APMA 0350 – FINAL EXAM – SOLUTIONS

1.

$$\begin{cases} y' = 4t\left(y^2+1\right) \\ y(0) = 1 \end{cases}$$

STEP 1:

$$\begin{aligned} \frac{dy}{dt} &= 4t\left(y^2+1\right) \\ \frac{dy}{y^2+1} &= 4tdt \\ \int \frac{dy}{y^2+1} &= \int 4tdt \\ \tan^{-1}(y) &= 2t^2 + C \\ y &= \tan\left(2t^2+C\right) \end{aligned}$$

STEP 2:

$$\begin{aligned} y(0) &= 1 \\ 1 &= \tan\left(2(0)^2+C\right) \\ \tan(C) &= 1 \\ C &= \frac{\pi}{4} \end{aligned}$$

STEP 3:

$$y = \tan\left(2t^2 + \frac{\pi}{4}\right)$$

Note: Technically it should be $C = \frac{\pi}{4} + \pi m$ but this will still give you the same value of y

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2.

$$\begin{cases} \left(\frac{2x}{y}+4\right)dx - \left(\frac{x^2}{y^2}\right)dy = 0 \\ y(2) = 1 \end{cases}$$

STEP 1: Check Exact

$$\left(\frac{2x}{y}+4\right)_y = -\frac{2x}{y^2}$$
$$\left(-\frac{x^2}{y^2}\right)_x = -\frac{2x}{y^2} \checkmark$$

STEP 2:

$$\langle P, Q\rangle = \nabla f = \langle f_x, f_y\rangle$$
$$f_x = P \Rightarrow f(x,y) = \int \frac{2x}{y} + 4dx = \frac{x^2}{y} + 4x + g(y)$$
$$f_y = Q \Rightarrow f(x,y) = \int -\frac{x^2}{y^2}dy = \frac{x^2}{y} + h(x)$$

$$f(x,y) = \frac{x^2}{y} + 4x$$

STEP 3: Solution:

$$\frac{x^2}{y} + 4x = C$$

STEP 4: Initial Condition:

$$y(2) = 1$$
$$\frac{2^2}{1} + 4(2) = C$$
$$C = 12$$

STEP 5: Explicit Form:

$$\frac{x^2}{y} + 4x = 12$$
$$\frac{x^2}{y} = 12 - 4x$$
$$y = \frac{x^2}{12-4x}$$

3.

$$\begin{cases} y'' = \lambda y \\ y'(0) = 0 \\ y'(4) = 0 \end{cases}$$

STEP 1: Auxiliary Equation: $r^2 = \lambda$

Case 1: $\lambda > 0$

Then $r^2 = \lambda = \omega^2$ and so $r = \pm\omega$

$$y = Ae^{\omega t} + Be^{-\omega t}$$
$$y' = A\omega e^{\omega t} - B\omega e^{-\omega t}$$
$$y'(0) = A\omega - B\omega = 0 \Rightarrow A\omega = B\omega \Rightarrow B = A$$
$$y = Ae^{\omega t} + Ae^{-\omega t}$$
$$y' = A\omega e^{\omega t} - A\omega e^{-\omega t}$$

$$\begin{aligned} y'(4) &= 0 \\ A\omega e^{4\omega} - A\omega e^{-4\omega} &= 0 \\ \cancel{A\omega} e^{4\omega} &= \cancel{A\omega} e^{-4\omega} \\ e^{4\omega} &= -e^{4\omega} \\ 4\omega &= -4\omega \\ \omega &= 0 \end{aligned}$$

Which is a contradiction $\Rightarrow\Leftarrow$

Case 2: $\lambda = 0$

Aux: $r^2 = 0 \Rightarrow r = 0$ (repeated twice)

$$y = A + Bt$$

$$y' = B$$
$$y'(0) = B = 0 \Rightarrow B = 0$$

$$y = A$$

But then automatically we have $y'(4) = 0$ so this is valid

Hence $\lambda = 0$ is an eigenvalue and $y = A$ an eigenfunction

Case 3: $\lambda < 0$

In this case $\lambda = -\omega^2$ where $\omega > 0$

Aux: $r^2 = \lambda = -\omega^2 \Rightarrow r = \pm \omega i$

$$y = A\cos(\omega t) + B\sin(\omega t)$$

$$y' = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

$$y'(0) = -A\omega\sin(0) + B\omega\cos(0) = B\omega = 0 \Rightarrow B = 0$$

$$y = A\cos(\omega t)$$

$$y' = -A\omega\sin(\omega t)$$

$$\begin{aligned} y'(4) &= 0 \\ \cancel{-A\omega}\sin(4\omega) &= 0 \\ \sin(4\omega) &= 0 \\ 4\omega &= \pi m \\ \omega &= \left(\frac{\pi}{4}\right)m \qquad m = 1, 2, \cdots \end{aligned}$$

STEP 2: Answer:

Eigenvalues:

$$\lambda = -\omega^2 = -\left(\frac{\pi}{4}m\right)^2 \qquad m = 0, 1, 2, \cdots$$

Eigenfunctions:

$$y = \cos(\omega t) = \cos\left(\frac{\pi}{4}mt\right) \qquad m = 0, 1, 2\cdots$$

Notice this includes Case 2 if we start with $m = 0$

4. **STEP 1: Homogeneous Solution:** $y'' + 4y = 0$

$$\textbf{Aux: } r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm 2i$$

$$y_0 = A\cos(2t) + B\sin(2t)$$

STEP 2: Particular Solution:

No resonance because the right-hand-side corresponds to $r = 0$ which doesn't coincide with the homogeneous root $r = \pm 2i$

$$y_p = At^2 + Bt + C$$

$$\begin{aligned} (y_p)'' + 4(y_p) &= 12t^2 + 20t + 30 \\ \left(At^2 + Bt + C\right)'' + 4\left(At^2 + Bt + C\right) &= 12t^2 + 20t + 30 \\ 2A + 4At^2 + 4Bt + 4C &= 12t^2 + 20t + 30 \\ 4At^2 + 4Bt + (2A + 4C) &= 12t^2 + 20t + 30 \end{aligned}$$

Comparing the coefficients, we get

$$\begin{cases} 4A &= 12 \\ 4B &= 20 \\ 2A + 4C &= 30 \end{cases}$$

Hence $A = 3$ and $B = 5$ and the third gives

$$2(3) + 4C = 30 \Rightarrow 4C = 30 - 6 = 24 \Rightarrow C = 6$$

$$y_p = 3t^2 + 5t + 6$$

STEP 3: General Solution:

$$y = y_0 + y_p = A\cos(2t) + B\sin(2t) + 3t^2 + 5t + 6$$

STEP 4: Initial Condition:

$$\begin{aligned} y(0) &= 8 \\ A\cos(0) + B\sin(0) + 6 &= 8 \\ A + 6 &= 8 \\ A &= 2 \end{aligned}$$

$$
\begin{aligned}
y'(t) = -2A\sin(2t) + 2B\cos(2t) + 6t + 5 \\
y'(0) &= -3 \\
-2A\sin(0) + 2B\cos(0) + 5 &= -3 \\
2B &= -8 \\
B &= -4
\end{aligned}
$$

STEP 5: Answer:

$$y = 2\cos(2t) - 4\sin(2t) + 3t^2 + 5t + 6$$

5.

$$\begin{cases} y'' + 2y' + 2y = f(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \quad \text{where } f(t) = \begin{cases} 0 & \text{if } 0 \leq t < \pi \\ 2 & \text{if } \pi \leq t < 2\pi \\ 0 & \text{if } t \geq 2\pi \end{cases}$$

STEP 1: Write f in terms of u_c

Start at 0 then jump by $2 - 0 = 2$ at $t = \pi$ then jump by $0 - 2 = -2$ at $t = 2\pi$ and so

$$f(t) = 2u_{\pi}(t) - 2u_{2\pi}(t)$$

$$\mathcal{L}\{f(t)\} = \frac{2e^{-\pi s}}{s} - \frac{2e^{-2\pi s}}{s} = \left(\frac{2}{s}\right)\left(e^{-\pi s} - e^{-2\pi s}\right)$$

STEP 2: Take Laplace transforms

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \left(\frac{2}{s}\right)\left(e^{-\pi s} - e^{-2\pi s}\right)$$

$$\left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) + 2\left(s\mathcal{L}\{y\} - y(0)\right) + 2\mathcal{L}\{y\} = \left(\frac{2}{s}\right)\left(e^{-\pi s} - e^{-2\pi s}\right)$$

$$\left(s^2 + 2s + 2\right)\mathcal{L}\{y\} = \left(\frac{2}{s}\right)\left(e^{-\pi s} - e^{-2\pi s}\right)$$

$$\mathcal{L}\{y\} = \left(\frac{1}{s^2 + 2s + 2}\right)\left(\frac{2}{s}\right)\left(e^{-\pi s} - e^{-2\pi s}\right)$$

STEP 3: Partial Fractions

$$\begin{aligned} \left(\frac{2}{s}\right)\left(\frac{1}{s^2 + 2s + 2}\right) &= \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2} \\ &= \frac{A\left(s^2 + 2s + 2\right) + (Bs + C)s}{s\left(s^2 + 2s + 2\right)} \\ &= \frac{(A + B)s^2 + (2A + C)s + 2A}{s\left(s^2 + 2s + 2\right)} \end{aligned}$$

$$\begin{cases} A + B = 0 \\ 2A + C = 0 \\ 2A = 2 \end{cases}$$

$$\begin{cases} A = 1 \\ B = -A = -1 \\ C = -2A = -2 \end{cases}$$

$$\left(\frac{2}{s}\right)\left(\frac{1}{s^2+2s+2}\right) = \frac{1}{s} - \frac{s+2}{s^2+2s+2}$$

STEP 4:

$$\frac{1}{s} = \mathcal{L}\left\{1\right\}$$

$$\frac{s+2}{s^2+2s+2} = \frac{s+2}{(s+1)^2+1} = \frac{(s+1)+1}{(s+1)^2+1}$$

This is a shifted version by -1 units of

$$\frac{s+1}{s^2+1} = \mathcal{L}\left\{\cos(t)+\sin(t)\right\}$$

$$\text{Hence } \frac{s+2}{s^2+2s+2} = \mathcal{L}\left\{e^{-t}\left(\cos(t)+\sin(t)\right)\right\}$$

$$\frac{1}{s} - \frac{s+2}{s^2+2s+2} = \mathcal{L}\left\{1 - e^{-t}\left(\cos(t)+\sin(t)\right)\right\}$$

STEP 4: Let $h(t) = 1 - e^{-t}\left(\cos(t)+\sin(t)\right)$

$$\begin{aligned} \mathcal{L}\left\{y\right\} =& \mathcal{L}\left\{h(t)\right\}\left(e^{-\pi s} - e^{-2\pi s}\right) \\ \mathcal{L}\left\{y\right\} =& \mathcal{L}\left\{h(t-\pi)u_\pi(t) - h(t-2\pi)u_{2\pi}(t)\right\} \end{aligned}$$

STEP 5: Answer:

$$y = h(t-\pi)u_\pi(t) - h(t-2\pi)u_{2\pi}(t)$$

$$\text{Where } h(t) = 1 - e^{-t}\left(\cos(t)+\sin(t)\right)$$

6.
$$\phi(t) + \int_0^t (t-\tau)\,\phi(\tau)d\tau = 1$$

This is of the form

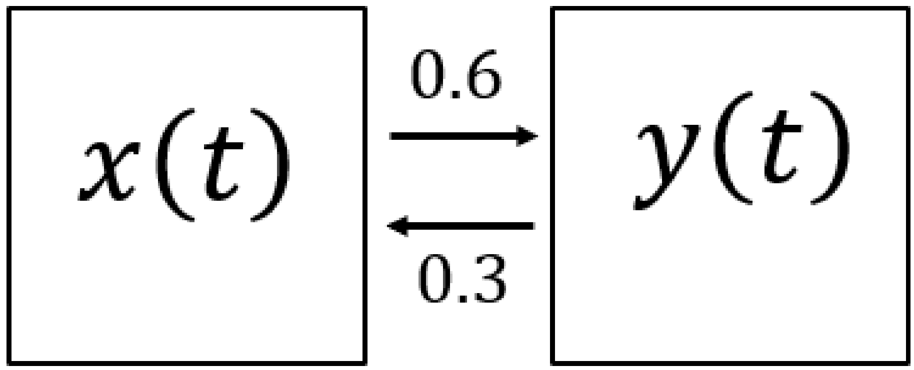
$$\phi + (t \star \phi) = 1$$

Take Laplace transforms

$$\begin{aligned}
\mathcal{L}\{\phi\} + \mathcal{L}\{t \star \phi\} =& \mathcal{L}\{1\} \\
\mathcal{L}\{\phi\} + \mathcal{L}\{t\}\,\mathcal{L}\{\phi\} =& \frac{1}{s} \\
\mathcal{L}\{\phi\} + \left(\frac{1}{s^2}\right)\mathcal{L}\{\phi\} =& \frac{1}{s} \\
\mathcal{L}\{\phi\}\left(1 + \frac{1}{s^2}\right) =& \frac{1}{s} \\
\mathcal{L}\{\phi\}\left(\frac{s^2+1}{s^2}\right) =& \frac{1}{s} \\
\mathcal{L}\{\phi\} =& \left(\frac{s^2}{s^2+1}\right)\frac{1}{s} \\
\mathcal{L}\{\phi\} =& \frac{s}{s^2+1} \\
\mathcal{L}\{\phi\} =& \mathcal{L}\{\cos(t)\}
\end{aligned}$$

$$\phi(t) = \cos(t)$$

7. (a)



$$x'(t) = \text{In} \;-\; \text{Out} \;= (0.3)y(t) - (0.6)x(t) = -(0.6)x(t) + (0.3)y(t)$$
$$y'(t) = \text{In} \;-\; \text{Out} \;= (0.6)x(t) - (0.3)y(t)$$

$$A = \begin{bmatrix} -0.6 & 0.3 \\ 0.6 & -0.3 \end{bmatrix}$$

(b)

$$\begin{aligned}(x(t) + y(t))' &= x'(t) + y'(t) \\ &= -\cancel{(0.6)x(t)} + \cancel{(0.3)y(t)} + \cancel{(0.6)x(t)} - \cancel{(0.3)y(t)} \\ &= 0\end{aligned}$$

Therefore the total population $x(t) + y(t)$ is constant

8.

$$\mathbf{x}' = A\mathbf{x} \qquad A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \qquad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

STEP 1: Eigenvalues

$$\begin{aligned} |A - \lambda I| = & \begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} \\ =& (5-\lambda)(1-\lambda) - (-1)(3) \\ =& 5 - 5\lambda - \lambda + \lambda^2 + 3 \\ =& \lambda^2 - 6\lambda + 8 \\ =& (\lambda-2)(\lambda-4) = 0 \end{aligned}$$

Which gives $\lambda = 2$ or $\lambda = 4$

STEP 2: $\lambda = 2$

$$\text{Nul }(A - 2I) = \left[\begin{array}{cc|c} 5-2 & -1 & 0 \\ 3 & 1-2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $3x - y = 0 \Rightarrow y = 3x$

$$\mathbf{v} = \begin{bmatrix} x \\ 3x \end{bmatrix} = x \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

STEP 3: $\lambda = 4$

$$\text{Nul }(A - 4I) = \left[\begin{array}{cc|c} 5-4 & -1 & 0 \\ 3 & 1-4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $x - y = 0 \Rightarrow y = x$

$$\mathbf{v} = \begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A = PDP^{-1} \text{ where } D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$$

STEP 4:

$$
\begin{aligned}
e^{At} =& P e^{Dt} P^{-1} \\
=& \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}^{-1} \\
=& \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix} \left(-\frac{1}{2}\right) \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix} \\
=& \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 3/2 & -1/2 \end{bmatrix} \\
=& \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -e^{2t}/2 & e^{2t}/2 \\ 3e^{4t}/2 & -e^{4t}/2 \end{bmatrix} \\
=& \begin{bmatrix} -e^{2t}/2 + 3e^{4t}/2 & e^{2t}/2 - e^{4t}/2 \\ -3e^{2t}/2 + 3e^{4t}/2 & 3e^{2t}/2 - e^{4t}/2 \end{bmatrix}
\end{aligned}
$$

STEP 5: General solution

$$
\mathbf{x}(t) = C_1 \begin{bmatrix} -e^{2t}/2 + 3e^{4t}/2 \\ -3e^{2t}/2 + 3e^{4t}/2 \end{bmatrix} + C_2 \begin{bmatrix} e^{2t}/2 - e^{4t}/2 \\ 3e^{2t}/2 - e^{4t}/2 \end{bmatrix}
$$

STEP 6: Initial Condition

$$
\mathbf{x}(0) = \begin{bmatrix} 4 \\ 8 \end{bmatrix}
$$

This gives $C_1 = 4$ and $C_2 = 8$ and so

$$
\mathbf{x}(t) = 4 \begin{bmatrix} -e^{2t}/2 + 3e^{4t}/2 \\ -3e^{2t}/2 + 3e^{4t}/2 \end{bmatrix} + 8 \begin{bmatrix} e^{2t}/2 - e^{4t}/2 \\ 3e^{2t}/2 - e^{4t}/2 \end{bmatrix} = \begin{bmatrix} -2e^{2t} + 6e^{4t} \\ -6e^{2t} + 6e^{4t} \end{bmatrix} + \begin{bmatrix} 4e^{2t} - 4e^{4t} \\ 12e^{2t} - 4e^{4t} \end{bmatrix}
$$

$$
\mathbf{x}(t) = \begin{bmatrix} (-2+4)e^{2t} + (6-4)e^{4t} \\ (-6+12)e^{2t} + (6-4)e^{4t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} + 2e^{4t} \\ 6e^{2t} + 2e^{4t} \end{bmatrix}
$$

$$
\mathbf{x}(t) = e^{2t} \begin{bmatrix} 2 \\ 6 \end{bmatrix} + e^{4t} \begin{bmatrix} 2 \\ 2 \end{bmatrix}
$$

9.

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \mathbf{f} = \begin{bmatrix} 6e^{4t} \\ 4e^{3t} \end{bmatrix}$$

STEP 1: Eigenvalues

$$\left|A - \lambda I\right| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0$$

Which gives $\lambda - 2 = 1$ or $\lambda - 2 = -1$ and so $\lambda = 3$ or $\lambda = 1$

STEP 2: $\lambda = 1$

$$\text{Nul } (A - 1I) = \left[\begin{array}{cc|c} 2-1 & 1 & 0 \\ 1 & 2-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $x + y = 0 \Rightarrow y = -x$

$$\mathbf{v} = \begin{bmatrix} x \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

STEP 3: $\lambda = 3$

$$\text{Nul } (A - 3I) = \left[\begin{array}{cc|c} 2-3 & 1 & 0 \\ 1 & 2-3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $x - y = 0 \Rightarrow y = x$

$$\mathbf{v} = \begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

STEP 4: Homogeneous Solution

$$\mathbf{x_0}(t) = C_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} e^t \\ -e^t \end{bmatrix} + C_2 \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$

STEP 5: Variation of Parameters

$$\mathbf{x_p}(t) = u(t) \begin{bmatrix} e^t \\ -e^t \end{bmatrix} + v(t) \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$

$$\begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 6e^{4t} \\ 4e^{3t} \end{bmatrix}$$

Denominator:

$$\begin{vmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{vmatrix} = e^{4t} + e^{4t} = 2e^{4t}$$

$$u'(t) = \frac{\begin{vmatrix} 6e^{4t} & e^{3t} \\ 4e^{3t} & e^{3t} \end{vmatrix}}{2e^{4t}} = \frac{6e^{7t} - 4e^{6t}}{2e^{4t}} = 3e^{3t} - 2e^{2t}$$

$$u(t) = \int 3e^{3t} - 2e^{2t} dt = e^{3t} - e^{2t}$$

$$v'(t) = \frac{\begin{vmatrix} e^t & 6e^{4t} \\ -e^t & 4e^{3t} \end{vmatrix}}{2e^{4t}} = \frac{4e^{4t} + 6e^{5t}}{2e^{4t}} = 2 + 3e^t$$

$$v(t) = \int 2 + 3e^t dt = 2t + 3e^t$$

$$\begin{aligned} \mathbf{x_p}(t) &= \left(e^{3t} - e^{2t}\right) \begin{bmatrix} e^t \\ -e^t \end{bmatrix} + \left(2t + 3e^t\right) \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \\ &= \begin{bmatrix} e^{4t} \\ -e^{4t} \end{bmatrix} + \begin{bmatrix} -e^{3t} \\ e^{3t} \end{bmatrix} + \begin{bmatrix} 2t\, e^{3t} \\ 2t\, e^{3t} \end{bmatrix} + \begin{bmatrix} 3e^{4t} \\ 3e^{4t} \end{bmatrix} \\ &= \begin{bmatrix} e^{4t} + 3e^{4t} \\ -e^{4t} + 3e^{4t} \end{bmatrix} + \begin{bmatrix} -e^{3t} + 2te^{3t} \\ e^{3t} + 2te^{3t} \end{bmatrix} \\ &= e^{4t} \begin{bmatrix} 4 \\ 2 \end{bmatrix} + e^{3t} \begin{bmatrix} 2t - 1 \\ 2t + 1 \end{bmatrix} \end{aligned}$$

STEP 6: Answer

$$\mathbf{x_p}(t) = e^{4t} \begin{bmatrix} 4 \\ 2 \end{bmatrix} + e^{3t} \begin{bmatrix} 2t - 1 \\ 2t + 1 \end{bmatrix}$$

10.

$$\begin{cases} x' = (y-x)(1-x-y) \\ y' = x(2+y) \end{cases}$$

STEP 1: Equilibrium Points

$$\begin{cases} x' = (y-x)(1-x-y) = 0 \\ y' = x(2+y) = 0 \end{cases}$$

Here it's a bit easier to deal with the second equation first.

Case 1: $x = 0$

Then either $y - x = 0 \Rightarrow y - 0 = 0 \Rightarrow y = 0 \rightsquigarrow (0,0)$

Or $1 - x - y = 0 \Rightarrow 1 - 0 - y = 0 \Rightarrow y = 1 \rightsquigarrow (0,1)$

Case 2: $2 + y = 0 \Rightarrow y = -2$

Either $y - x = 0 \Rightarrow (-2) - x = 0 \Rightarrow x = -2 \rightsquigarrow (-2,-2)$ or

$1-x-y = 0 \Rightarrow 1-x-(-2) = 0 \Rightarrow 3-x = 0 \Rightarrow x = 3 \rightsquigarrow (3,-2)$

Equilibrium Points: $(0,0), (0,1), (-2,-2), (3,-2)$

STEP 2: Classification

Totally fine to use the product rule to calculate the partial derivatives, but it might be easier to expand everything out first

$$\begin{cases} x' = (y-x)(1-x-y) = y - \cancel{xy} - y^2 - x + x^2 + \cancel{xy} = -x + y + x^2 - y^2 \\ y' = x(2+y) = 2x + xy \end{cases}$$

$$\nabla F(x,y) = \begin{bmatrix} 2x-1 & 1-2y \\ 2+y & x \end{bmatrix}$$

Case 1: $(0,0)$

$$\nabla F(0,0) = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$$

Eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} \\ &= (-1-\lambda)(-\lambda) - (1)(2) \\ &= \lambda^2 + \lambda - 2 \\ &= (\lambda+2)(\lambda-1) = 0 \end{aligned}$$

$\lambda = -2 < 0$ and $\lambda = 1 > 0$ so $(0,0)$ is a **saddle**

Case 2: $(0,1)$

$$\nabla F(0,1) = \begin{bmatrix} -1 & 1-2 \\ 2+1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}$$

Eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -1-\lambda & -1 \\ 3 & -\lambda \end{vmatrix} \\ &= (-1-\lambda)(-\lambda) - (-1)(3) \\ &= \lambda^2 + \lambda + 3 = 0 \end{aligned}$$

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4(3)}}{2} = \frac{-1 \pm \sqrt{-11}}{2} = -\frac{1}{2} \pm \frac{\sqrt{11}}{2} i$$

Because $-\frac{1}{2} < 0$ $(0,1)$ is **stable**

Case 3: $(-2,-2)$

$$\nabla F(-2,-2) = \begin{bmatrix} 2(-2)-1 & 1-2(-2) \\ 2-2 & -2 \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ 0 & -2 \end{bmatrix}$$

$\lambda = -5 < 0$ and $\lambda = -2 < 0$ so $(-2,-2)$ is **stable**

Case 4: $(3,-2)$

$$\nabla F(3,-2) = \begin{bmatrix} 2(3)-1 & 1-2(-2) \\ 2-2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 0 & 3 \end{bmatrix}$$

$\lambda = 5 > 0$ and $\lambda = 3 > 0$ so $(3, -2)$ is **unstable**

STEP 3: Answer:

$$\begin{aligned} (0,0) &\text{ saddle} \\ (0,1) &\text{ stable} \\ (-2,-2) &\text{ stable} \\ (3,-2) &\text{ unstable} \end{aligned}$$