepsilon}}^{r/\sqrt{\varepsilon}} e^{-by^2/2} dy \\ & \leq C \sqrt{\varepsilon} r^{N+1}. \end{aligned}$$

In the end, we get

$$\left| I[\varepsilon] - \sum_{k=0}^{N} \frac{a^{(k)}(0)}{k!} C_k \varepsilon^{\frac{k+1}{2}} \right| \lesssim \sqrt{\varepsilon} r^{N+1} + e^{-\frac{br^2}{4\varepsilon}}.$$

Then the desired result follows by choosing $r = \varepsilon^{\sigma}$, where $\frac{N}{2(N+1)} < \sigma < 1/2$. $\square$

In fact, the general case also holds. The result follows from the Morse lemma and the previous case.

Theorem 2.2. *We have*

$$I[\varepsilon] \sim \sum_{k=0}^{\infty} (L_{2k}a)(0)\varepsilon^{\frac{k+1}{2}},$$

where

$$(L_{2k}a)(0) = b^0 a(0) + b^1 a'(0) + \cdots + b^{2k} a^{(2k)}(0)$$

and $b^0, b^1, \ldots, b^{2k} \in \mathbb{R}$.

Remark. In the previous case, we note that

$$b^0 = 0,\, b^1 = 0,\, b^2 = 0, \quad b^k = \frac{c_k}{k!}, \ldots, b^{2k} = 0.$$

Proof. Let ψ (and U and V) be given by the Morse lemma. Let η be a smooth support function satisfying $\eta = 0$ outside V and $m = 1$ on $(-\delta, \delta) \subset V$ for some $\delta > 0$. Decompose

$$I[\varepsilon] = \int_{-\infty}^{\infty} \eta(x)a(x)e^{\phi(x)/\varepsilon}\, dx + \int_{-\infty}^{\infty} (1 - \eta(x))a(x)e^{\phi(x)/\varepsilon}\, dx.$$

We will show that the second integral is exponentially small. Since $\eta = 1$ on $(-\delta, \delta)$, it follows that

$$\left| \int_{-\infty}^{\infty} (1 - \eta(x))a(x)e^{\phi(x)/\varepsilon}\, dx \right| \leq \int_{\mathbb{R}\setminus(-\delta,\delta)} |a(x)|e^{\phi(x)/\varepsilon}\, dx.$$

Also, $\phi(x) \leq -\gamma$ on $\mathbb{R} \setminus (-\delta, \delta)$ for some $\gamma > 0$. This implies that

$$\left| \int_{-\infty}^{\infty} (1 - \eta(x))a(x)e^{\phi(x)/\varepsilon}\, dx \right| \leq e^{-\gamma/\varepsilon} \int_{\mathbb{R}\setminus(-\delta,\delta)} |a(x)| dx$$

It remains for us to estimate the first integral. Since $\operatorname{supp} \eta \subset V$, a change of variable gives

$$\begin{aligned} \int_{\mathbb{R}} \eta(x)a(x)e^{\phi(x)/\varepsilon}\, dx &= \int_{\psi^{-1}(V)} \eta(\psi(y))a(\psi(y))e^{\phi(\psi(y))/\varepsilon}\psi'(y) dy \\ &= \int_{\mathbb{R}} a(\psi(y))\eta(\psi(y))\psi'(y)e^{-\frac{y^2}{2\varepsilon}}\, dy. \end{aligned}$$

Define $\tilde{a}(y) = a(\psi(y))\eta(\psi(y))\psi'(y)$. Then $\tilde{a}$ is compactly supported and so the above integral becomes

$$= \int_{-\infty}^{\infty} \tilde{a}(y)e^{-y^2/2\varepsilon} dy \sim \sum_{k=0}^{\infty} \frac{(\tilde{a})^{(k)}(0)}{k!} c_k \varepsilon^{\frac{k+1}{2}}.$$

Here we used Theorem 2.1. If we define $(L_{2k}a)(0) = \frac{(\tilde{a})^{(k)}(0)}{k!}$, then we get the desired result. $\square$

Example 2.3. Let us calculate the first temr.

$$I[\varepsilon] \sim L_0(a)(0)\sqrt{\varepsilon} + o(\sqrt{\varepsilon}).$$

Recall that

$$L_0(a)(0) = \tilde{a}(0)c_0 = a(\psi(0))\eta(\psi(0))\psi'(0)c_0.$$

It is easy to see that

$$c_0 = \int_{-\infty}^{\infty} y^0 e^{-y^2/2} dy = \sqrt{2\pi}.$$

Since $\phi(\psi(y)) = -y^2/2$ and $\phi(y) = 0$ implies $y = 0$, it follows that $\psi(0) = 0$. By chain rule, we have

$$\phi'(\psi(y))\psi'(y) = -y.$$

Since $\phi'(0) = 0$, we cannot extract any information from this. By taking differentiation, we have

$$\phi''(\psi(y))(\psi'(y))^2 + \psi'(\psi(y))\phi''(y) = -1.$$

Plugging $y = 0$ in the expression, we get

$$\phi''(0)\psi'(0)^2 = -1,$$

i.e.,

$$\psi'(0)^2 = -\frac{1}{\phi''(0)}.$$

Hence if we choose $\psi'(0) > 0$, then

$$\psi'(0) = \frac{1}{\sqrt{|\phi''(0)|^2}}.$$

From this, we conclude that

$$L_0(a)(0) = a(0)\eta(0)\frac{1}{\sqrt{|\psi''(0)|}}\sqrt{2\pi} = a(0)\sqrt{\frac{2\pi}{|\phi''(0)|}}.$$

Remark. (i) If $\phi(0) \neq 0$, then we can write

$$I[\varepsilon] = e^{\phi(0)/\varepsilon} \int_{-\infty}^{\infty} a(x) e^{\frac{\phi(x)-\phi(0)}{\varepsilon}} dx$$

and we apply the our previous result.

(ii) If ϕ attains a maximum at x_0 instead of 0, then by using a change of variable with translations, we get

$$I[\varepsilon] = \int_{-\infty}^{\infty} a(y + x_0) e^{\phi(y+x_0)/\varepsilon} dy,$$

where $\phi(y + x_0)$ becomes a function which attains at a maximum at $y = 0$. Hence if ϕ attains a global maximum at x_0, $\phi'(x_0) = 0$, and $\phi''(x_0) < 0$, then

$$I[\varepsilon] \sim \sum_{k=0}^{\infty} (L_{2k}a)(x_0) e^{\phi(x_0)/\varepsilon} \varepsilon^{\frac{k+1}{2}} = \sqrt{\frac{2\pi\varepsilon}{|\phi''(x_0)|}} a(x_0) e^{\frac{\phi(x_0)}{\varepsilon}} + o(\sqrt{\varepsilon}).$$

Observe that the above result can be generalized to higher dimensional as well. To do this, we introduce some notations which involve multi-indices. We write $\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$. $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_N$ is the order of index, and $\boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_N!$. Also, we write

$$\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_N^{\alpha_N} \quad \text{and} \quad D^{\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.$$

In particular, we will use Taylor's formula in high-dimension:

$$f(\boldsymbol{x}) = \sum_{|\alpha| \le N} \frac{1}{\alpha!} (D^\alpha f)(0) \boldsymbol{x}^{\boldsymbol{\alpha}} + o(|\boldsymbol{x}|^N)$$

as $|\boldsymbol{x}| \to 0$.

Let $a : \mathbb{R}^N \to \mathbb{R}$ be a smooth and compactly supported and $\phi : \mathbb{R}^N \to \mathbb{R}$ which has a global matrix at 0 with $D\phi(0) = (\phi_{x_1}(\mathbf{0}), \dots, \phi_{x_d}(\mathbf{0})) = \mathbf{0}$. We also assume that $D^2\phi(\mathbf{0})$ is negative definite, i.e., the Hessian $D^2\phi(\mathbf{0})$ has all strictly negative eigenvalues.

Theorem 2.4. *Suppose that* $\phi(\boldsymbol{x}) = -\frac{1}{2}\sum_{i=1}^N a_i x_i^2$ *with* $a_i > 0$. *Then*

$$I[\varepsilon] \sim \sum_{\boldsymbol{\alpha}} \frac{D^{\boldsymbol{\alpha}} a(0)}{\boldsymbol{\alpha}!} c_{\boldsymbol{\alpha}} \varepsilon^{\frac{|\alpha|+N}{2}}$$

as $\varepsilon \to 0$, *where*

$$c_{\boldsymbol{\alpha}} = \int_{\mathbb{R}^N} \boldsymbol{y}^{\boldsymbol{\alpha}} e^{-\phi(\boldsymbol{y})} d\boldsymbol{y}.$$

By using the Morse lemma as well, we can prove the general version as well.

$$I[\varepsilon] \sim \sum_{k=0}^{\infty} (L_{2k}a)(x_0) e^{\phi(x_0)/\varepsilon} \varepsilon^{(k+N)/2} = \frac{(2\pi\varepsilon)^{N/2}}{\sqrt{|\det D^2\phi(x_0)|}} a(x_0) e^{\phi(x_0)/\varepsilon} + o(\sqrt{\varepsilon}).$$

2.2 Stationary phase method

We assume that $a = a(\boldsymbol{x})$ and $\phi = \phi(\boldsymbol{x})$ are smooth, a has compact support. We are interested in evaluating the following integral

$$I[\varepsilon] = \int_{\mathbb{R}^N} a(\boldsymbol{x}) e^{\frac{i\phi(\boldsymbol{x})}{\varepsilon}} \, dx.$$

We call ϕ as *phase*. We will find an asymptotic behavior for $I[\varepsilon]$.

We first consider the rapid decay case.

Theorem 2.5. *If* $\nabla\phi \neq 0$ *everywhere on the support of* a, *then*

$$I[\varepsilon] = o(\varepsilon^M)$$

for all M.

Proof. Given any ψ, define

$$L\psi := \frac{\varepsilon}{i|\nabla\phi|^2}\sum_{j=1}^{N}\phi_{x_j}\psi_{x_j}.$$

It is easy to see that $L(e^{i\phi/\varepsilon}) = e^{i\phi/\varepsilon}$. Hence it follows that

$$\int_{\mathbb{R}^N} f(Lg)\,dx = \varepsilon\int_{\mathbb{R}^N}(Sf)g\,dx,$$

where

$$Sf = -\sum_{j=1}^{N}\left(\frac{1}{i|\nabla\phi|^2}f\phi_{x_j}\right)_{x_j}.$$

Hence for any M, we have

$$I[\varepsilon] = \int_{\mathbb{R}^N} aL^M(e^{i\phi/\varepsilon})\,dx = \varepsilon^M\int_{\mathbb{R}^N} S^M(f)e^{i\phi/\varepsilon}\,dx,$$

which implies that

$$|I[\varepsilon]| \lesssim \varepsilon^M.$$

□

Hence it is legitimate to consider the case $\nabla\phi(x_0) = 0$ for some x_0. We first consider the case $\phi(x) = \frac{b}{2}x^2$, where $b \neq 0$. In this case, $\phi'(0) = 0$. So the previous theorem cannot be applied.

To study this type of integral, we review notions of the Fourier transform and its properties.

Definition 2.6. For sufficiently good function f, we define

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\cdot\xi}\,dx.$$

We list several properties of the Fourier transform.

Proposition 2.7. *For sufficiently good function f, we have*

(i) *(Fourier Inversion) we have*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{ix\cdot\xi}d\xi.$$

(ii) *Plancherel*

$$\int_{-\infty}^{\infty} f\overline{g}dx = \int_{-\infty}^{\infty}\hat{f}\overline{\hat{g}}\,d\xi.$$

(iii) *For all $c \neq 0$, we have*

$$\left(e^{\frac{icx^2}{2}}\right)^{\wedge} = \sqrt{\frac{2\pi}{|c|}}e^{\frac{\pi}{4}\operatorname{sgn}(c)}e^{-\frac{i}{2c}\xi^2}$$

in the sense of distribution.

(iv) *For all* $\widehat{f^{(k)}}(\xi) = \xi^k i^k \hat{f}(\xi)$.

Theorem 2.8 (Stationary Phase)**.** *We have*

$$I[\varepsilon] \sim 2\pi\sqrt{\frac{2\pi\varepsilon}{|b|}} e^{i\frac{\pi}{4}\operatorname{sgn}(b)} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\varepsilon}{2}\right)^k \frac{1}{b} a^{(2k)}(0) \right).$$

Proof. We may assume that $b = 1$. By Plancherel's identity and Proposition 2.7 (iii), we have

$$I[\varepsilon] = \int_{-\infty}^{\infty} a(x)\overline{e^{-ix^2 2\varepsilon}}\, dx = e^{i\frac{\pi}{4}}\sqrt{2\pi\varepsilon}\int_{-\infty}^{\infty} \hat{a}(\xi) e^{-\frac{i}{2}\varepsilon|\xi|^2}\, d\xi = e^{i\frac{\pi}{4}}\sqrt{2\pi\varepsilon} J(\varepsilon).$$

It remains for us to estimate $J(\varepsilon)$. For each N, we have

$$\begin{aligned} J(\varepsilon) &= \sum_{k=0}^{N} \frac{\varepsilon^k}{k!} J^{(k)}(0) + o(\varepsilon^N) \\ &= \sum_{k=0}^{N} \left(-\frac{i}{2}\right)^k \int_{-\infty}^{\infty} \hat{a}(\xi)\xi^{2k}\, d\xi + o(\varepsilon^N). \end{aligned}$$

By Proposition 2.7 (iv), we get

$$= \frac{i^k}{2^k} 2\pi a^{(2k)}(0),$$

which implies the desired result. □

Now we move to the general case. The proof is an immediate consequence of the previous proposition with Morse's lemma.

Theorem 2.9. *Suppose that* $\phi'(x_0) = 0$ *and* $\phi''(x_0) \neq 0$*, and* x_0 *is the only critical point of* ϕ*. Then*

$$I[\varepsilon] \sim e^{\frac{i\phi(x_0)}{\varepsilon}} \sum_{k=0}^{\infty} (L_{2k}a)(x_0)\varepsilon^{k+1/2},$$

where $(L_{2k}a)(x_0)$ *is a some linear combination of* a *and its derivatives.*

Remark. (a) In particular,

$$I[\varepsilon] = \sqrt{\frac{2\pi\varepsilon}{|\phi''(x_0)|}} e^{i\frac{\pi}{4}\operatorname{sgn}(\phi''(x_0))} e^{i\frac{\phi(x_0)}{\varepsilon}} a(x_0) + o(\sqrt{\varepsilon}).$$

(b) The multi-dimensional case is as follows: if $\nabla\phi(\boldsymbol{x}_0) = 0$ and $D^2\phi(\boldsymbol{x}_0)$ is nonsingular, then

$$I[\varepsilon] \sim e^{\frac{i\phi(\boldsymbol{x}_0)}{\varepsilon}} \sum_{k=0}^{\infty} (L_{2k}a)(\boldsymbol{x}_0)\varepsilon^{k+N/2}.$$

In particular,

$$I[\varepsilon] = \sqrt{\frac{(2\pi\varepsilon)^{N/2}}{|\det[D^2\phi(\boldsymbol{x}_0)]|}} e^{i\frac{\pi}{4}\operatorname{sgn}(\phi''(\boldsymbol{x}_0))} e^{i\frac{\phi(\boldsymbol{x}_0)}{\varepsilon}} a(\boldsymbol{x}_0) + o(\varepsilon^{N/2}).$$

2.3 Applications: group and phase velocity

In this section, we give several applications of Laplace method and stationary phase method. Let us consider the following Airy equation:

$$u_t + u_{xxx} = 0. \tag{2.1}$$

Definition 2.10. A *plane wave solution* is a solution of the form

$$u(x,t) = V(\xi x - \sigma(\xi)t),$$

where $V = V(s) : \mathbb{C} \to \mathbb{C}$ is given, ξ a fixed, and $\sigma = \sigma(\xi)$ is to be found.

The origin of the word comes from the following observation: note that u is constant on planes of the equation $\xi x - \sigma(\xi)t = c$.

Put $u(x,t) = e^{i(\xi x - \sigma(\xi)t)}$, where we chose $V(s) = e^{is}$. If we plug into the expression

$$u_t + u_{xxx} = 0,$$

then one can easily verify that

$$(-i\sigma(\xi) + (i\xi)^3)e^{i(\xi x - \sigma(\xi)t} = 0.$$

Hence $\sigma(\xi) = -\xi^3$. Note that σ is real.

Definition 2.11. If σ is real in Definition 2.10, then the PDE is called *dispersive.* Note that $\sigma(\xi)/|\xi|$ is called the *phase speed.*

It can be shown that the solution of

$$u_t + u_{xxx} = 0 \quad \text{in } \mathbb{R} \times (0,\infty), \quad u(x,0) = g(x) \quad \text{on } \mathbb{R}$$

is given by

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x\xi - \sigma(\xi)t)}\hat{g}(\xi)d\xi. \tag{2.2}$$

Up to so far, there is no connection with the stationary phase method. Surprisingly, there is a connection when we study the asymptotic behavior of u as $t \to \infty$.

Consider u on the line $x = ct$. Then

$$u(ct,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(ct\xi - \sigma(\xi)t)}\hat{g}(\xi)d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{t(c\xi - \sigma(\xi))}\hat{g}(\xi)d\xi.$$

If we define $a(\xi) = \frac{1}{2\pi}\hat{g}(\xi)$, $\phi(\xi) = c\xi - \sigma(\xi)$, and $\varepsilon = 1/t$, then we can apply the method of stationary phase to the above integral. Note that

$$D\phi(\xi) = c - 3D\sigma(\xi) = 0.$$

$D\sigma(\xi)$ is sometimes called the group velocity.

Remark. In general, group velocity and phase velocity are not equal.

3

Multiple scales

3.1 Rapidly oscillating coefficients

Consider the following ODE on $(0,1)$:

$$\begin{cases} -\left(a\left(\frac{x}{\varepsilon}\right)u_\varepsilon'(x)\right)' = f(x), \\ \qquad\qquad u_\varepsilon(0) = u_\varepsilon(1), \end{cases} \tag{3.1}$$

where $a = a(y)$, $a > 0$, is periodic of period 1. We are interested in the behavior of u_ε as $\varepsilon \to 0$. However, the problem is that $a(x/\varepsilon)$ oscillates wild. Think about $\sin(2000x)$. Hence it is not clear what the limiting ODE looks like.

From (3.1), we have

$$-a(x/\varepsilon)u_\varepsilon'' - \frac{1}{\varepsilon}a_y\left(\frac{x}{\varepsilon}\right)u_\varepsilon' = f. \tag{3.2}$$

We first try to put the ansatz $u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$ into (3.2), and find $u_0 = \lim_{\varepsilon\to 0} u_\varepsilon$. Then

$$-au_0'' - \varepsilon a u_1'' + \cdots - \frac{1}{\varepsilon}a_y u_0' - a_y u_1' = f.$$

Observe that in $O(1/\varepsilon)$-term

$$-a_y u_0' = 0$$

From this, we see that u_0 is constant. Next, we analyze $O(1)$-term. Then

$$-a_y u_1' = f, \quad \text{i.e.,} \quad u_1'(x) = \frac{-f(x)}{a_y(x/\varepsilon)}.$$

The above identity shows that this is not a good ansatz.

Next natural trial is

$$u_\varepsilon = u_0(x, x/\varepsilon) + \varepsilon u_1(x, x/\varepsilon) + \varepsilon^2 u_2(x, x/\varepsilon) + \ldots$$

to capture the high frequency. Then put this ansatz into (3.2). Then we have

$$-a\left(\frac{x}{\varepsilon}\right)\left[(u_0(x,x/\varepsilon))'' + \varepsilon(u_1(x,x/\varepsilon))'' + \varepsilon^2(u_2(x,x/\varepsilon))' + \cdots\right].$$

Then we have

$$-\frac{1}{\varepsilon}a_y(x/\varepsilon)\left[(u_0(x,x/\varepsilon))' + \varepsilon(u_1(x,x/\varepsilon))' + \varepsilon^2(u_2(x,x/\varepsilon))'\right] = f(x). \tag{3.3}$$

Note that

$$[u^k(x,x/\varepsilon)]' = u_x^k(x,x/\varepsilon) + \frac{1}{\varepsilon}u_y^k(x,x/\varepsilon)$$

and

$$[u^k(x,x/\varepsilon)]'' = u_x x^k + \frac{2}{\varepsilon}u_{xy}^k + \frac{1}{\varepsilon^2}u_{yy}^k.$$

Hence (3.3) becomes

$$\begin{aligned}&-a\left[u^0_{xx}+\frac{2}{\varepsilon}u^0_{xy}+\frac{1}{\varepsilon^2}u^0_{yy}+\varepsilon u^1_{xx}+2u^1_{xy}+\frac{1}{\varepsilon}u^1_{yy}+\varepsilon^2u^2_{xx}+2\varepsilon u^2_{xy}+u^2_{yy}+\cdots\right]\\&-\frac{1}{\varepsilon}a_y\left[u^0_x+\frac{1}{\varepsilon}u^0_y+\varepsilon u^1_x+u^1_y+\varepsilon^2u^2_x+\varepsilon u^2_y+\cdots\right]\\&=f(x).\end{aligned}\tag{3.4}$$

The ultimate goal is to find u_0. We will check that by analyzing $O(1/\varepsilon^2)$-term, we will show that u_0 does not depend on y. Then by analyzing $O(1/\varepsilon)$-term, we will write u_1 in terms of u_0, and finally we will find u_0 from $O(1)$.

By comparing $O(1/\varepsilon^2)$-term, we have $-au^0_{yy}-a_yu^0_y=0$, i.e.,

$$-(au^0_y)_y=0. \tag{3.5}$$

Multiplying u^0 and integrating it over $[0,1]$, we have

$$\int_0^1 a|u^0_y|^2\,dy=0.$$

by the periodicity. Hence $a(u^0_y)=0$ and $a>0$, i.e., $u^0_y=0$ on $[0,1]$. Hence u^0 depends only on x. The limiting function u^0 has no oscillations.

Next, we analyze $O(1/\varepsilon)$ term. We have

$$-2au^0_{xy}-au^1_{yy}-a_yu^0_x-a_yu^1_1=0.$$

Observe that $u^0_{xy}=0$. From this, we can rewrite it into

$$-(au^1_y)_y=a_yu^0_x, \tag{3.6}$$

which is the PDE for u^1. We will rewrite it in terms of u^0.

To solve (3.6), introduce auxiliary function $w=w(y)$ satisfying

$$-(a(y)w_y)_y=a_y(y),\quad w(0)=w(1).$$

Such a solution exists by using the Fredholm alternative. Define $u^1(x,y)=w(y)u^0_x(x)$. Then, it is a solution (3.6). From this, we see the $O(1)$-term:

$$-au^0_{xx}-2au^1_{xy}-au^2_{yy}-a_yu^1_x-a_yu^2_y=f. \tag{3.7}$$

Put all u^2 on the left, so

$$-au^2_{yy}-a_yu^2_y=au^0_{xx}+2au^1_{xy}+a_yu^1_x+f.$$

In other words,

$$-(au^2_y)_y=au^0_{xx}+2au^1_{xy}+a_yu^1_x+f.$$

By using $u^1(x,y)=w(y)u^0_x(x)$, we can rewrite it into

$$-(au^2_y)_y=u^0_{xx}(a+w_ya+wa_y)+f. \tag{3.8}$$

By taking integration over $(0,1)$, we have

$$0=\int_0^1 -(au_y^2)_y dy=\int_0^1 u^0_{xx}(a+2w_y a+wa_y)dy+\int_0^1 f(x)dy$$

by the periodicity of a and u_y. Since u^0 depends on x, we have

$$0=u^0_{xx}(x)\int_0^1(a+2w_y a+wa_y)dy+f(x).$$

If we write

$$\overline{a}=\int_0^1(a+2w_y a+wa_y)dy,$$

then we get

$$-\overline{a}u^0_{xx}=f(x).$$

3.2 Oscillator with damping (Duffing's equation)

Consider the ODE

$$\begin{cases} u_\varepsilon''+u_\varepsilon+\varepsilon(u_\varepsilon)^3=0,\\ u_\varepsilon(0)=1, \quad u_\varepsilon'(0)=0,\end{cases} \tag{3.9}$$

where $u_\varepsilon=u_\varepsilon(t)$ and $t\geq 0$. We will find a 'good' approximation of u^ε and find a 'nice' function $f=f(t,\varepsilon)$ with $u^\varepsilon=f+o(\varepsilon)$.

Note that this ODE has a conserverd quantity: let

$$g(t)=\frac{(u_\varepsilon'(t))^2}{2}+\frac{(u_\varepsilon(t))^2}{2}+\frac{\varepsilon(u_\varepsilon(t))^4}{4}$$

which is called the first integral of the system which came from Noether's theorem. Then

$$g'=u_\varepsilon'u_\varepsilon''+u_\varepsilon u_\varepsilon'+\varepsilon u_\varepsilon^3u_\varepsilon'=u_\varepsilon'(u_\varepsilon''+u_\varepsilon+\varepsilon(u_\varepsilon)^3)=0.$$

So g is constant. In particular,

$$\frac{(u_\varepsilon(t))^2}{2}\leq\frac{(u_\varepsilon'(t))^2}{2}+\frac{(u_\varepsilon(t))^2}{2}+\frac{\varepsilon(u_\varepsilon)^4}{4}=g(t)+C,$$

which proves that u_ε is bounded and similarly, u_ε' is bounded (the bound could depend on ε).

We first try the following ansatz

$$u_\varepsilon(t)=u_0(t)+\varepsilon u_1(t)+\varepsilon^2u_2(t)+\cdots$$

Then we have

$$u_0''+\varepsilon u_1''+\cdots+u_0+\varepsilon u_1+\cdots+\varepsilon(u_0)^3+o(\varepsilon)=0.$$

Note that $u_0(t)=\cos t$ and u_1 satisfies

$$u_1''+u_1=-(u_0)^3=-\cos^3 t.$$

We assume $u_1(0) = 0$ and $u_1'(0) = 0$ (for simplicity). Then one can easily show that

$$u_1(t) = -\frac{3}{8} t \sin t + \frac{1}{32} \left(\cos(3t) - \cos(t) \right).$$

Note that u_1 is unbounded. So for fixed ε, $u_\varepsilon = u_0 + \varepsilon u_1 + \cdots$ is unbounded but this contradicts our previous observation on u_ε. This shows that our ansatz is not appropriate for studying this problem.

Next, we try

$$u^\varepsilon(t) = u^0(t, \varepsilon t) + \varepsilon u^1(t, \varepsilon t) + \varepsilon^2 u^2(t, \varepsilon t) + ...$$

Write $u_k(t) = u^k(t, \varepsilon t)$. Then

$$u_k' = u_t^k(t, \varepsilon t) + \varepsilon u_\tau^k(t, \varepsilon t).$$

Similarly, we have

$$u_k'' = u_{tt}^k + 2\varepsilon u_{t\tau}^k + \varepsilon^2 u_{\tau\tau}^k.$$

Finally, we get

$$\begin{aligned} u_{tt}^0 + 2\varepsilon u_{t\tau}^0 + \varepsilon^2 u_{\tau\tau}^0 + \varepsilon u_{tt}^1 + 2\varepsilon^2 u_{t\tau}^1 + \varepsilon^3 u_{\tau\tau}^1 \\ + u^0 + \varepsilon u^1 + \varepsilon (u^0)^3 + 3\varepsilon^2 (u^0)^2 u_1 + \cdots = 0. \end{aligned}$$

We first look the $O(1)$-term: note that the general solution of

$$u_{tt}^0 + u^0 = 0$$

is

$$u^0(t, \tau) = A(\tau) \cos t + B(\tau) \sin t. \tag{3.10}$$

Next we look the $O(\varepsilon)$-term:

$$2u_{t\tau}^0 + u_{tt}^1 + u^1 + (u^0)^3 = 0.$$

In other words,

$$u_{tt}^1 + u^1 = -(u_0)^3 - 2(u_{t\tau}^0).$$

If we put (3.10) in the expression, then

$$\begin{aligned} u_{tt}^1 + u^1 &= -(A \cos t + B \sin t)^3 - 2(A \cos t + B \sin t)_{t\tau} \\ &= (-A^3 \cos^3 t - 3A^2 B \cos^2 t \sin t - 3AB^2 \cos t \sin^2 t - B^3 \sin^3 t) \\ &\quad + (2A' \sin t + B' \cos t) \\ &= \left[-\frac{3}{4} A^3 - \frac{3}{4} AB^2 - 2B' \right] \cos t \\ &\quad + \left[-\frac{1}{4} A^3 + \frac{3}{4} AB^2 \right] \cos(3t) \\ &\quad + \left[-\frac{3}{4} A^2 B - \frac{3}{4} B^3 + +2A' \right] \sin t \\ &\quad + \left[-\frac{3}{4} A^2 B + \frac{1}{4} B^3 \right] \sin(3t). \end{aligned}$$

To remove the effect of resonance, we seek A and B so that

$$\begin{cases} A' = \frac{3}{8}(A^2 B + B^3), \\ B' = -\frac{3}{8}(A^3 + AB^2) \end{cases} \tag{3.11}$$

which is called the *modulation equations*. We impose $A(0) = 1$ and $B(0) = 0$ from the problem. In fact, this is Hamiltonian ODE, which guarantees the solution A and B to be global. Even though we are interested in the behavior of solutions for small ε, t could be large, so it might have an issue to guarantee the limit. Hence the global existence is important to justify all calculation. Hence we get the following

$$u^{\varepsilon}(t) = A(\varepsilon t)\cos t + B(\varepsilon t)\sin t + O(\varepsilon).$$

Now we are going to study what Hamiltonian ODE is. Recall that there is a classical example that has a local solution but does not have a global solution.

Example 3.1. Consider

$$y' = y^2, \quad y(0) = 1.$$

Note that $y(t) = 1/(1-t)$ is a solution to the ODE but it blows up at $t = 1$.

Definition 3.2. Let $H = H(x,p) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We say that $(x(t), p(t))$ is a Hamiltonian ODE associated to H if

$$x'(t) = H_p(x(t), p(t)), \quad p'(t) = -H_x(x(t), p(t)).$$

Lemma 3.3. *If $(x(t), p(t))$ is a Hamiltonian ODE associated with H, then $H(x(t), p(t))$ is conserved.*

Proof. By the chain rule, we have

$$\frac{d}{dt}H(x(t)), p(t)) = (H_x)x'(t) + (H_p)p'(t) = (H_x)(H_p) + (H_p)(-H_x) = 0. \qquad \square$$

Remark. Moreover, if H is coercive, in the sense that

$$\lambda(|x(t)| + |p(t)|)^{\mu} \le H(x(t), p(t))$$

for some $\mu > 0$, then $(x(t), p(t))$ is global.

Note that if we define

$$H(A, B) = \frac{3}{32}(A^2 + B^2)^2,$$

and if (A, B) is a solution to (3.11), then one can easily see that (A, B) is Hamiltonian ODE associated with H. Hence by the remark, the solution (A, B) is global.

3.3 Wentzel-Kramers-Brillouin's method

We consider the following linear ODE

$$u_\varepsilon'' + (\omega(\varepsilon t))^2 u_\varepsilon = 0 \tag{3.12}$$

where $\omega = \omega(\tau) > 0$ and $u_\varepsilon = u_\varepsilon$. One example is a pendulum of varying length.

We try the following ansatz

$$u_\varepsilon(t) = u^0(t, \varepsilon t) + \varepsilon u^1(t, \varepsilon t) + \cdots,$$

where $u^k = u^k(t, \tau)$. Then we get

$$(u_0'' + \varepsilon u_1'') + \omega^2(\varepsilon t)(u_0 + \varepsilon u_1 + \cdots) = 0.$$

So

$$\begin{aligned}(u^0_{tt} &+ 2\varepsilon u^0_{t\tau} + \varepsilon^2 u^0_{\tau\tau} + \varepsilon u^1_{tt} + 2\varepsilon^2 u^1_{t\tau} + \varepsilon^3 u^1_{\tau\tau}) \\ &+ (\omega(\tau))^2 u_0 + \varepsilon(\omega(\tau))^2 u_1 = 0.\end{aligned}$$

Let us ignore $\omega(\tau)$-term for a moment even though $\tau = \varepsilon t$. Then $O(1)$-term is

$$u^0_{tt} + (\omega(\tau))^2 u_0 = 0,$$

which gives

$$u_0(t) = A(\tau)\cos(\omega(\tau)t) + B(\tau)\sin(\omega(\tau)t).$$

Similarly, $O(\varepsilon)$ term is

$$2u^0_{t\tau} + u^1_{tt} + (\omega(\tau))^2 u_1 = 0.$$

So

$$\begin{aligned}u^1_{tt} + \omega^2 u^1 &= -2u^0_{t\tau} = -2(A\cos(\omega t) + B\sin(\omega t))_{t\tau} \\ &= 2[(A\omega)_\tau - B\omega\omega'(t)]\sin(\omega t) - 2[(A\omega)_\tau - B\omega\omega'(t)]\cos(\omega t)\end{aligned}$$

To avoid resonance, we need to find A and B so that the coefficients are zero. However, unlike the previous example, this ODE might not have a global solution.

Hence we need to use another ansatz to solve the problem. Define

$$u^\varepsilon = u^0(\sigma^\varepsilon(t), \varepsilon t) + \varepsilon u^1(\sigma^\varepsilon(t), \varepsilon t) + \cdots,$$

where $u^k = u^k(s, \tau)$ and σ^ε is to be determined. We will impose $\sigma^\varepsilon(0) = 0$, $(\sigma^\varepsilon)'(t) > 0$, $(\sigma^\varepsilon)' = O(1)$, $(\sigma^\varepsilon)'' = O(\varepsilon)$.

By chain rule, we have

$$(u^k)' = u^k_s(\sigma^\varepsilon, \varepsilon t)(\sigma^\varepsilon)' + u^k_\tau(\sigma^\varepsilon, \varepsilon t)\varepsilon$$

and

$$\begin{aligned}(u^k)'' &= (u^k_{ss})(\sigma')^2 + (u^k_{s\tau})\varepsilon(\sigma^\varepsilon)' + u^k_s(\sigma^\varepsilon)'' + (u^k_{\tau s})(\sigma^\varepsilon)'\varepsilon + (u^k_{\tau\tau})\varepsilon^2 \\ &= (u^k_{ss})(\sigma')^2 + 2(u^k_{s\tau})\varepsilon(\sigma^\varepsilon)' + u^k_s(\sigma^\varepsilon)'' + (u^k_{\tau\tau})\varepsilon^2\end{aligned}$$

Now we put this expression into the equation. Then

$$\begin{aligned} &u^0_{ss}(\sigma')^2 + 2(u^0_{s\tau})\varepsilon(\sigma') + u^0_{\tau\tau}\varepsilon^2 \\ &\quad + u^0_s(\sigma'') + \varepsilon u^1_{ss}(\sigma')^2 + 2u^1_{s\tau}\varepsilon^2(\sigma') + u^1_{\tau\tau}\varepsilon^3 \\ &\quad + u^1_s(\sigma'')\varepsilon + \omega^2 u_0 + \varepsilon\omega^2 u_1 = 0. \end{aligned}$$

By considering the restriction $\sigma' = O(1)$ and $\sigma'' = O(\varepsilon)$, we get

$$O(1) : u^0_{ss}(\sigma')^2 + \omega^2 u_0 = 0.$$

To solve this equation, we choose $\sigma' = \omega$. Then

$$\sigma^\varepsilon(t) = \int_0^t \omega(\varepsilon\tau)d\tau.$$

Since ω and its derivatives are bounded, it follows that

$$(\sigma^\varepsilon)(0) = 0, \quad (\sigma^\varepsilon)' = \omega > 0, \quad (\sigma^\varepsilon)'(t) = \omega(\varepsilon t) = O(1),$$

and

$$\sigma'' = \varepsilon\omega_\tau(\varepsilon t) = O(\varepsilon).$$

From this construction, one can get

$$(u^0)_{ss} + u^0 = 0, \quad u^0 = u^0(s, \tau).$$

From this, we get

$$u^0 = A(\varepsilon t)\cos(\sigma(t)) + B(\varepsilon t)\sin(\sigma(t)).$$

Now we will find A and B so that it has good dynamics to control. Note that the $O(\varepsilon)$-term is

$$2u^0_{s\tau}(\sigma') + \omega^2 u_1 + u^1_{ss}\omega^2 + u^0_s\omega_\tau = 0$$

We first estimate

$$\begin{aligned} \omega^2 u^1_{ss} + \omega^2 u^1 &= -2u^1_{s\tau} - u^0_{ss}\omega = -2[A(\tau)\cos s + B(\tau)\sin(s)]_{s\tau} - \omega_J[A(\tau)\cos(s) + B(\tau)\sin s]_{ss'} \\ &= -2\omega[-A'(\tau)\sin s + B'(\tau)\cos s] \\ &\quad - w_\tau[-A(\tau)\sin s - B(\tau)\cos s] \\ &= [-2B'\omega + B\omega']\cos s + [2A'\omega + A\omega']\sin s. \end{aligned}$$

In order to avoid the resonance, we assume A and B so that

$$-2B'\omega + B\omega' = 0 \quad \text{and} \quad 2A'\omega + A\omega' = 0.$$

Note that we put (εt) in the parameter. Then we can solve the equation by separation of variables:

$$A(\tau) = c_1\omega^{-1/2}(\tau) \quad \text{and} \quad B(\tau) = c_2\omega^{-1/2}(\tau).$$

Therefore,

$$u^\varepsilon(t) \approx \frac{c_1}{\sqrt{\omega(\varepsilon t)}} \cos\left(\int_0^t \omega(\varepsilon r)dr\right) + \frac{c_2}{\sqrt{\omega(\varepsilon t)}} \sin\left(\int_0^t \omega(\varepsilon r)dr\right).$$

On the other hand, if we define $\Theta(\tau) = \int_0^\tau \omega(s)ds$, then a change of variable gives

$$u^\varepsilon(t) = \frac{c_1}{\sqrt{\omega(\tau)}} \cos\left(\frac{\Theta(\varepsilon t)}{\varepsilon}\right) + \frac{c_2}{\sqrt{\omega(\tau)}} \sin\left(\frac{\Theta(\varepsilon t)}{\varepsilon}\right),$$

where $\tau = \varepsilon t$.

This is of the form

$$U\left(\frac{\Theta(\varepsilon t)}{\varepsilon}, \varepsilon t\right),$$

where

$$U(m,\tau) = \frac{c_1}{\sqrt{\omega(\tau)}} \cos(m) + \frac{c_2}{\sqrt{\omega(\tau)}} \sin(m).$$

3.4 Nonlinear oscillator with damping

Consider

$$(u_\varepsilon)'' + \Phi'(u_\varepsilon) + \varepsilon u_\varepsilon' = 0, \tag{3.13}$$

where

$$u_\varepsilon = u_\varepsilon(t), \quad u_\varepsilon' = \frac{du_\varepsilon}{dt}, \quad \Phi = \Phi(s), \quad \Phi' = \frac{d\Phi}{ds}.$$

Here Φ is some nonlinear function or potential with $\Phi(0) = 0$. Although this seems complicated, we will use a modified ansatz motivated from the previous section. We write

$$u^\varepsilon = u\left(\frac{\Theta(\varepsilon t, \varepsilon)}{\varepsilon}, \varepsilon t, \varepsilon\right)$$

for some $u = u(\eta, \tau, \varepsilon)$ and $\Theta = \Theta(\tau, \varepsilon)$ which will be determined later.

Note that

$$(u_\varepsilon)' = \frac{du^\varepsilon}{dt} = U_\eta\left(\frac{\Theta(\varepsilon t, \varepsilon)}{\varepsilon}, \varepsilon t, \varepsilon\right)\Theta_\tau(\varepsilon t, \varepsilon) + U_\tau\left(\frac{\Theta(\varepsilon t, \varepsilon)}{\varepsilon}, \varepsilon t, \varepsilon\right)\varepsilon$$

and

$$(u_\varepsilon)'' = (u_{\eta\eta})(\Theta_\tau)^2 + u_{\eta\tau}\Theta_\tau\varepsilon + u_\eta\Theta_{\tau\tau}\varepsilon + u_{\eta\tau}(\Theta_\tau)\varepsilon + u_{\tau\tau}\varepsilon^2.$$

If we put these expression into (3.13), then

$$(u_{\eta\eta}(\Theta_\tau)^2 + 2u_{\eta\tau}\Theta_\tau\varepsilon + u_\eta\Theta_{\tau\tau}\varepsilon + u_{\tau\tau}\varepsilon^2 + \Phi'(u) + \varepsilon(u_\eta\Theta_\tau + \varepsilon u_\tau) = 0.$$

Now we put

$$u = u^0 + \varepsilon u^1 + \cdots, \quad \Theta = \Theta^0 + \varepsilon\Theta^1 + ...,$$

where

$$u^k = u^k(\eta,\tau) \quad \text{and} \quad \Theta^k = \Theta^k(\tau).$$

We impose $u_k(\eta,\tau)$ to be 2π-periodic in η for all k. This is expectable since our model exhibits periodic orbits.

We will find appropriate u^0 and Θ^0 because then we have

$$u^\varepsilon(t) \approx u^0\left(\frac{\Theta^0(\varepsilon t,\varepsilon)}{\varepsilon}, \varepsilon t, \varepsilon\right) + O(\varepsilon).$$

If we plug the ansatz, then

$$\begin{aligned}(u^0_{\eta\eta} + \varepsilon u^1_{\eta\eta})(\theta^0_\tau + \varepsilon\Theta^1_\tau)^2 + 2(u^0_{\eta\tau})(\Theta^0_\tau)\varepsilon + (u^0_\eta)(\Theta^0_{\tau\tau})\varepsilon \\ + \varepsilon^2\text{-term} + \Phi'(u^0+\varepsilon u^1) + \varepsilon(u^0_\eta\Theta^0_\tau) + O(\varepsilon^2) = 0.\end{aligned}$$

The $O(1)$-term is

$$u^0_{\eta\eta}(\Theta^0_\tau)^2 + \Phi'(u^0) = 0.$$

We write $\omega^0 = \Theta^0_\tau$ (which means an angular momentum). Then

$$u^0_{\eta\eta}(\omega^0)^2 + \Phi'(u^0) = 0.$$

This is an ODE in η with an interesting conserved quantity:

$$E[\eta,\tau] = \frac{1}{2}(\omega^0)^2(u^0_\eta)^2 + \Phi(u^0)$$

By using chain rule, one can see that

$$\frac{\partial E}{\partial\eta} = \frac{1}{2}(\omega^0)^2 2u^0_\eta u^0_{\eta\eta} + \Phi'(u^0)u^0_\eta = 0,$$

which means that E depends only on τ. We will express the system by this energy.

We write $v = \omega^0 u^0_\eta$. Then

$$E[\tau] = \frac{1}{2}v^2 + \Phi(u^0).$$

For fixed τ, we have $v = \pm\sqrt{2(E-\Phi(u^0))}$. Because of Φ, we assume that $\Phi(u^0) = E$ for some u^0. We write $a(E)$ and $b(E)$ for such values as the turning points of the energy.

If we fix a point τ, then $u^0 = u^0(\eta,\tau) = u^0(\eta)$. Let $a = u^0(0)$ and $b^0 = u^0(\pi)$. By definition of v, we have

$$\omega^0\frac{du^0}{d\eta} = \sqrt{2(E-\Phi(u^0))}.$$

Write $s = u^0$. Then

$$\omega^0\frac{ds}{d\eta} = \sqrt{2(E-\Phi(s))}.$$

By using a separation of variable, we have

$$\int_a^b \frac{\omega^0}{\sqrt{2(E-\Phi(s))}}ds = \int_a^b\left(\frac{d\eta}{ds}\right)ds.$$

If we write $s = u^0(\eta)$, then

$$\omega^0 \int_a^b \frac{ds}{\sqrt{2(E-\Phi(s))}} = \int_0^\pi d\eta.$$

Hence

$$\omega^0(E) = \left(\int_{a(E)}^{b(E)} \frac{ds}{\sqrt{2(E-\Phi(s))}}\right)^{-1} \pi. \tag{3.14}$$

Similarly, integrating from a to $u^0(t)$, we get

$$\omega^0(E) = \left(\int_{a}^{u_0(t)} \frac{ds}{\sqrt{2(E-\Phi(s))}}\right)^{-1} t. \tag{3.15}$$

So if we know E and ω^0, this gives us an implicit formula for u^0. Moreover, we can also figure out Θ^0 because $\Theta^0_\tau = \omega^0$ which implies that

$$\int_0^\tau \Theta^0_\tau(s)ds = \int_0^\tau \omega^0(s)ds,$$

and hence

$$\Theta^0(\tau) = \Theta^0(0) + \int_0^\tau \omega^0(s)ds. \tag{3.16}$$

Let $w = u^0_\eta$. Then we have

$$(w_\eta)(\omega^0)^2 + \Phi'(u^0) = 0.$$

If we differentiate it in w, then

$$w_{\eta\eta}(\omega^0)^2 + \Phi''(u^0)w = 0. \tag{3.17}$$

Observe that it is a linear differential equation in w.

Then we look at $O(\varepsilon)$-terms. Then

$$2w_\eta\omega^0\omega^1 + u^1_{\eta\eta}(\omega^0)^2 + 2w_\tau\omega^0 + w\omega^0_\tau + \Phi''(u^0)u^1 + w\omega^0 = 0.$$

So

$$(\omega^0)^2u^1_{\eta\eta} + \Phi''(u^0)u^1 = -2\omega^0\omega^1w_\eta - w\omega^0_\tau - 2w_\tau\omega^0 - w\omega^0.$$

We multiply it by $w = u^0_\eta$ and integrate it. Recall that the function is periodic. Then on the left-hand side, it becomes

$$\int_0^{2\pi} (\omega^0)u^1_{\eta\eta}w + \Phi''(u^0)u^1w\, d\eta = 0$$

by (3.17). On the right hand side,

$$\int_0^{2\pi} (-2\omega^0\omega^1w_\eta - w\omega^0_\tau - 2w_\tau\omega^0 - w\omega^0)w\, d\eta.$$

Since ω^0 and ω^1 does not depend on η, we get

$$\int_0^{2\pi} -w^2\omega_\tau^0 - 2\omega^0 w_\tau w \, d\eta = \int_0^{2\pi} -w^2\omega_\tau^0 - \omega^0 (w^2)_\tau d\eta = -\int_0^{2\pi} (w^2\omega^0)_\tau d\eta.$$

If we let

$$A(E(\tau)) = \int_0^{2\pi} \omega^0(E(\tau)) w^2 d\eta,$$

then it follows from the above observation that A satisfies

$$(A(E(\tau)))_\tau = -A(E(\tau)).$$

So

$$A(E(\tau)) = A(E(0))e^{-\tau}.$$

From the relationship, we can figure out what E is, and then we can figure out ω^0 from (3.14), and figure out u^0 from (3.15), and finally we figured out Θ^0 from (3.16).

We will recall the following fact: if we write

$$A = \text{area of orbit} = \{\frac{v^2}{2} + \Phi(u^0) \leq E\}.$$

Then

$$\frac{dA}{dE} = -\frac{2\pi}{\omega^0(E)}.$$

3.5 Nonlinear wave equation

Consider the following Klein-Gordon equation:

$$u_{tt}^\varepsilon - u_{xx}^\varepsilon + \Phi'(u^\varepsilon) = 0, \tag{3.18}$$

where $\Phi = \Phi(s)$ satisfies $\Phi(0) = 0$.

We are looking solutions of the form

$$u^\varepsilon(x,t) = u\left(\frac{\theta(\varepsilon x, \varepsilon t, \varepsilon)}{\varepsilon^2}, \varepsilon x, \varepsilon t, \varepsilon\right),$$

where $u = u(\eta, \xi, \tau, \varepsilon)$ and $\theta = \theta(\zeta, \tau, \varepsilon)$ which will be found later.

Note that

$$u_t^\varepsilon = u_\eta \left(\frac{\theta_\tau}{\varepsilon}\right) \varepsilon + u_\tau(\varepsilon),$$

and

$$u_{tt}^\varepsilon = u_{\eta\eta}(\theta_\tau)^2 + 2u_{\eta\tau}\theta_\tau\varepsilon + u_\eta\theta_{\tau\tau}\varepsilon + u_{\tau\tau}\varepsilon^2.$$

In other words,

$$u_{tt}^\varepsilon = u_{\eta\eta}\omega^2 - 2u_{\eta\tau}\varepsilon\omega - u_\eta\omega_\tau\varepsilon + u_{\tau\tau}\varepsilon^2,$$

where $\omega = -\theta_\tau$ (which is called the local frequency). Similarly,

$$u_{xx}^\varepsilon = u_{\eta\eta}\kappa^2 + 2u_{\eta\tau}\varepsilon\kappa + u_\eta\kappa_\xi\varepsilon + u_{\xi\xi}\varepsilon^2,$$

where $\kappa = \theta_\xi$ (which is called the local wave number).

Hence if we plug u^ε into (3.18), then

$$(\omega^2 - \kappa^2)u_{\eta\eta} + \Phi'(u) - \varepsilon(w_\tau u_\eta + 2wu_{\eta\tau} + \kappa_\xi u_\eta + 2\kappa u_{\eta\xi}) - \varepsilon^2(u_{\xi\xi} - u_{\tau\tau}) = 0. \quad (3.19)$$

Now we expand

$$u = u_0 + \varepsilon u_1 + \cdots \quad \text{and} \quad \theta = \theta_0 + \varepsilon\theta_1 + \cdots$$

for $u_k = u_k(\eta, \xi, \tau)$ which is 2π-periodic in η and $\theta_k = \theta_k(\eta, \xi)$.

Observe that

$$-\theta_\tau = \theta^0_\tau - \varepsilon\theta^1_\tau$$

and

$$\omega = \omega_0 + \varepsilon\omega_1.$$

Focusing only on $O(1)$ and $O(\varepsilon)$-terms, we get

$$\begin{aligned}&((\omega_0)^2 - (\kappa_0)^2)u^0_{\eta\eta} + \varepsilon((\omega^0)^2 - (\kappa^0)^2)u^1_{\eta\eta}\\ &\quad + \Phi'(u_0) + \varepsilon u_1\Phi''(u_0) - \varepsilon\omega^0_\tau u^0_\eta - 2\varepsilon\omega^0 u^0_{\eta\tau} - \varepsilon\kappa^0_\xi u^0_\eta - 2\varepsilon\kappa^0 u^0_{\eta\xi} = 0.\end{aligned}$$

For $O(1)$-term, we have

$$\Phi'(u_0) + ((\omega_0)^2 - (\kappa_0)^2)u^0_{\eta\eta} = 0.$$

Like before, if

$$E[\eta, \xi, \tau] = \left(\frac{(\omega_0)^2 - (\kappa_0)^2}{2}\right)(u^0_\eta)^2 + \Phi(u^0),$$

then one can easily see that $E_\eta = 0$ and hence $E = E[\xi, \tau]$. So if we define

$$v = \sqrt{(\omega_0)^2 - (\kappa_0)^2}\, u^0_\eta,$$

then we have

$$\frac{v^2}{2} + \Phi(u_0) = E, \quad v = \pm\sqrt{2(E - \Phi'(u_0))}.$$

Note that $v^2/2 + \Phi(u^0) = E$ forms a closed curve in uv-plane. We recall that

$$\frac{dA}{dE} = \frac{2\pi}{\omega(E)}.$$

Let

$$A = \sqrt{(\omega_0)^2 - (\kappa_0)^2}\int_0^{2\pi} (u^0_\eta)^2\, d\eta.$$

Then the formula for this becomes

$$\frac{dA}{dE} = \frac{2\pi}{\sqrt{(\omega_0)^2 - (\kappa_0)^2}},$$

which implies that

$$E'(A) = \frac{dE}{dA} = \frac{1}{2\pi}\sqrt{(\omega_0)^2 - (\kappa_0)^2}.$$

Just as before, $O(1)$ implies that $w = u^0_\eta$. Then w solves

$$((\omega_0)^2 - (\kappa_0)^2)w_{\eta\eta} + \Phi''(u_0)w = 0. \tag{3.20}$$

On the other hand, if we observe $O(\varepsilon)$-term, then

$$\begin{aligned}&((\omega_0)^2 - (\kappa_0)^2)u^1_{\eta\eta} + u^1\Phi''(u_0)\\ &= (\omega^0_\tau)^2 w + 2\omega^0 w_\tau + \kappa^0_\xi w + 2\kappa^0 w_\xi.\end{aligned}$$

Multiplying it by w and taking integration on $[0, 2\pi]$ with respect to η, we get

$$\int_0^{2\pi} ((\omega_0)^2 - (\kappa_0)^2)u^1_{\eta\eta}w + u^1\Phi''(u_0)w\, d\eta = \int_0^{2\pi} [[(\omega_0)^2 - (\kappa_0)^2]w_{\eta\eta} + \Phi''(u^0)]u^1 w d\eta = 0$$

from equation (3.20). On the other hand, we have

$$\int_0^{2\pi} (\omega^0_\tau)w^2 + \omega^0 2w_\tau w + (\kappa^0_\xi)w^2 + \kappa^0 2ww_\xi d\eta.$$

Since

$$(w^2)_\tau = 2w_\tau w \quad \text{and} \quad (w^2\omega^0)_\tau = \omega^0_\tau + w^2 + \omega^0(w^2)_\tau$$

and

$$(w^2\kappa^0)_\xi = (\kappa^0_\xi)w^2 + \kappa^- 2ww_\xi,$$

it follows that

$$\left(\omega_0 \int_0^{2\pi} w^2\right)_\tau + \left(\kappa_0 \int_0^{2\pi} w^2\right)_\xi = 0.$$

Recall that

$$A = \sqrt{(\omega_0)^2 - (\kappa_0)^2} \int_0^{2\pi} w^2 d\eta = 2\pi E'(A) \int_0^{2\pi} w^2 d\eta.$$

From this, we get

$$0 = \left(\omega_0 \frac{A}{2\pi E'(A)}\right) + \left(\kappa^0 \frac{A}{2\pi E'(A)}\right)_\xi.$$

Here A is independent of η. Also, recall that $\kappa^0 = \theta^0_\xi$ and $\omega^0 = -\theta^0_\tau$. This implies that

$$\kappa^0_\tau + \omega^0_\xi = \theta^0_{\xi\tau} - \theta^0_{\tau\xi} = 0.$$

Therefore, we obtained three PDEs for A, ω_0, and κ_0 in terms of ξ and τ.

$$\begin{cases} \left(\dfrac{\omega^0 A}{E'(A)}\right)_\tau + \left(\dfrac{\kappa^0 A}{E'(A)}\right)_\xi = 0, \\ \sqrt{(\omega^0)^2 - (\kappa^0)^2} = 2\pi E'(A) \\ \kappa^0_\tau + \omega^0_\xi = 0. \end{cases} \tag{3.21}$$

We can solve for A, ω^0, and κ^0 and therefore, we can solve for θ^0 using $\omega_0 = -\theta^0_\tau$ and $\kappa_0 = \theta^0_\xi$.

3.6 A diffusion-transport PDEs

Consider the following diffusion-transport PDEs

$$u^\varepsilon_t + (w(x)u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}, \tag{3.22}$$

where $u^\varepsilon = u^\varepsilon(x,t)$ which is 2π-periodic in x and $w = w(x) > 0$ and 2π-periodic in x. We want to know what happens when $\varepsilon \to 0$.

We put

$$u^\varepsilon(x,t) = u^0(x,t,\varepsilon t) + \varepsilon u^1(x,t,\varepsilon t) + \cdots$$

and we assume that $u^k = u^k(x,t,\tau)$ are 2π-periodic in x. So

$$(u^k(x,t,\varepsilon t))_t = u^k_t + \varepsilon u^k_\tau$$

and

$$(u^k(x,t,\varepsilon t))_x = u^k_x.$$

So

$$u^\varepsilon_t + (wu^\varepsilon)_x = \varepsilon u^\varepsilon_{xx},$$

and this implies that

$$(u^0 + \varepsilon u^1)_t + (w(x)(u^0 + \varepsilon u^1))_x = \varepsilon(u^0 + \varepsilon u^1)_{xx}.$$

In other words, we have

$$u^0_t + \varepsilon u^0_\tau + \varepsilon u^1_t + \varepsilon^2 u^1_\tau + (w(x)u^0)_x + \varepsilon(w(x)u^1)_x = \varepsilon u^0_{xx} + \varepsilon^2 u^1_{xx}.$$

Hence

$$u^0_t + (w(x)u^0)_x = 0,$$

which is the first order linear PDEs. Write $v^0 = w(x)u^0$, where $v^0 = v^0(x,t,\tau)$. Then

$$\left(\frac{v^0}{w(x)}\right)_t + (v^0)_x = 0.$$

From this, we have

$$(v^0)_t + wv^0_x = 0,$$

which can be solved by using the method of characteristic.

Consider

$$\theta'(t) = w(\theta(t)), \quad \theta(0) = 0.$$

Then w is Lipschitz since w is continuous and 2π-periodic. Hence the solution exists globally. By a chain rule, one can see that $v^0(\theta(t),t)$ is constant. Indeed, we have

$$\frac{d}{dt}v^0(\theta(t),t) = v^0_x(\theta'(t)) + v^0_t = v^0_x w + v^0_t = 0.$$

Also, one can easily see that v^0 is constant along the curve $(\theta(t-a),t)$ for any $a \in \mathbb{R}$. Hence, given (x,t), we need to figure out on which translate of θ (x,t) is on. Given (x,t) define $s = s(x,t)$ by $\theta(t-s) = x$. The s-translate that goes through (x,t). Sometimes, it is called the *foliation of curves*.

Hence the value of s

1. completely determines which curve we are on;

2. completely determines value of v^0.

In particular, $v^0(x,t) = v^0(\theta(t-s), t)$ depends only on s since v^0 is constant on curves. We write this by $\tilde{v}^0(s)$. However, for simplicity, we write v^0 instead of $\tilde{v}^0$. So

$$v^0(x,t,\tau) = v^0(s,\tau)$$

and hence

$$u^0(x,t,\tau) = \frac{v^0(x,t,\tau)}{w(x)} = \frac{v^0(s,\tau)}{w(\theta(t-s))}.$$

Let us compute $O(\varepsilon)$-term. Recall that

$$u^0_\tau + u^1_t + (wu^1)_x = u^0_{xx}$$

and then

$$\left(\frac{v^0}{w}\right)_\tau + \left(\frac{v^1}{w}\right)_t + v^1_x = \left(\frac{v^0}{w}\right)_{xx}.$$

If we multiply it by w, then

$$v^0_\tau + v^1_t + wv^1_x = w\left(\frac{v^0}{w}\right)_{xx}.$$

Note that

$$\theta(t-s) = x \quad \text{and so} \quad \frac{d\theta(t-s)}{dx} = 1.$$

In other words,

$$\theta'(t-s)\left(-\frac{ds}{dx}\right) = 1.$$

On this trajectory, $\theta' = w(\theta) = w(x)$. From this, one can see that

$$\left(\frac{d}{dx}\right) = \left(-\frac{1}{w}\right)\left(\frac{d}{ds}\right).$$

In particular, we have

$$w\left(\frac{v^0}{w}\right)_{xx} = w\frac{d}{dx}\left(-\frac{1}{w}\frac{ds}{d}\left(\frac{v^0}{w}\right)\right) = \frac{d}{ds}\left(\frac{1}{w}\frac{d}{ds}\left(\frac{v^0}{w}\right)\right).$$

Hence

$$\begin{aligned} v^1_t + wv^1_x &= \frac{d}{ds}\left(\frac{1}{w}\frac{d}{ds}\left(\frac{v^0}{w}\right)\right) - v^0_\tau \\ &= \frac{v^0_{ss}}{w^2} + \frac{3}{2}\left(\frac{1}{w^2}\right)_s v^0_s + \frac{1}{2}\left(\frac{1}{w^2}\right)_{ss} v^0 - v^0_\tau =: f, \end{aligned}$$

which is the first-order linear PDE. We can solve this PDE by a method of characteristic.

We will use this to find a simple PDE for v^0. We want solutions that do not blow up. The following criteria can be proved by using the Fredholm alternative theorem.

Proposition 3.4. *v^1 becomes unbounded unless $\int_0^T f dt = 0$.*

Since $w \geq c > 0$, $\theta' \geq c$. Hence it follows that $\theta(t) \to \infty$ as $t \to \infty$. Hence there exists the smallest T so that $\theta(T) = 2\pi$. Since w is 2π-periodic and solutions to the characteristic ODE areq unite, it follows that $\theta(t+T) = \theta(t) + 2\pi$ for all t.

Now we note that

$$\begin{aligned} \int_0^T \frac{d}{ds}\left(\frac{1}{w^2}\right) dt &= \int_0^T \frac{d}{dt}\left(\frac{1}{w^2}\right) dt \\ &= \int_0^T \left(\frac{1}{w^2(\theta(t-s))}\right)_t dt = 0 \end{aligned}$$

since $w(\theta(t-s))$ is T-periodic. Indeed, note that

$$w(\theta(t+T-s)) = w(\theta(t-s)+2\pi) = w(\theta(t-s)).$$

Similarly, one can show that

$$\int_0^T \left(\frac{1}{w^2}\right)_{ss} dt = 0.$$

From these, we see that

$$\begin{aligned} \int_0^T f dt &= v^0_{ss} \int_0^T \left(\frac{1}{w^2}\right) dt + \frac{3}{2} v^0_s \int_0^T \left(\frac{1}{w^2}\right)_s dt + \frac{1}{2} \int_0^T \left(\frac{1}{w^2}\right)_{ss} dt - v^0_\tau \int_0^T 1 dt \\ &= v^0_{ss} \int_0^T \left(\frac{1}{w^2}\right) dt - v^0_\tau T, \end{aligned}$$

i.e.,

$$v^0_\tau = v^0_{ss} \left(\frac{\int_0^T \frac{1}{w^2} dt}{T}\right) =: \bar{a} v^0_{ss},$$

where $v^0 = v^0(s,\tau)$ and

$$\bar{a} =: \frac{1}{T} \int_0^T \frac{1}{(w(\theta(t-s))^2} dt.$$

In the limit, we get a diffusion equation.

3.7 Interlude: the calculus of variations

Consider the following minimal surface equation

$$\sum_{i=1}^N - \left(\frac{u_{x_i}}{\sqrt{1+|Du|^2}}\right)_{x_i} = 0,$$

where $u = u(x_1, \ldots, x_N)$, $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$.

It is hard to find a solution to the above equation by solving the equation directly. Instead, we could think a minimization of

$$I[u] = \int_W \sqrt{1+|Du|^2}dx,$$

where W is an open set. It is quite handy because it means the surface area of the graph of u. It turns out that the minimizer u of $I[u]$ solves the PDE above.

To be rigorous, we introduce some notions. Given $L = L(p, z, x)$, the Lagrangian, let

$$I[u] = \int_W L(Du(x), u(x), x)dx$$

which is called the energy functional. For instance, if

$$L(p, z, x) = \frac{1}{2}|p|^2,$$

then

$$I[u] = \frac{1}{2}\int_W |Du|^2\, dx,$$

which is the Dirichlet energy.

Another example is

$$L(p, z, x) = \sqrt{1+|p|^2}, \quad I[u] = \int_W \sqrt{1+|Du|^2}dx.$$

We want to find u that minimizes $I[u]$ among all functions u. Why do we care about the minimizer?

Theorem 3.5. *If u minimizes $I[u]$, then u solves the Euler-Lagrange PDE*

$$-\text{div}\,(D_pL(Du, u, x)) + L_z(Du, u, x) = 0.$$

Proof. For simplicity, let us consider the case $N = 1$. Suppose that u minimizes I and let v be arbitrary. Define

$$g(h) = I[u + hv] = \int_W L(u' + hv', u + hv, x)dx.$$

Since g attains a minimum at $h = 0$, $g'(0) = 0$. Then

$$g'(h) = \int_W L_p(u' + hv', u + hv, x)v' + L_z(u' + hv', u + hv, x)v\, dx.$$

So

$$\begin{aligned} 0 = g'(0) &= \int_W [L_p(u', u, x)v' + L_z(u', u, x)v]dx \\ &= \int_W [-(L_p(u', u, x))' + L_z(u', u, x)]vdx. \end{aligned}$$

Since the above identity is true for all v, we finally get

$$-(L_p(u', u, x))' + L_z(u', u, x) = 0.$$

□

Example 3.6. Consider

$$L(p,z,x) = \frac{1}{2}|p|^2 \quad \text{and} \quad I[u] = \int_W \frac{1}{2}|Du|^2\,dx.$$

Then the corresponding Euler-Lagrange equation is

$$-\Delta u = -\mathrm{div}\,(Du) = 0.$$

Example 3.7. Consider

$$L(p,z,x) = \frac{1}{2}\sqrt{1+|p|^2} \quad \text{and} \quad I[u] = \int_W \frac{1}{2}L(Du)^2\,dx.$$

Then the corresponding Euler-Lagrange equation is

$$-\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0,$$

which is the minimal surface equation.

Hence the minimizer problem produces a PDE. Conversely, given a PDE, if you can write the PDE into Euler-Lagrange equation for some functional I, then the PDE is called *variational* and this is good.

Example 3.8. The nonlinear Poisson equation

$$-\Delta u = f(u)$$

is variational. Indeed,

$$I[u] = \int_W \frac{1}{2}|Du|^2 - F(u)dx,$$

where $F(t) = \int_0^t f(s)ds$.

3.8 An Eikonal and Continuity equation

Consider

$$-\varepsilon^2\Delta u^\varepsilon + V(x)u^\varepsilon = 0, \tag{3.23}$$

where $u^\varepsilon = u^\varepsilon(x)$, $x = (x_1,\dots,x_N) \in \mathbb{R}^N$ and $v = v(x)$ is given and real. Here $u^\varepsilon \in \mathbb{C}$.

In fact, the equation is variational. Indeed, if

$$I^\varepsilon[u^\varepsilon] = \int_{\mathbb{R}^N} \frac{\varepsilon^2}{2}|Du^\varepsilon|^2 + \frac{1}{2}V(x)|u^\varepsilon|^2\,dx,$$

then the corresponding Euler-Lagrange equation is (3.23).

We choose the ansatz by

$$u^\varepsilon(x) = a^\varepsilon(x)e^{i\theta^\varepsilon(x)/\varepsilon},$$

where $a^\varepsilon = a^\varepsilon(x)$ and $\theta^\varepsilon = \theta^\varepsilon(x)$ are real. Note that

$$|u^\varepsilon|^2 = |a^\varepsilon|^2.$$

Recall that $|z|^2 = z\overline{z}$. Using this, we note that

$$\begin{aligned}
Du^\varepsilon &= (Da^\varepsilon)e^{i\theta^\varepsilon/\varepsilon} + a^\varepsilon e^{i\theta^\varepsilon/\varepsilon}\left(\frac{D\theta^\varepsilon}{\varepsilon}i\right)\\
\overline{Du^\varepsilon} &= (Da^\varepsilon)e^{-i\theta^\varepsilon/\varepsilon} + a^\varepsilon e^{-i\theta^\varepsilon/\varepsilon}\left(\frac{D\theta^\varepsilon}{\varepsilon}(-i)\right)
\end{aligned}$$

Then of our interest is

$$|Du^\varepsilon|^2 = |Da^\varepsilon|^2 + \frac{(a^\varepsilon)^2}{\varepsilon^2}|D\theta^\varepsilon|^2.$$

So

$$\begin{aligned}
I^\varepsilon[u^\varepsilon] &= \int_{\mathbb{R}^N} \frac{\varepsilon^2}{2}\left(|Da^\varepsilon|^2 + |a^\varepsilon|^2\frac{|D\theta^\varepsilon|^2}{\varepsilon^2}\right) + \frac{1}{2}V(x)|a^\varepsilon|^2\,dx\\
&= I^\varepsilon[a^\varepsilon, \theta^\varepsilon].
\end{aligned}$$

Now put $a^\varepsilon = a^0 + \varepsilon a^1 + \cdots$ and $\theta^\varepsilon = \theta^0 + \varepsilon\theta^1 + \cdots$. We will find a PDE for a^0 and θ^0. Then

$$\begin{aligned}
&I[a^\varepsilon, \theta^\varepsilon]\\
&= \int_{\mathbb{R}^N} \frac{\varepsilon^2}{2}\left(|Da^0 + \varepsilon Da^1|^2 + |(a^0)^2 + \varepsilon(a^1)|^2 \times \frac{|D\theta^0 + \varepsilon D\theta^1|^2}{\varepsilon^2}\right) + \frac{1}{2}V(x)|a^0 + \varepsilon a^1|^2\,dx.
\end{aligned}$$

Expanding this, we have

$$= \int_{\mathbb{R}^N} \frac{\varepsilon^2}{2}(|Da^0|^2 + \cdots) + \frac{\varepsilon^2}{2}\frac{|a^0|^2|D\theta^0|^2}{\varepsilon^2} + \frac{1}{2}v(x)|a^0|^2\,dx.$$

The $O(1)$ term is

$$I^0[a^0, \theta^0] = \int_{\mathbb{R}^N} \frac{1}{2}|a^0|^2|D\theta^0|^2 + \frac{1}{2}V(x)|a^0|^2\,dx.$$

Since we want to minimize $I^\varepsilon[a^\varepsilon, \theta^\varepsilon]$, select a^0 and θ^0 to minimize $I^0[a^0, \theta^0]$. Let a be arbitrary and let

$$g(h) = I^0[a^0 + ha, \theta^0] = \int_{\mathbb{R}^N} \frac{1}{2}|a^0 + ha|^2|D\theta^0|^2 + \frac{1}{2}V(x)|a^0 + ha|^2\,dx.$$

Then

$$g'(h) = \int_{\mathbb{R}^N} (a^+ ha)|D\theta^0|^2 a + V(x)(a^0 + ha)a\,dx.$$

Since $g'(0) = 0$, it follows that

$$a^0|D\theta^0|^2 + V(x)a^0 = 0.$$

This implies that

$$|D\theta^0|^2 + V(x) = 0.$$

Similarly, if we do a variation in θ, then

$$-\mathrm{div}((a^0)^2 D\theta^0) = 0$$

can solve for θ^0 and then for a^0.

3.9 Homogenization

Consider

$$-\left(a\left(\frac{x}{\varepsilon}\right)u_\varepsilon'\right) = f(x) \quad \text{in } W, \quad u^\varepsilon = 0 \quad \text{on } \partial W,$$

where $W \subset \mathbb{R}$. Here $u^\varepsilon = u^\varepsilon(x)$ and $a = a(y) > 0$ and 1-periodic. So $a(x/\varepsilon)$ is ε-periodic.

Problem 3.9. One can show that $u^\varepsilon \to u^0$ where $u^0 = u^0(x)$. What PDE does u^0 satisfy?

We use the following exotic ansatz

$$u^\varepsilon(x) = u^0(x) + \varepsilon u^1\left(x, \frac{x}{\varepsilon}\right) + \cdots$$

where $u^1 = u^1(x, y)$ and $u^1(x, y)$ is 1-periodic in y.

Note that this PDE is variational with

$$I[u^\varepsilon] = \int_W \frac{1}{2}\left(a\left(\frac{x}{\varepsilon}\right)u_\varepsilon'\right)u_\varepsilon' - fu^\varepsilon\, dx.$$

Observe that

$$\frac{d}{dx}u^1\left(x, \frac{x}{\varepsilon}\right) = u_x^1 + \left(\frac{1}{\varepsilon}\right)u_y^1.$$

Then the result is

$$I[u^\varepsilon] = \int_W \frac{1}{2}a\left(\frac{x}{\varepsilon}\right)[u_x^0 + u_y^1]^2 - fu^0\, dx + o(1).$$

Note that the above function is still a function of y. By taking integral on $[0, 1]$, we get

$$I[u^0, u^1] = \int_0^1 \int_W \frac{1}{2}a(y)[u_x^0 + u_y^1]^2 - fu^0\, dxdy.$$

If we take the variation in u^1, then we get the following Euler-Lagrange equation

$$-[a(y)u_y^1]_y = [u_x^0 a(y)]_y.$$

Here u^0 is given. To solve this PDE, define $u^1(x, y) = w(y)u_x^0(x)$. Then if we put this into the PDE, then

$$-[a(y)w'(y)]' = a'(y).$$

Then we can solve for w, and so does for u^1. Then if we plug u^1 into the functional $I[u^0, u^1]$, then we get

$$I^0[u^0, wu^0_x] = \int_W \frac{1}{2}\overline{a}(u^0_x)^2 - fu^0\,dx,$$

where $\overline{a} = \int_0^1 a(y)(1 + w'(y))^2 dy$.

Now if we do the variation in u^0, then

$$-(\overline{a}u^0_x)_x = f.$$

(compare this to $-\overline{a}u^0_{xx} = f$).

Boundary layers

4

4.1 Introduction

We give some examples of boundary layers. Consider two different chemicals A and B and react to each other. Then there would be a layer between two chemicals. We could also see the seashores between sand and sea.

We will find an approximation of $uvarepsilon$ that takes into account the boundary layer. First, we move the usual ansatz far from the boundary layer. In the inner solution, we use a change of variable $y = x/varepsilon$ to open up the boundary layer, making the system more manageable. Then do ansatz on smoother solution $\overline{u}^\varepsilon$, and we combine the two to get our approximation u^*.

Example 4.1. Consider

$$\varepsilon u^\varepsilon_{xx} + 2u^\varepsilon_x + 2u^\varepsilon = 0, \quad u^\varepsilon(0) = 0, \quad u^\varepsilon(1) = 1,$$

where $u^\varepsilon = u^\varepsilon(x)$ and $0 \le x \le 1$. It turns out (numerically) that there is a boundary layer at $x = 0$. We will find an approximation of u^ε that takes into account the boundary layer.

We put $u^\varepsilon(x) = u^0(x) + \varepsilon u^1(x) + \varepsilon^2 u^2(x) + \cdots$. Then

$$\varepsilon(u^0_{xx} + 2u^1_x + 2u^1) + \varepsilon^2(u^1_{xx}) + 2u^0_x + 2u^0 = 0.$$

Comparing this with $O(1)$-term, we have $u^0(x) = Ae^{-x}$. Since $u^0(1) = 1$, $A = e$, and hence $u^0(x) = e^{1-x}$.

Next, we choose the change of variable. Let $y = x/\varepsilon^\alpha$, where α will be determined later. Define $\overline{u}^\varepsilon(y) = u^\varepsilon(x)$. Chain rule gives

$$u^\varepsilon_x = \frac{d\overline{u}^\varepsilon}{dy}\frac{dy}{dx} = \frac{1}{\varepsilon^\alpha}\overline{u}^\varepsilon_y$$
$$u^\varepsilon_{xx} = \frac{1}{\varepsilon^{2\alpha}}\overline{u}^\varepsilon_{yy}.$$

This implies that

$$\varepsilon\left(\frac{1}{\varepsilon^{2\alpha}}\overline{u}^\varepsilon_{yy}\right) + 2\left(\frac{1}{\varepsilon^\alpha}\overline{u}^\varepsilon_y\right) + 2\overline{u}^\varepsilon = 0.$$

In other words, we have

$$\varepsilon^{1-2\alpha}\overline{u}^\varepsilon_{yy} + 2\varepsilon^{-\alpha}\overline{u}^\varepsilon_y + 2\overline{u}^\varepsilon = 0.$$

We write

$$A = \varepsilon^{1-2\alpha}u^\varepsilon_{yy}, \quad B = 2\varepsilon^{-\alpha}\overline{u}^\varepsilon_y, \quad C = 2\overline{u}^\varepsilon.$$

We divide several cases. Suppose that $B \sim C$(same order) and A is smaller. This means that $\varepsilon^{-\alpha} = \varepsilon^0$, so $\alpha = 0$. But then $y = x$ but there is no boundary layer. Suppose that $A \sim C$ and B is smaller. Then $\alpha = 1/2$. But $B \sim \varepsilon^{-1/2}$ is not small. Finally, let us suppose that $A \sim B$, and C is smaller. Then $\alpha = 1$. Note that ε^{-1} is bigger than constant as $\varepsilon \to 0+$. So our ODE becomes

$$\overline{u}^\varepsilon_{yy} + 2\overline{u}^\varepsilon_y + 2\varepsilon\overline{u}^\varepsilon = 0.$$

Now we put $\overline{u}^\varepsilon(y) = \overline{u}^0(y) + \varepsilon\overline{u}^1(y) + \cdots$ into the ODE. Then

$$\overline{u}^0_{yy} + \varepsilon\overline{u}^1_{yy} + 2\overline{u}^0_y + 2\varepsilon\overline{u}^1_y + 2\varepsilon\overline{u}^0 + 2\varepsilon^2\overline{u}^1 = 0.$$

So

$$\overline{u}^0_{yy} + 2\overline{u}^0_y = 0, \quad \overline{u}^0(y) = A + Be^{-2y}.$$

If we impose $\overline{u}^0(0) = 0$, then

$$\overline{u}^0(0) = A + B = 0, \quad \text{and so } B = -A,$$

and hence

$$\overline{u}^0(y) = A - Ae^{-2y} = A(1 - e^{-2y}).$$

Now, we match two solutions to find A.

Method 1: Matching in asymptotic limit as $x \to 0+$ and $y \to \infty$. By using this, $A = e$. This methodology does not always work.

Method 2: Matching in overlapping regions. Suppose that the overlap region is $(\varepsilon^{\alpha_1}, \varepsilon^{\alpha_2})$, where $\alpha_2 < \alpha_1 < 1$. Let $x \in (\varepsilon^{\alpha_1}, \varepsilon^{\alpha_2})$ and let $z = x/\varepsilon^\beta$ be an intermediate variable in between x and $x/\varepsilon = y$, where $0 < \beta < 1$ is to be determined. Note that

$$u^0(x) = e^{1-\varepsilon^\beta z} \quad \text{and} \quad \overline{u}^0(y) = A(1 - e^{-2x/\varepsilon}) = A(1 - e^{-2\varepsilon^\beta z/\varepsilon}).$$

We want to claim

$$\lim_{\varepsilon \to 0+} [u(x) - \overline{u}^0(y)] = 0$$

under appropriate value on A and β. If $\varepsilon \to 0+$, then what happens to z? It turns out that as $\varepsilon \to 0+$, we have $\varepsilon^\beta z \to 0$ and $\varepsilon^{\beta-1}z \to \infty$. Therefore, we get

$$\lim_{\varepsilon \to 0+} [u^0(x) - \overline{u}^0(y)] = 0,$$

which implies that $A = e$.

Now it remains to show that as $\varepsilon \to 0+$, we have $\varepsilon^\beta z \to 0$ and $\varepsilon^{\beta-1}z \to \infty$. Since $\varepsilon^{\alpha_1} < x < \varepsilon^{\alpha_2}$ and $x = \varepsilon^\beta z$, it follows that

$$\varepsilon^{\alpha+1} < \varepsilon^\beta z < \varepsilon^{\alpha_2}.$$

Since $\varepsilon^{\alpha_1} \to 0$ and $\varepsilon^{\alpha_2} \to 0$, it follows that $\varepsilon^\beta z \to 0$. Also, since $\varepsilon^{\beta-1}z > \varepsilon^{\alpha_1 - 1}$, we have $\varepsilon^{\beta-1}z \to \infty$ as $\varepsilon \to 0+$ since $\alpha_1 < 1$.

Now we are ready to construct solution u^*. Define

$$u^*(x) = u^0(x) + \overline{u}^0(y) - \text{common part}.$$

Then

$$u^*(x) = e^{1-x} + e(1 - e^{-2y}) - e = e^{1-x} + e(1 - e^{-2x/\varepsilon}) - e = e^{1-x} - e^{1-2x/\varepsilon}.$$

This solution reflects our intuition.

Example 4.2. We study the same problem but we will pay attention to higher order on ε:

$$\varepsilon u^\varepsilon_{xx} + 2u^\varepsilon_x + 2u^\varepsilon = 0, \quad u^\varepsilon(0) = 0, \quad u^\varepsilon(1) = 1,$$

where $u^\varepsilon = u^\varepsilon(x)$ and $0 \le x \le 1$.

We will look at $O(\varepsilon)$-terms to get a better approximation of u^ε. We put the usual ansatz to the equation:

$$(\varepsilon u^0_{xx} + \varepsilon^2 u^1_{xx}) + (2u^0_x + 2\varepsilon u^1_x) + (2u^0 + 2\varepsilon u^1) + ... = 0.$$

By looking at $O(1)$-terms, we get $u^0(x) = e^{1-x}$ if we impose $u^0(1) = 1$. By looking at $O(\varepsilon)$-terms, we get

$$u^0_{xx} + 2u^1_x + 2u^1 = 0$$

and so

$$u^1_x + u^1 = -\frac{1}{2}u^0_{xx} = -\frac{1}{2}e^{1-x}.$$

We can solve this differential equation using undetermined coefficients to get

$$u^1(x) = Ae^{-x} - \frac{e}{2}xe^{-x}.$$

From the boundary condition, we can put $u^1(1) = 0$. Then one can see that $A = e/2$ and so

$$u^1(x) = \frac{1}{2}(1-x)e^{1-x}.$$

Next, we find an inner solution and let $y = x/\varepsilon^\alpha$, where α is to be determined. Define $\overline{u}^\varepsilon(y) = u^\varepsilon(x)$. We rewrite the ODE in terms of y:

$$\varepsilon^{1-2\alpha}\overline{u}^\varepsilon_{yy} + 2\varepsilon^{-\alpha}\overline{u}^\varepsilon_y + 2\overline{u}^\varepsilon = 0.$$

Then by a method of dominated balance, we see that $\alpha = 1$.

Therefore, we get

$$\overline{u}^\varepsilon_{yy} + 2\overline{u}^\varepsilon_y + 2\varepsilon\overline{u}^\varepsilon = 0.$$

By using the usual ansatz for $\overline{u}^\varepsilon(y)$, we get

$$(\overline{u}^0_{yy} + \varepsilon\overline{u}^1_{yy}) + (2\overline{u}^0_y + 2\varepsilon\overline{u}^1_y) + (2\varepsilon\overline{u}^0 + 2\varepsilon^2\overline{u}^1) = 0.$$

By considering $O(1)$ terms, we get

$$\overline{u}^0(y) = A + Be^{-2y}.$$

Since $\overline{u}^0(0) = 0$, we see that $B = -A$. Hence

$$\overline{u}^0(y) = A(1 - e^{-2y}).$$

By using a matching method, one can see that $A = e$. By looking at $O(\varepsilon)$-term, we get

$$\overline{u}^1_{yy} + 2\overline{u}^1_y = -2\overline{u}^0 = -2e(1 - e^{-2y}).$$

We impose $\overline{u}^1(0) = 0$ and hence we can easily get

$$\overline{u}^1(y) = A(1 - e^{-2y}) - ye(1 + e^{-2y}).$$

Note that method 1 is not applicable since $\overline{u}^1(y)$ does not converge.

To use the second method, let $z = x/\varepsilon^\beta$ and let $x \in (\varepsilon^{\alpha_1}, \varepsilon^{\alpha_2})$ be an intermediate variable.

Write $u^0(x) + \varepsilon u^1(x)$ and $\overline{u}^0(y) + \varepsilon \overline{u}^1(y)$ in terms of z. By direct computation, we get

$$u^0(x) + \varepsilon u^1(x) = e^{1-\varepsilon^\beta z} + \frac{\varepsilon}{2}(1 - \varepsilon^\beta z)e^{1-\varepsilon^\beta z}.$$

On the other hand, we note that

$$\begin{aligned}\overline{u}^0(y) + \varepsilon\overline{u}^1(y) &= e(1 - e^{-2y}) + \varepsilon(A(1 - e^{-2y}) - ye(1 + e^{2-y})) \\ &= e\left(1 - e^{-2\varepsilon^{\beta-1}z}\right) - \varepsilon^{\beta-1}ze\left(1 + e^{-2\varepsilon^{\beta-1}}\right) \\ &\quad + \varepsilon\left(A\left(1 - e^{-2\varepsilon^{\beta-1}z}\right)\right) - \varepsilon^{\beta-1}ze(1 + e^{-2\varepsilon^{\beta-1}z})\Big).\end{aligned}$$

By comparing coefficients of ε and sending a limit, we get

$$A = \frac{e}{2}.$$

Now we construct u^* by

$$\begin{aligned}u^*(x) &= u^0(x) + \varepsilon u^1(x) + \overline{u}^0(y) + \varepsilon\overline{u}^1(y) \\ &= e^{1-x} - e^{1-2x/\varepsilon} + \frac{\varepsilon}{2}(1-x)e^{1-x} - \frac{\varepsilon}{2}e^{1-2x/\varepsilon} - xe(1 + e^{-x/\varepsilon}).\end{aligned}$$

Note that this solution contains the first example as well.

Example 4.3 (An internal layer)**.** Consider

$$\begin{cases}\varepsilon u^\varepsilon_{xx} + xu^\varepsilon_x + x^2 u^\varepsilon = 0, \\ u^\varepsilon(-1) = \alpha \quad \text{and} \quad u^\varepsilon(1) = \beta.\end{cases} \tag{4.1}$$

Here $u^\varepsilon = u^\varepsilon(x)$ with $-1 \le x \le 1$ and α, β are given. We expect the boundary layer at $x = 0$.

We put $u^\varepsilon(x) = u^0(x) + \varepsilon u^1(x) + \cdots$ into the equation. Then

$$(\varepsilon u^0_{xx} + \varepsilon^2 u^1_{xx}) + (xu^0_x + \varepsilon x u^1_x) + (x^2 u^0 + \varepsilon x^2 u^1) = 0.$$

By considering $O(1)$-term, we get

$$u^0_x + xu^0 = 0.$$

It is easy to see that $u^0(x) = Ae^{-x^2/2}$ is a general solution of the equation.

Since we are dealing with 2 domains $(-1, 0)$ and $(0, 1)$, we actually have

$$u^0(x) = \begin{cases}Ae^{-x^2/2} & \text{if } -1 \le x < 0, \\ Be^{-x^2/2} & \text{if } 0 < x \le 1.\end{cases}$$

Since $u^0(-1) = \alpha$ and $u^0(1) = \beta$, we see that $A = \alpha e^{1/2}$ and $B = \beta e^{1/2}$ and hence

$$u^0(x) = \begin{cases}\alpha e^{\frac{1-x^2}{2}} & \text{if } -1 \le x < 0, \\ \beta e^{\frac{1-x^2}{2}} & \text{if } 0 < x \le 1.\end{cases}$$

Let us look at the inner solution. Let $y = x/\varepsilon^\delta$ and $\overline{u}^\varepsilon(y) = u^\varepsilon(x)$. Then a tedious calculation gives

$$\varepsilon^{1-2\delta}\overline{u}^\varepsilon_{yy} + y\overline{u}^\varepsilon_y + \varepsilon^{2\delta}y^2\overline{u}^\varepsilon = 0.$$

By using a dominated balance principle, one can easily see that $\delta = 1/2$. So we get

$$\overline{u}^\varepsilon_{yy} + y\overline{u}^\varepsilon_y + \varepsilon y^2\overline{u}^\varepsilon = 0.$$

If we put our usual ansatz, then we get

$$\begin{aligned}(\overline{u}^0_{yy} + \varepsilon\overline{u}^1_{yy}) + (y\overline{u}^0_y + \varepsilon y\overline{u}^1_y)\\ + (\varepsilon y^2\overline{u}^0 + \varepsilon^2 y^2\overline{u}^1) = 0.\end{aligned}$$

Note that $O(1)$ term becomes $\overline{u}^0_{yy} + y\overline{u}^0_y = 0$. Then by using a method of integrating factor, we get

$$\overline{u}^0(y) = A\int_0^y e^{-t^2/2}dt + B.$$

Now we need to do match

$$\lim_{x\to 0+} u^0(x) = \lim_{y\to\infty} \overline{u}^0(y)$$

and

$$\lim_{x\to 0-} u^0(x) = \lim_{y\to-\infty} \overline{u}^0(y).$$

Then we get

$$\beta\sqrt{e} = A\int_0^\infty e^{-t^2/2}dt = A\left(\frac{\sqrt{2\pi}}{2}\right) + B.$$

Similarly, we have

$$\alpha\sqrt{e} = -A\frac{\sqrt{2\pi}}{2} + B.$$

Hence we get

$$A = \left(\frac{\beta-\alpha}{\sqrt{2\pi}}\right)\sqrt{e} \quad \text{and} \quad B = \left(\frac{\beta+\alpha}{\sqrt{2\pi}}\right)\sqrt{e}.$$

Therefore,

$$u^*(x) = u^0(x) + \overline{u}^0(y) - \text{common part}.$$

So

$$u^*(x) = \begin{cases}\alpha e^{(1-x^2)/2} + \left(\dfrac{\beta-\alpha}{\sqrt{2\pi}}\right)\sqrt{e}\displaystyle\int_0^{x/\sqrt{\varepsilon}} e^{-t^2/2}dt + \left(\dfrac{\beta+\alpha}{2}\right)\sqrt{e}\\ \beta e^{(1-x^2)/2} + \left(\dfrac{\beta-\alpha}{\sqrt{2\pi}}\right)\sqrt{e}\displaystyle\int_0^{x/\sqrt{\varepsilon}} e^{-t^2/2}dt + \left(\dfrac{\alpha-\beta}{2}\right)\sqrt{e}.\end{cases}$$

Example 4.4 (Earth-Moon spacecraft)**.** Suppose that we launch the spacecraft from earth $(0,0)$ with the initial angle $\varepsilon\kappa$ to the moon $L(1,0)$ but deflect. We want to calculate the deflection in terms of κ.

Let M be the mass of earth and εM mass of moon, and let G be the gravitational constant (for simplicity, we may assume that $GM = 1$) and let μ be the mass of spacecraft.

Let $r(t)$ be the position of the spacecraft. By Newton's second law, we have

$$\mu \ddot{r} = -\frac{GM\mu r}{|r|^2} - \frac{G(\varepsilon M)\mu(r-(1,0))}{|r-(1,0)|^3}.$$

If we write $r(t) = (x_\varepsilon(t), y_\varepsilon(t))$, then x_ε and y_ε satisfy

$$\begin{cases} x_\varepsilon''(t) = -\dfrac{x_\varepsilon}{((x_\varepsilon)^2+(y_\varepsilon)^2)^{3/2}} - \dfrac{\varepsilon(x_\varepsilon - 1)}{((x_\varepsilon-1)^2+(y_\varepsilon)^2)^{3/2}}, \\ y_\varepsilon''(t) = -\dfrac{y_\varepsilon}{((x_\varepsilon)^2+(y_\varepsilon)^2)^{3/2}} - \dfrac{\varepsilon(y_\varepsilon)}{((x_\varepsilon-1)^2+(y_\varepsilon)^2)^{3/2}}. \end{cases} \tag{4.2}$$

If we put $x_\varepsilon(t) = x_0(t) + \varepsilon x_1(t)$ and $y_\varepsilon(t) = \varepsilon y_1(t)$ (since $y_0(t) = 0$), then we get

$$x^0_{tt} + \varepsilon(\cdots) = -\frac{x^0(t)+\varepsilon(\cdots)}{[(x^0(t)+\varepsilon(\cdots)]^2 + (\varepsilon y_1(t)+\cdots)^2]^{3/2}} + \varepsilon(\cdots).$$

So we get

$$x^0_{tt} = -\frac{x^0(t)}{(x^0)^3} = -\frac{1}{(x^0)^2}.$$

Note that

$$E(t) = \frac{(x^0_t)^2}{2} - \frac{1}{(x^0)}$$

is constant. So

$$\frac{(x^0_t)^2}{2} - \frac{1}{x^0} = 0.$$

by choosing initial condition $x^0(0) = 2/(x^0_t(0))^2$. Then by solving this ODE, we get

$$x^0(t) = \left(\frac{3}{\sqrt{2}}t + \frac{3}{2}C\right)^{2/3}.$$

Since $x^0(0) = 0$, we have $x^0(t) = (9/2)^{1/3}t^{2/3}$. So if we write $t^* = \sqrt{2}/3$, then $x^0(\sqrt{2}/3) = 0$.

Let us observe the ODE for y^ε. Then one can get

$$y^1_{tt} = -\frac{y^1}{(x^0)^3}.$$

We impose $y^1(0) = 0$ and $y^\varepsilon_t(0)/x^\varepsilon_t(0) = \varepsilon\kappa$. If we put the standard perturbation, then we get

$$\varepsilon y^1_t(0) = \varepsilon\kappa x^0_t(0) + \varepsilon^2\kappa x^1_t(0).$$

So

$$y^1_t(0) = \kappa x^0_t(0)$$

We claim that

$$y^1(t) = \kappa x^0(t).$$

Indeed, $y^1(t)$ and $\kappa x^0(t)$ both solve

$$w_{tt} = -\frac{w}{(x^0)^3}, \quad w(0) = 0, \quad w_t(0) = \kappa x_t^0(0).$$

From (4.2), we derive an ODE for x^1:

$$x_{tt}^1 = -\frac{x^1}{(x^0)^3} - \frac{1}{(x^0-1)^2}.$$

As $t \to t^* = \frac{\sqrt{2}}{3}$, we have $x^0 \to 1$ and so $x_{tt}^1 \to -\infty$. This calculate indicates that the deflection could happen near $t = t^*$.

Next, we look at the inner solution (near $t = t^*$). We define $y = x/\varepsilon^\alpha$ and let $\tau = (t - t^*)/\varepsilon$, $\xi = (1 - x^\varepsilon)/\varepsilon$, and $\eta = y^\varepsilon/\varepsilon$. Write our ODE in terms of ξ, η, and τ:

$$-\frac{1}{\varepsilon}\xi_{\tau\tau} = \frac{1-\varepsilon\xi}{((1-\varepsilon\xi)^2 + (\varepsilon\eta)^2)^{3/2}} - \frac{\varepsilon(-\varepsilon\xi)}{[(-\varepsilon\xi)^2 + (\varepsilon\eta)^2]^{3/2}}.$$

If we put our standard perturbation $\xi = \xi^0 + \varepsilon\xi^1 + ...$ and $\eta = \eta^0 + \varepsilon\eta^1 + \cdots$, then we look at $O(1/\varepsilon)$-terms to get

$$\begin{cases} \xi_{\tau\tau}^0 = -\dfrac{\xi^0}{((\xi^0)^2 + (\eta^0)^2)^{3/2}}, \\ \eta_{\tau\tau}^0 = -\dfrac{\eta^0}{((\xi^0)^2 + (\eta^0)^2)^{3/2}}. \end{cases} \tag{4.3}$$

Those are the standard equations for Kepler's motion. If $r(\tau) = (\xi^0(\tau), \eta^0(\tau))$, then $r(\tau)$ solves $r''(\tau) = -r(\tau)/|r(\tau)|^3$. Then

$$r(\theta) = \frac{(1+\varepsilon)r_0}{1 + e\cos\theta},$$

where $r_0 = r(0)$ and $e = |r_0||v_0|^2 - 1$ with $v_0 = r'(0)$.

Next, we rotate the plane by α to get a horizontal asymptote. Then $r(\theta)$ becomes

$$r(\theta) = \frac{(1+e)r_0}{1 + e\cos(\theta - \alpha)}.$$

We choose α so that $\lim_{\theta\to\pi} r(\theta) = \infty$, i.e., find α so that

$$\lim_{\theta\to\pi}[1 + e\cos(\theta - \alpha)] = 0,$$

and hence

$$e = \frac{1}{\cos\alpha}.$$

This gives

$$r(\theta) = \frac{(\cos\alpha + 1)r_0}{\cos\alpha + \cos(\theta - \alpha)} = \frac{b\sin\alpha}{\cos\alpha + \cos(\theta - \alpha)},$$

where

$$b = \left(\frac{\cos\alpha + 1}{\sin\alpha}\right) r_0.$$

The quantity b denotes the impact parameter, the height of the horizontal asymptote. We call the number 2α the deflection. We will calculate 2α in terms of κ.

Let

$$E(\theta) = \frac{1}{2}|r'(\theta)|^2 - \frac{1}{|r(\theta)|}$$

which is a conserved quantity. From this, we have

$$\frac{|r'(\theta)|^2}{2} - \frac{1}{|r(\theta)|} = \frac{|r'(\alpha)|^2}{2} - \frac{1}{|r(\alpha)|} = \frac{|v_0|^2}{2} - \frac{1}{|r_0|}.$$

Letting $\theta \to \pi$ so that $|r(\theta)| \to \infty$ and $|r'(\theta)| \to v_\infty$, and so

$$\frac{(v_\infty)^2}{2} - 0 = \frac{|v_0|^2}{2} - \frac{1}{r_0}$$

and hence

$$(v_\infty)^2 = \frac{\tan\alpha}{b}.$$

We will match outer solution and inner solution:

$$\lim_{t \to t^*-} x_t^0(t) = \lim_{\tau \to -\infty} -\xi_\tau^0(\tau)$$

and

$$\lim_{t \to t^*-} y_t^1(t) = \lim_{\tau \to -\infty} \eta^0(\tau).$$

We used $x_t^0 = -\xi_\tau^0$ and $\tau = (t - t^*)/\varepsilon$ and $\eta = y^\varepsilon/\varepsilon$, and so $\eta^0 + \varepsilon\eta^1 = (y^0 + \varepsilon y^1)/\varepsilon$, and so $\eta^0 = y^1$.

In the first equation, the left side is $\sqrt{2}$. In the second equation, the left hand side is κ. To calculate the right hand side, recall that

$$r(\tau) = (\xi^0(\tau), \eta^0(\tau)),$$

and so $\eta^0(-\infty) = b$. Since $|r'(\tau)| = (\xi_\tau^0, \eta_\tau^0)$ and if we think about the limit, then $r'(\tau) \approx ((\xi_\tau^0)^2)^{1/2}$, and so $-(\xi_\tau^0)$. Hence

$$\lim_{\tau \to -\infty} -(\xi_\tau^0) = \lim_{\tau \to -\infty} |r'(\tau)| = v^\infty.$$

Putting everything together, we get $\sqrt{2} = v_\infty$ and $\kappa = b$. Hence

$$\alpha = \tan^{-1}(\sqrt{2k}).$$

4.2 Singular variational problem

Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a double-well potential function with $\Phi(\pm) = 0$ and $\Phi(0) = 1$. We want to minimize

$$I^\varepsilon[u] = \int_W \varepsilon \frac{|Du|^2}{2} + \frac{1}{\varepsilon}\Phi(u)\, dx.$$

among all $U : W \to \mathbb{R}$ such that $U = g$ on ∂W. Let U^ε be a minimzer. We will study $\lim_{\varepsilon\to 0+} U^\varepsilon(x) =: U^0(x)$.

For example, we might consider

$$W = W^- \cup W^+ \cup \Gamma^\varepsilon,$$

where

$$W^- = \{x : U^\varepsilon \to -1\}, \quad W^+ = \{x : U^\varepsilon \to 1\}$$

and a boundary layer region Γ^ε between.

Note that the Lagrangian is

$$L(p, z, x) = \varepsilon\frac{|p|^2}{2} + \frac{1}{\varepsilon}\Phi(z).$$

So the Euler-Lagrange equation associated with this Lagrangian is

$$-\varepsilon^2 \Delta U^\varepsilon + \Phi'(U^\varepsilon) = 0 \quad \text{in W} \quad \text{on} \qquad U^\varepsilon = g \quad \text{on } \partial W.$$

We first seek an outer solution. We put the ansatz

$$U^\varepsilon(x) = U^0(x) + \varepsilon U^1(x) + \cdots,$$

and then

$$-\varepsilon^2 \Delta U^0 + \Phi(U^0 + \varepsilon U^1) = 0.$$

Taylor expansion gives

$$-\varepsilon^2 \Phi(U^0) + \Phi'(U^0) + \varepsilon \Phi''(U^0)U^1 = 0.$$

So $\Phi'(U^0) = 0$ in W, which implies that U^0 has values in -1, 0, 1.

Since U is a minimizer of

$$I[U] = \int_W \frac{\varepsilon}{2}|DU|^2 + \frac{1}{\varepsilon}\Phi(U)\, dx,$$

U^0 has values in $\{-1, 1\}$. This implies that U^0 is 1 in W^+ and -1 in W^- because of the continuity of U^0.

To study the inner solution, let $\Gamma^\varepsilon = \{-1+\varepsilon < U^\varepsilon < 1-\varepsilon\}$ the thicker version of Γ. Now we work on an analog of dominant balance.

Suppose that the width of Γ_ε is $O(\varepsilon^\alpha)$, where α is to be determined. Recall that $W = W^+ \cap W^- \cup \Gamma^\varepsilon$, where

$$I[U] = \int_W \frac{\varepsilon}{2}|DU|^2 + \frac{1}{\varepsilon}\Phi(U)\, dx.$$

On W^+, $U^\varepsilon \approx 1$. So $\Phi(U^\varepsilon) \approx 0$, and $|DU^\varepsilon| \approx 0$. So

$$\int_{W^+} \varepsilon \frac{|DU|^2}{2} + \frac{1}{\varepsilon}\Phi(U)\, dx \approx 0.$$

SImilarly, we can argue it for W^-. Hence

$$I[U] \approx \int_{\Gamma^\varepsilon} \frac{\varepsilon}{2}|DU|^2 + \frac{1}{\varepsilon}\Phi(U)\, dx.$$

On the region Γ^ε, we note that

$$|DU| \approx \frac{2}{\varepsilon^\alpha}.$$

Also, on Γ^ε, $\Phi(U) \approx \Phi(0) = 1$. Hence

$$\begin{aligned} I[U] &\approx \int_{\Gamma^\varepsilon} \frac{\varepsilon}{2}\left(\frac{2}{\varepsilon^\alpha}\right)^2 + \frac{1}{\varepsilon}\, dx \\ &\approx (2\varepsilon^{1-2\alpha} + \varepsilon^{-1})|\Gamma^\varepsilon| \\ &\approx (2\varepsilon^{1-\alpha} + \varepsilon^{\alpha-1}). \end{aligned}$$

If $\alpha = 1$, then $I[U]$ does not blow up.

Let $\Gamma = \{U^\varepsilon = 0\}$. Then we choose an appropriate change of variables to turn Γ into a graph $x^N = s(x')$.

After translating, we may assume that $s^\varepsilon(0) = 0$ and after rotating, we may assume that $Ds(0) = (0, \ldots, 0)$. Finally, change coordinates so that we are on s^ε and boundary layer has width $O(1)$.

Let $y = (y^1, \ldots, y^N)$ be such that $y_i = x_i$ for $i = 1, \ldots, N-1$, and $y_N = (x_N - s^\varepsilon)/\varepsilon$ so that we can straighthen the boundary. Now let

$$\overline{U}^\varepsilon(y) = U^\varepsilon(x) = \overline{U}^\varepsilon\left(x', \frac{x_n - s^\varepsilon(x')}{\varepsilon}\right).$$

Now we seek a PDE that $\overline{U}^\varepsilon$ satisfies.

By a chain rule, we have

$$\begin{aligned} U_{x_i} &= \frac{dU}{dx_i} = \overline{U}_{y_i} + \overline{U}_{y_N}\left(\frac{-s_{x_i}}{\varepsilon}\right), \\ U_{x_i x_i} &= \overline{U}_{y_i y_i} + 2\overline{U}_{y_i y_N}\left(-\frac{s_{x_i}}{\varepsilon}\right) + \overline{U}_{y_N y_N}\left(-\frac{s_{x_i}}{\varepsilon}\right)^2 + \overline{U}_{y_N}\left(-\frac{s_{x_i x_i}}{\varepsilon}\right). \end{aligned}$$

Finally, when $i = N$, we have

$$U_{x_N x_N} = \overline{U}_{y_N y_N}\left(\frac{1}{\varepsilon^2}\right).$$

These calculation show that $\overline{U}$ satisfies

$$-\varepsilon^2\left(\sum_{i=1}^{N_1} \overline{U}_{y_i y_i} - \frac{2}{\varepsilon}\overline{U}_{y_i y_N}(s_{x_i}) + \frac{1}{\varepsilon^2}\overline{U}_{y_N y_N}(s_{x_i})^2 - \frac{1}{\varepsilon}\overline{U}_{y_N} s_{x_i x_i}\right) - \overline{U}_{y_N y_N} + \Phi'(\overline{U}) = 0. \tag{4.4}$$

However, we cannot perform a standard ansatz to $\overline{U}$ since $s = s^\varepsilon$ also depends on ε. So we put $s^\varepsilon(x') = s^0(x') + \varepsilon s^1(x')$ and put this ansatz to compute the PDE. Then $O(1)$ term gives

$$-\sum_{i=1}^{N-1} \overline{U}^0_{y_N y_N}(s^0_{x_i})^2 - \overline{U}^0_{y_N y_N} + \Phi'(\overline{U}^0) = 0.$$

But recall that $s(0) = 0$ and $Ds(0) = (s_{x_1}(0), \ldots, s_{x_{N-1}}(0)) = (0, \ldots, 0)$.

If you evaluate this at $y = (0, \ldots, y_N)$, then $s^0_{x_i} = 0$ and we get

$$-\overline{U}^0_{y_N y_N}(0, \ldots, 0, y_N) + \Phi'(\overline{U}^0(0, \ldots, 0, y_N)) = 0.$$

Example 4.5 (Singular perturbation of eigenfunctions)**.** Consider

$$-\Delta u^0 = \lambda_0 u^0 \quad \text{in } W \subset \mathbb{R}^3, \quad u^0 = 0 \quad \text{on } \partial W.$$

By a standard PDE theory, there exists $\lambda_0 > 0$ which is called the principal eigenvalue such that the problem has a non-trivial solution $u^0 > 0$ in W and $\int_W (u^0)^2\, dx = 1$. This λ_0 is called the *principal harmonic* and any other eigenvalues are called the *overtones*.

An interesting question is "can you hear the shape of a drum"? No in 2D if your instrument has corners. Yes in 2D if instrument is smooth and convex. For the higher dimension, there is a 16-dimensional counterexample.

Now we study a perturbation on the domain. Let $W^\varepsilon = W \setminus \overline{B}_\varepsilon$. Consider

$$-\Delta u^\varepsilon = \lambda_\varepsilon u^\varepsilon \quad \text{in } W^\varepsilon \quad \text{and} \quad u^\varepsilon = 0 \quad \text{on } \partial W^\varepsilon = \partial W \cup \partial B_\varepsilon.$$

We will build λ^ε as a perturbation of λ_0. Note that there is a boundary layer on ∂B_ε.

We put the ansatz $u^\varepsilon = u^0 + \varepsilon u^1 + \cdots$ and $\lambda^\varepsilon = \lambda^0 + \varepsilon\lambda^1 + \cdots$. Then

$$-\Delta u^0 - \varepsilon\Delta u^1 + \cdots = \lambda^0 u^0 + \varepsilon(\lambda^1 u^0 + \lambda^0 u^1) + \cdots$$

So

$$-\Delta u^0 = \lambda^0 u^0 \quad \text{and} \quad -\Delta u^1 = \lambda^1 u^0 + \lambda^0 u^1.$$

Then

$$-\Delta u^0 = \lambda^0 u^0 \quad \text{and} \quad -\Delta u^1 - \lambda_0 u^1 = \lambda_1 u^0.$$

If we specify $u^1 = 0$ on ∂B_ε, then $u^1 = 0$ which is not interesting. Observe that λ_0, λ_1, u_0, and u_1 do not depend on ε.

Letting $\varepsilon \to 0+$, we get

$$-\Delta u^1 - \lambda^0 u^1 = \lambda^1 u^0 \quad \text{in } W \setminus \{0\} \quad \text{and} \quad u^1 = 0 \quad \text{on } \partial W.$$

By a PDE theory(?), u^1 blows up at 0.

Starting from here, we assume that $|u^1(x)||x|$ is bounded. This is for the outer solution.

To study the inner solution near 0, let $y = x/\varepsilon$. Define $\overline{u}^\varepsilon(y) = u^\varepsilon(x)$. Then

$$-\Delta u^\varepsilon = \lambda^\varepsilon u^\varepsilon$$

becomes

$$-\frac{1}{\varepsilon^2}\Delta\overline{u}^\varepsilon = \lambda^\varepsilon\overline{u}^\varepsilon \quad \text{in } (1/\varepsilon)W \setminus B_1 \quad \overline{u}^\varepsilon = 0 \quad \text{on } \partial B_1.$$

Now we put the ansatz

$$\overline{u}^\varepsilon = \overline{u}^0 + \varepsilon\overline{u}^1 + \cdots$$
$$\lambda^\varepsilon = \lambda^0 + \varepsilon\lambda^1 + \cdots$$

Note that $O(\varepsilon^{-2})$-term becomes $-\Delta\overline{u}^0 = 0$ in $(1/\varepsilon)W$ and $\overline{u}^0 = 0$ on ∂B_1.

Letting $\varepsilon \to 0$, we get

$$-\Delta u^0 = 0 \quad \text{in } \mathbb{R}^3 \setminus B_1 \quad \overline{u}^0 = 0 \quad \text{on } \partial B_1.$$

One can construct that $\overline{u}^0(y) = A + B/|y|$ is a solution to the problem and one can show that $B = -A$ since $\overline{u}^0(y) = 0$ on ∂B_1. Hence

$$\overline{u}^0(y) = A\left(1 - \frac{1}{|y|}\right).$$

To match the solution, we need

$$\lim_{|y|\to\infty} \overline{u}^0(y) = \lim_{x\to 0} u^0(x).$$

So

$$A = \lim_{x\to 0+} u^0(x) = u^0(0).$$

Hence

$$u^*(x) = u^0(x) + \overline{u}^0(y)\left(1 - \frac{\varepsilon}{|x|}\right) - u^0(0) = u^0(x) + \varepsilon\left(\frac{u^0(x)}{|x|}\right).$$

On the one hand, $u^\varepsilon(x) \approx u^*(x) = u^0(x) + \varepsilon\left(-u^0(x)/|x|\right)$. On the other hand, $u^\varepsilon(x) = u^0(x) + \varepsilon u^1(x)$. We guess that $u^1(x) = -u^0(0)/|x|$.

Now we try to find λ_1. We try to find λ_1 so that

$$-\Delta u^1 - \lambda_0 u^1 = \lambda_1 u^0 \quad \text{in } W \setminus \{0\} \quad \text{and} \quad u^1 = 0 \quad \text{on } \partial W.$$

Fix $\delta > 0$ and we work on $W_\delta = W \setminus B_\delta$. We multiply the above equation by u^0 and integrate it on W_δ. Then

$$-\int_{W_\delta} (\Delta u^1) u^0 \, dx - \lambda_0 \int_{W_\delta} u^1 u^0 \, dx = \lambda_1 \int_{W_\delta} u_0^2 \, dx.$$

Integrating by parts gives

$$\begin{aligned}
-\int_{W_\delta} (\Delta u^1) u^0 \, dx &= -\int_{\partial W} \frac{\partial u^1}{\partial \nu} u^0 \, d\sigma - \int_{\partial B_\delta} \frac{\partial u^1}{\partial \nu} u^0 \, d\sigma + \int_{W_\delta} \nabla u^1 \cdot \nabla u^0 \, dx \\
&= -\int_{\partial B_\delta} \frac{\partial u^1}{\partial \nu} u^0 \, d\sigma + \int_{\partial W_\delta} u^1 \left(\frac{\partial u^0}{\partial \nu}\right) d\sigma - \int_{W_\delta} u^1 (\Delta u^0) \, dx \\
&= -\int_{\partial B_\delta} \left(\frac{\partial u^1}{\partial \nu}\right) u^0 \, dx + \int_{\partial B_\delta} u^1 \left(\frac{\partial u^0}{\partial \nu}\right) dx \\
&\quad - \int_{W_\delta} u^1 (-\lambda_0 u^0) \, dx.
\end{aligned}$$

Hence

$$-\int_{\partial B_\delta}\left(\frac{\partial u^1}{\partial \nu}\right)u^0+\int_{\partial B_\delta}u^1\left(\frac{\partial u^0}{\partial \nu}\,d\sigma+\lambda_0\int_{W_\delta}u^0u^1-\lambda\right)\int_{W_\delta}u^0u^1\,dx=\lambda_1\int_{W_\delta}(u^0)^2\,dx.$$

Hence

$$\lambda_1\int_{W_\delta}(u^0)^2\,dx=-\int_{\partial B_\delta}\frac{\partial u^1}{\partial \nu}u^0+\int_{\partial B_\delta}u^1\frac{\partial u^0}{\partial \nu}.$$

The LHS becomes λ_1 as $\delta\to 0+$. One can show that

$$\int_{\partial B_\delta}u^1\frac{\partial u^0}{\partial \nu}\to 0$$

as $\delta\to 0$.

Note that

$$\frac{\partial u^1}{\partial \nu}=(Du^1)\cdot\nu.$$

On ∂B_δ, we note that

$$\nu=-\frac{x}{|x|}=-\frac{x}{\delta}.$$

So

$$Du^1=-u^0(0)\left(-\frac{1}{|x|^2}\right)\frac{x}{|x|}=\frac{u^0(0)x}{|x|^3}=\frac{u^0(0)}{\delta^3}x.$$

Hence

$$\frac{\partial u^1}{\partial \nu}=-\frac{u^0(0)}{\delta^2}.$$

So

$$-\int_{\partial B_\delta}\frac{\partial u^1}{\partial \nu}u^0\approx\frac{u^0(0)}{\delta^2}\int_{\partial B_\delta}u^0(0)=4\pi(u^0(0))^2$$

as $\delta\to 0+$. Therefore, $\lambda_1=4\pi(u^0(0))^2$.

Example 4.6 (Crushed ice problem)**.** Consider a glass of water W. In this region, put N ice cubes, modeled by balls of radius ε. Assume that $N=C_0/\varepsilon$. As smaller radius ε, the larger number of ice cubes. The number of balls in V is $\frac{1}{\varepsilon}\int_V\rho(x)\,dx$ for some density ρ.

Let $W^\varepsilon=W\setminus\bigcup_{i=1}^N B_\varepsilon(x_i)$. Consider

$$-\Delta u^\varepsilon=\lambda^\varepsilon u^\varepsilon\quad\text{in } W^\varepsilon\quad\text{and}\quad u^\varepsilon=0\quad\text{on }\partial W^\varepsilon.$$

Assume that $\lambda^\varepsilon\to\lambda^0$ and $u^\varepsilon\to u^0$ as $\varepsilon\to 0+$, where u^0 is smooth.

Let V be any subregion of W and let $V^\varepsilon=V\setminus\bigcup_{i=1}^N B_\varepsilon(x_i)$. Integrate $-\Delta u^\varepsilon=\lambda^\varepsilon u^\varepsilon$ over V^ε. Then we get

$$\int_V^\varepsilon-\Delta u^\varepsilon\,dx=\lambda_\varepsilon\int_{V^\varepsilon}u^\varepsilon.$$

It is easy to see that the RHS converges to $\lambda^0\int_V u^0$ as $\varepsilon\to 0+$.

On the other hand, then LHS gives

$$
\begin{aligned}
&-\int_{\partial V^\varepsilon}\left(\frac{\partial u^\varepsilon}{\partial \nu}\right) d\sigma \\
&= -\int_{\partial V}\frac{\partial u^\varepsilon}{\partial \nu} - \sum_{i=1}^{N}\int_{\partial B_\varepsilon(x_i)}\frac{\partial u^\varepsilon}{\partial \nu}.
\end{aligned}
$$

Since $u^\varepsilon \to u^0$, we see that

$$
-\int_{\partial V}\frac{\partial u^\varepsilon}{\partial \nu} \to -\int_V \Delta u^0\, dx.
$$

as $\varepsilon \to 0$.

Fix i. Note that

$$
u^\varepsilon(x) \approx u^0(x) - \varepsilon\left(\frac{u^0(0)}{|x|}\right).
$$

Here we assume that

$$
u^\varepsilon(x) \approx u^0(x) - \varepsilon\frac{u^0(x_i)}{|x-x_i|} \quad \text{on } B_\varepsilon(x_i).
$$

So

$$
\frac{\partial u^\varepsilon}{\partial \nu} \approx \frac{\partial u^0}{\partial \nu} - \varepsilon u^0(x_i)\frac{\partial}{\partial \nu}\left(\frac{1}{|x-x_i|}\right) = -\frac{\partial u^0}{\partial \nu} - \frac{1}{\varepsilon}u^0(x_i).
$$

Hence

$$
\int_{\partial B_\varepsilon(x_i)}\frac{\partial u^\varepsilon}{\partial \nu} \approx O(1)4\pi\varepsilon^2 - 4\pi\varepsilon u^0(x_i).
$$

Hence

$$
\sum_{i=1}^{N}\int_{\partial B_\varepsilon(x_i)}\frac{\partial u^\varepsilon}{\partial \nu} = -4\pi\varepsilon^2 N O(1) + 4\pi\varepsilon\sum_{i=1}^{N}u^0(x_i).
$$

Since $N = C_0/\varepsilon$, we have

$$
\sum_{i=1}^{N}\int_{\partial B_\varepsilon(x_i)}\frac{\partial u^\varepsilon}{\partial \nu} = -C_0 4\pi\varepsilon O(1) + 4\pi\varepsilon\sum_{i=1}^{N}u^0(x_i) \to 4\pi\int_V u^0(x)\rho(x)dx.
$$

Therefore, we get

$$
\lambda_0\int_V u^0\, dx = -\int_V \Delta u^0 + 4\pi\int_V \rho u^0\, dx,
$$

and hence

$$
-\Delta u^0 + 4\pi\rho u^0 - \lambda_0 u^0 = 0 \quad \text{in } V.
$$

In particular, if $\rho = c$, then

$$
-\Delta u^0 = (\lambda - 0 - 4\pi c)u^0 \quad \text{in } W, \quad u^0 = 0 \quad \text{on } \partial W.
$$

If we choose λ_0' as the principal eigenvalue for the Laplace operator, then $\lambda_0 = \lambda_0' + 4\pi c$, where c is an extra cooling factor.