APMA 0350 – HOMEWORK 10 – SOLUTIONS

Problem 1: Setting x' and y' equal to 0, we find

$$x' = 0 \Rightarrow 0 = y - x^2y = y(1-x^2) \Rightarrow \text{ either } y = 0 \text{ or } x = \pm 1$$
$$y' = 0 \Rightarrow 0 = y^2x - 4x^2 \Rightarrow y^2x = 4x^2 \Rightarrow \text{ either } x = 0 \text{ or } y = \pm 2\sqrt{x}$$

Thus, the points that work are $(0,0),(1,2),(1,-2)$

Problem 2: Find and classify the equilibrium points of

$$\begin{cases} x' = x - y + x^2 \\ y' = x + y \end{cases}$$

Setting these both to zero, we see that

$$0 = x - y + x^2 \tag{1}$$

$$0 = x + y \tag{2}$$

From (2), we see that any points will satisfy $x = -y$. Solving (1) gives $-y - y + y^2 = 0$. Thus, $y = 0$ or $y = 2$. Finally, we have $(0,0)$ and $(-2,2)$. We are left to classify the points. We have the linearzation

$$\nabla F(x,y) = \begin{bmatrix} \frac{\partial(x')}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial(y')}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}(x - y + x^2) & \frac{\partial}{\partial y}(x - y + x^2) \\ \frac{\partial}{\partial x}(x+y) & \frac{\partial}{\partial y}(x+y) \end{bmatrix} = \begin{bmatrix} 1+2x & -1 \\ 1 & 1 \end{bmatrix}$$

For $(0,0)$, we thus have

$$A = \nabla F(0,0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

The eigenvalues of A are $1 \pm i$

Since the real part is positive, $(0,0)$ is unstable.

For $(-2,2)$, we thus have

$$A = \nabla F(-2,2) = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

The eigenvalues are $-1-\sqrt{3} < 0$ and $-1+\sqrt{3} > 0$. Because of the mixed sign, we see this is a saddle.

Problem 3: Setting both equations equal to 0 gives us:

$$0 = x(3 - x - 2y) \tag{3}$$

$$0 = y(2 - x - y) \tag{4}$$

From equation $(3): x(3-x-2y) = 0 \implies x = 0$ or $x = 3 - 2y$

From equation $(2): y(2-x-y) = 0 \implies y = 0$ or $y = 2 - x$

This gives us $(0,0), (0,2), (3,0), (1,1)$

Now, we must classify the points. The linearization gives us

$$\nabla F(x,y) = \begin{bmatrix} \frac{\partial(x')}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial(y')}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}(3x - x^2 - 2yx) & \frac{\partial}{\partial y}(3x - x^2 - 2yx) \\ \frac{\partial}{\partial x}(2y - yx - y^2) & \frac{\partial}{\partial y}(2y - yx - y^2) \end{bmatrix}$$

$$\nabla F(x,y) = \begin{bmatrix} 3-3x-2y & -2x \\ -y & 2-x-2y \end{bmatrix}$$

For $(0,0)$, we find $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ This has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$, thus it is unstable

For $(0,2)$, we find $\begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}$ This has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -1$, thus it is stable

For $(3,0)$ we find $\begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}$ This has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -1$ which are stable

For $(1,1)$ we find $\begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$
This has eigenvalues $\lambda_1 = -1-\sqrt{2}$ and $\lambda_2 = -1+\sqrt{2}$. Thus, the eigenvalues are positive and negative, so it is a saddle point.

Problem 4:

(a) We find the system

$$\begin{cases} x'(t) = -0.4x(t) + 0.2y(t) \\ y'(t) = 0.4x(t) - 0.2y(t) \end{cases}$$

(b)

$$\begin{cases} x'(t) = -0.4x(t) + 0.2y(t) = 0 \\ y'(t) = 0.4x(t) - 0.2y(t) = 0 \end{cases}$$

$$\implies 0.2y(t) = 0.4x(t)$$

$$\implies y(t) = 2x(t)$$

Thus, we've found the equilibrium point $(x, 2x)$

We classify the stability:

$$\nabla F(x,y) = \begin{bmatrix} -0.4 & 0.2 \\ 0.4 & -0.2 \end{bmatrix}$$

Taking

$$\begin{aligned} \left|\nabla F(x,2x) - \lambda I\right| =& (-0,4-\lambda)(-0.2-\lambda) - (0.4)(0.2) \\ =& \lambda^2 + 0.6\lambda \\ =& \lambda(\lambda + 0.6) = 0 \end{aligned}$$

Thus, $\lambda = 0$ and $\lambda = -0.6$

Since one of the eigenvalues is 0, the equilibrium points $(x, 2x)$ are "neither stable nor unstable"

Note: Here we also accept the answer "stable"