A SECOND-ORDER BOUND-PRESERVING EXPONENTIAL SCHEME FOR DEGENERATE PARABOLIC EQUATIONS *

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Abstract. This paper proposes a second-order numerical method for solving nonlinear parabolic equations with degenerate mobility. The intrinsic degenerate mobility in the equation yields a globally bounded solution. A pivotal feature of our methodology is an appropriate reformulation of the equation into an equivalent form. After applying a discontinuous Galerkin spatial discretization method, we derive a fully nonlinear ordinary differential equation (ODE) with a splitting structure. By introducing a linear term into the ODE, an exponential temporal discretization method, which involves only linear solvers, is proposed based on integrating factors and strong stability preserving (SSP) Runge-Kutta methods. Our approach is proven to exhibit second-order accuracy, ensures bound preservation and mass conservation, and demonstrates a favorable CFL condition $\tau \sim h$, where τ and h are the temporal and spatial mesh sizes respectively. Comprehensive numerical tests validate the second-order accuracy and bound-preserving behaviors of our method.

Key words. bound preservation, discontinuous Galerkin method, exponential scheme, SSP Runge-Kutta, degenerate parabolic equations

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1. Introduction. In this paper, we focus on designing numerical schemes for solving a class of degenerate parabolic equations of the form

$$\begin{cases} \rho_t = \nabla f(\rho) \nabla \left(H'(\rho) + V(\boldsymbol{x}) + W * \rho \right), & \boldsymbol{x} \in \Omega, \quad t > 0, \\ \rho(\boldsymbol{x}, 0) = \rho_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^d$ with $d = 1$ or 2, and $\rho(\boldsymbol{x}, t) : \Omega \times [0, +\infty) \to \mathbb{R}$ is the unknown particle density; $f(\rho)$ is a nonlinear degenerate mobility function that may cause solutions to exhibit boundedness or non-smoothness; the $H(\rho)$, $V(\boldsymbol{x})$, and $W(\boldsymbol{x})$ are given functions with various meanings depending on the specific context [8, 30]. In this work, we assume that $H(\rho)$ is convex and $W(\boldsymbol{x}) = W(-\boldsymbol{x})$. The equation (1.1) can be written as a gradient flow

$$\rho_t = \nabla f(\rho) \nabla \left(\frac{\delta E(\rho)}{\delta \rho} \right)$$

with respect to the free energy functional

$$E(\rho) = \int_\Omega H(\rho) + V\rho + \frac{1}{2}(W * \rho)\rho \, d\boldsymbol{x}.$$

The presence of the degenerate mobility $f(\rho)$ introduces a complex layer of intricacy to the equation. Unlike its constant case, this type of mobility exerts a substantial influence over the global bound and continuity of solutions (see Section 4). Consider the mobility function $f(\rho) = \rho(1 - \rho)$. It remains positive within the interval (0,1) and becomes degenerate at the endpoints 0 and 1. This characteristic ensures that

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the solution to equation (1.1) with this specific mobility always stays confined within the range $[0, 1]$ (such as [16]). For our subsequent analysis, we make the assumption, without loss of generality, that $f(\rho) \geq 0$ for all $\rho \in [0, 1]$ and the exact solution ρ of equation (1.1) adheres to the condition that

$$\rho(\boldsymbol{x}, t) \in [0, 1] \text{ for all } (\boldsymbol{x}, t) \in \Omega \times [0, +\infty). \tag{1.2}$$

Given this framework, the importance of a numerical scheme ensuring bound preservation becomes evident when attempting to resolve (1.1).

There are extensive bound-preserving and high-order spatial numerical methods for equation (1.1), such as the finite-difference method [26], finite-volume method [2, 3], discontinuous Galerkin method [23, 29] and references therein. For temporal discretization of (1.1), the implicit methods are commonly used to maintain stability and preserve physical properties [2, 9, 27]. However, implicit bound-preserving schemes are usually limited to first-order in time and necessitate the solution of a large non-linear algebraic system at each time-step. Explicit temporal discretizations are computationally efficient at each iteration, but a parabolic CFL condition $\tau \sim h^2$, where τ and h are temporal and spatial mesh sizes respectively, is normally required for explicit schemes [29]. To strike a balance between efficiency and stability, implicit-explicit (IMEX) methods are frequently utilized when the equation exhibits a splitting structure. The work [6] rewrote the model (1.1) as a splitting form

$$\rho_t = \Delta\Phi(\rho) + \nabla f(\rho)\nabla \left(V(\boldsymbol{x}) + W * \rho\right) = \mathcal{C}(\rho) + \mathcal{D}(\rho) \tag{1.3}$$

with $\Phi(\rho) = \int_0^\rho f(s)H''(s)\, ds$, and they dealt implicitly with $\mathcal{C}(\rho)$ and explicitly with $\mathcal{D}(\rho)$ following the idea as in [5]. However, nonlinear systems were still required to be solved using this splitting, and the preservation of positivity was only proven for the first-order temporal scheme. This issue of limited accuracy is pervasive in the development of bound-preserving schemes using classical IMEX-RK schemes, owing to the infeasibility of implicit SSP-RK schemes with order higher than one [18]. Recently, the IMEX-RK incorporating multi-derivatives [11, 17, 22] has offered the potential to obtain high-order bound-preserving schemes. However, the assumption of high-order derivatives is not easily applicable to the equation (1.1). Even if (1.1) is simplified to the heat equation $\rho = \Delta\rho$, the second-order derivative Δ^2 no longer satisfies the maximum bound principle. Furthermore, a variety of novel techniques have been developed for bound preservation, including Lagrange multiplier approaches [10, 31] and the energy variational approach [14]. These approaches primarily confront challenges in solving fully nonlinear implicit schemes or conducting rigorous error analysis for general models.

An alternative prevalent way to explore high-order bound-preserving schemes is by employing the exponential integrator (see [19] for a review). To the best of our knowledge, no previous work has studied an exponential bound-preserving scheme for (1.1), but numerous research studies have explored similar models from the insights of exponential integrators. In contrast to (1.3), we reformulate (1.1) with a distinct splitting structure

$$\rho_t = \nabla F(\rho)\nabla\rho + \nabla f(\rho)\nabla \left(V(\boldsymbol{x}) + W * \rho\right) = \mathcal{L}(\rho)\rho + \mathcal{N}(\rho), \tag{1.4}$$

where $F(\rho) = f(\rho)H^{''}(\rho) \geq 0$. By introducing a term $L\rho$ as an approximation of $\mathcal{L}(\rho)\rho$, the equation (1.4) can be further written as

$$\left(e^{\mathcal{T}(t)}\rho\right)_t = e^{\mathcal{T}(t)}\left((\mathcal{L}(\rho) - L)\rho + \mathcal{N}(\rho)\right). \tag{1.5}$$

where $\mathcal{T}(t) = \int_0^t L(s)\, ds$ is the integrating factor associated with L. The work [23] took $L = \mu I$ with a constant $\mu > 0$, which makes its computation similar to that of explicit methods. However, their scheme required the parabolic CFL condition $\tau \sim h^2$ analogous to explicit methods when attempting to apply it to our equation (1.4). In cases where $\mathcal{L}(\rho)$ is independent of ρ, several exponential linear schemes employing $L = \mathcal{L}$ have been effectively utilized for (1.4) to generate bounded numerical solutions [13, 24, 25]. In these works, the operator $\mathcal{L}$ possesses a specific special structure, resulting in a symmetry of either Toeplitz matrices or circulant matrices. Consequently, FFT-based algorithms are adequate to achieve enhanced efficiency. Moreover, the very recent study [7] also implemented a fixed constant approximation to $\mathcal{L}$ that depends only on the spatial vector $\boldsymbol{x}$, although the issue of bound-preservation has not been addressed in their context. For $\mathcal{L}(\rho)$ depending on ρ, the work [21] introduced a second-order exponential scheme that incorporates $L = \mathcal{L}(\rho)$. This scheme needed three to four evaluations of the fully nonlinear equation $\rho_t = L(\rho)\rho$ per iteration. When applied to our model (1.4), the computational burden generally resembles that of implicit methods, despite their provision of detailed techniques for distinct stiff kinetic equations.

Above all, it remains a significant challenge to compute a bound-preserving solution to (1.1) while achieving both high-order accuracy and efficiency. In this paper, we adopt the splitting (1.4) and present a second-order scheme using an exponential integrator associated with $L = \mathcal{L}(\rho^*)$, where ρ^* is explicitly given. At each iteration, our scheme only requires three times of computation for the linear equation $\rho = L(\rho^*)\rho$, which can be efficiently evaluated by numerous existing algorithms (see the review [28]). As evidenced in prior studies, the Krylov- or Leja-based methodologies often surpass implicit schemes in many practical applications [12, 15]. Importantly, we demonstrate that our method is mass-conservative, bound-preserving and benefits from a favorable CFL condition $\tau \sim h$.

2. Spatial discretization. In this section, we present a spatial discretization method for parabolic equations (1.1) using a discontinuous Galerkin (DG) approach. We only consider the one-dimensional case ($d = 1$) as an demonstration and the two-dimensional case ($d = 2$) can be derived in the same way (see [29, Section 3]). In our following statement, we denote $x = \boldsymbol{x}$ as a spatial variable in one-dimensional space.

Let $I_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ and $I = \cup_{i=1}^N I_i$ be a partition of the domain Ω. For simplicity, we consider uniform meshes $h = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$, but this assumption is not essential. The discontinuous piecewise polynomial space is defined as

$$V_h = \{v_h : v_h|_{I_i} \in P^k(I_i), i = 1, 2 \cdots, N\},$$

where $P^k(I_i)$ is the space of k-th order polynomial. We use the notation v_h^+ and v_h^- as the right and left limit of $v_h \in V_h$ respectively. For a function $s = s(\rho)$ or $s = s(\rho, x)$, we denote by $s_h = s(\rho_h)$ or $s_h = s(\rho_h, x)$ respectively. Furthermore, the notation s_h^+ stands for $s(\rho_h^+)$ or $s(\rho_h^+, x)$, and s_h^- stands for $s(\rho_h^-)$ or $s(\rho_h^-, x)$. For the purpose of this paper, we only consider the second-order DG scheme ($k = 1$) and choose

$$\phi_i^{(1)} = \frac{x_{i+\frac{1}{2}} - x}{h}, \quad \phi_i^{(2)} = \frac{x - x_{i-\frac{1}{2}}}{h}, \tag{2.1}$$

as the basis for $P^1(I_i)$. Note that $x_{i\pm\frac{1}{2}}$ are the $k+1$ Gauss–Lobatto quadrature points

on I_i when $k = 1$. The Gauss–Lobatto quadrature on $x_{i\pm\frac{1}{2}}$ can be defined by

$$\widetilde{\int}_{I_i} \eta\zeta \, dx = \frac{h}{2}\left((\eta\zeta)^+_{i-\frac{1}{2}} + (\eta\zeta)^-_{i+\frac{1}{2}}\right)$$

and

$$\widetilde{\int}_{I_i} \eta\partial_x\zeta \, dx = \frac{h}{2}\left((\eta\partial_x(\mathcal{I}\zeta))^+_{i-\frac{1}{2}} + (\eta\partial_x(\mathcal{I}\zeta))^-_{i+\frac{1}{2}}\right),$$

where the operator $\mathcal{I}$ returns the first-order polynomial interpolating at $x_{i\pm\frac{1}{2}}$. As a convention, $\widetilde{\int}_\Omega$ stands for $\sum_i \widetilde{\int}_{I_i}$.

To define the DG method, we first introduce auxiliary variables to split the original problem (1.4) into the following system of first-order equations:

$$\begin{aligned}
\rho_t &= \partial_x\left(F(\rho)\eta\right) + \partial_x\left(f(\rho)u\right),\\
\eta &= \partial_x\rho,\\
u &= \partial_x\xi,\\
\xi &= V(x) + W * \rho.
\end{aligned}$$

A periodic or compactly supported boundary condition is considered in this work, but our work can extend to more general types of boundary conditions, such as zero-flux boundary conditions. Then our DG approximation can be described as: Find $\rho_h, \eta_h, u_h, \xi_h \in V_h$ such that for any $\phi_h, \psi_h, \varphi_h \in V_h$,

(2.2)

$$\begin{aligned}
\widetilde{\int}_{I_i} (\rho_h)_t\phi_h \, dx &= -\widetilde{\int}_{I_i} (F_h\eta_h)\partial_x\phi_h \, dx + (\widehat{F_h\eta_h})_{i+\frac{1}{2}}(\phi_h)^-_{i+\frac{1}{2}} - (\widehat{F_h\eta_h})_{i-\frac{1}{2}}(\phi_h)^+_{i-\frac{1}{2}}\\
&\quad - \widetilde{\int}_{I_i} (f_hu_h)\partial_x\phi_h \, dx + (\widehat{f_hu_h})_{i+\frac{1}{2}}(\phi_h)^-_{i+\frac{1}{2}} - (\widehat{f_hu_h})_{i-\frac{1}{2}}(\phi_h)^+_{i-\frac{1}{2}}\\
\widetilde{\int}_{I_i} \eta_h\psi_h \, dx &= -\widetilde{\int}_{I_i} \rho_h\partial_x\psi_h \, dx + (\widehat{\rho_h})_{i+\frac{1}{2}}(\psi_h)^-_{i+\frac{1}{2}} - (\widehat{\rho_h})_{i-\frac{1}{2}}(\psi_h)^+_{i-\frac{1}{2}}\\
\widetilde{\int}_{I_i} u_h\varphi_h \, dx &= -\widetilde{\int}_{I_i} \xi_h\partial_x\varphi_h \, dx + (\widehat{\xi_h})_{i+\frac{1}{2}}(\varphi_h)^-_{i+\frac{1}{2}} - (\widehat{\xi_h})_{i-\frac{1}{2}}(\varphi_h)^+_{i-\frac{1}{2}}\\
(\xi_h)_i &= V(x_i) + (W * \rho_h)(x_i)
\end{aligned}$$

When W is smooth, the convolution can be approximated by

$$(W * \rho_h)(x_i) \approx \widetilde{\int}_\Omega W(x_i - y)\rho_h(y) \, dy.$$

The numerical fluxes are chosen in the following way

$$\widehat{F_h\eta_h} = (F_h\eta_h)^+, \widehat{\rho_h} = \rho_h^- \quad \text{or} \quad \widehat{F_h\eta_h} = (F_h\eta_h)^-, \widehat{\rho_h} = \rho_h^+, \tag{2.3a}$$

$$\widehat{f_hu_h} = \frac{1}{2}\left((f_hu_h)^+ + (f_hu_h)^- + \alpha(g^+ - g^-)\right), \quad \alpha = \max\{|u_h|^+, |u_h|^-\}, \tag{2.3b}$$

$$\xi_h = \frac{1}{2}\left(\xi_h^+ + \xi_h^-\right), \tag{2.3c}$$

where g is chosen to satisfy $\text{sign}[g_h] = \text{sign}[f_h]$ or $\text{sign}[g_h] = 0$, and $\alpha g \pm fu \in [0, \alpha]$ when all $\rho \in [0, 1]$. For the case $f = \rho(1 - \rho)$, we can take $g = \rho$. Using the fact $\alpha \geq |u|$, it is straightforward to confirm that $\alpha g \pm fu \in [0, \alpha]$ for $\rho \in [0, 1]$.

Remark 2.1. If only positivity preservation is required, g in (2.3) can be relaxed to satisfy $\alpha g \pm fu \geq 0$ as suggested by [29].

Subsequently, we reformulate (2.2) into a vector ODE corresponding to the values on all Gauss-Lobatto nodes. Let

$$\rho_h(x,t) = \rho_i^{(1)}(t)\phi_i^{(1)}(x) + \rho_i^{(2)}(t)\phi_i^{(2)}(x), \quad x \in I_i,\ t \geq 0, \tag{2.4}$$

where $\phi_i^{(1)}$, $\phi_i^{(2)}$ are given in (2.1). Define $\boldsymbol{\rho}_h(t) = (\rho_1^{(1)}(t), \rho_1^{(2)}(t), \ldots, \rho_N^{(1)}(t), \rho_N^{(2)}(t))^\top$ as the vector representing the values of ρ_h on all Gauss-Lobatto nodes. By employing the basis functions $\phi_i^{(1)}$ and $\phi_i^{(2)}$ in the DG discretization (2.2), we can derive an ODE of the following form:

$$(\boldsymbol{\rho}_h)_t = \mathcal{L}_h(\boldsymbol{\rho}_h)\boldsymbol{\rho}_h + \mathcal{N}_h(\boldsymbol{\rho}_h), \tag{2.5}$$

where $\mathcal{L}_h(\boldsymbol{\rho}_h) \in \mathbb{R}^{2N\times 2N}$ is a matrix linked to the discretization of $\partial_x F(\rho)\nabla$, and $\mathcal{N}_h(\boldsymbol{\rho}) \in \mathbb{R}^{2N}$ is a vector associated with the discretization of $\partial_x(f(\rho)u)$. Employing the flux (2.3a) along with periodic boundary conditions, two asymmetric operator $\mathcal{L}_h(\boldsymbol{\rho})$ are obtained with off-diagonal elements involving difference $F(\rho_i^{(1)}) - F(\rho_i^{(2)})$ or $F(\rho_i^{(2)}) - F(\rho_i^{(1)})$, which can be negative. In this work, we average the two asymmetric operators to obtain a symmetric one

(2.6)

$$\mathcal{L}_h(\boldsymbol{\rho}_h) = \frac{1}{h^2}\begin{pmatrix}
d_1^1 & 0 & F_1^2 & & & & & & F_N^2 & a_1 \\
0 & d_1^2 & a_2 & F_2^1 & & & & & & F_1^1 \\
F_1^2 & a_2 & d_2^1 & 0 & F_2^2 & & & & & \\
 & F_2^1 & 0 & d_2^2 & a_3 & F_3^1 & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & F_{i-1}^2 & a_i & d_i^1 & 0 & F_i^2 & & \\
 & & & & F_i^1 & 0 & d_i^2 & a_{i+1} & F_{i+1}^1 & \\
 & & & & & \ddots & \ddots & \ddots & \ddots & \\
F_N^2 & & & & & & F_{N-1}^2 & a_{N-1} & d_N^1 & 0 \\
a_1 & F_1^1 & & & & & & F_N^1 & 0 & d_N^2
\end{pmatrix}_{2N\times 2N}$$

where $F_i^\ell = F(\rho_i^{(\ell)})$ and

$$d_i^1 = -(F_i^2 + F_i^1 + 2F_{i-1}^2), \quad d_i^2 = -(2F_{i+1}^1 + F_i^2 + F_i^1), \quad a_i = F_i^1 + F_{i-1}^2.$$

Consequently, using the fact $F = fH'' \geq 0$, one obtains that $\mathcal{L}_h(\boldsymbol{\rho}_h) = (\ell_{ij})_{2N\times 2N}$ is symmetric and has the following properties:

- Zero row sums: $\mathcal{L}_h(\boldsymbol{\rho}_h)\mathbf{1} = 0$ with $\mathbf{1} = (1, 1, \cdots, 1)^\top \in \mathbb{R}^{2N}$.
- Pattern of signs: $\ell_{ij} \geq 0$ if $i = j$, and $\ell_{ij} \leq 0$ if $i \neq j$.

Matrices exhibiting both of these characteristics are known as graph Laplacians [4], which lead to an ODE that guarantees both mass conservation and bound preservation. For a vector $\boldsymbol{\rho}$, we denote $\boldsymbol{\rho} \in [0,1]$ or $0 \leq \boldsymbol{\rho} \leq 1$ to indicate that every element of $\boldsymbol{\rho}$ falls within the range of 0 to 1. Based on this notation, we present the following classical conclusion:

LEMMA 2.2 ([4, Proposition 1.1]). *Let the matrix $\mathcal{L}_h(\boldsymbol{\rho}_h)$ be defined as* (2.6) *and $\boldsymbol{u}(t)$ be a solution to the ODE*

$$\boldsymbol{u}_t = \mathcal{L}_h(\boldsymbol{\rho}_h)\boldsymbol{u}, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0 \tag{2.7}$$

with $\boldsymbol{u}_0 \in [0,1]$. Then for all $t \geq 0$, $\boldsymbol{u}(t) \in [0,1]$ and $\mathbf{1}^\top\boldsymbol{u}(t) = \mathbf{1}^\top\boldsymbol{u}_0$.

Alternatively, we can express the solution of the ODE (2.7) using an exponential matrix as

$$\boldsymbol{u}(t) = e^{t\mathcal{L}_h(\boldsymbol{\rho}_h)}\boldsymbol{u}_0.$$

Therefore, Lemma 2.2 demonstrates that the exponential matrix $e^{t\mathcal{L}_h(\boldsymbol{\rho}_h)}$ ensures both mass conservation and bound preservation.

3. Temporal discretization. In this section, we present first- and second-order bound-preserving temporal discretizations for the ODE (2.5). Our schemes are designed by incorporating an auxiliary linear term independent of ρ_h and then employing an exponential integrator. For simplicity, the time domain is discretized using equispaced points with time-stepping $\tau > 0$, and we define the nth point given by $t_n = n\tau (n = 0, 1, 2, \cdots)$. Considering the ODE (2.5) on the interval $[t_n, t_{n+1}]$, we reformulate it as

$$(\boldsymbol{\rho}_h)_t = L_p(t)\boldsymbol{\rho}_h + \mathcal{N}_h(\boldsymbol{\rho}_h) + (\mathcal{L}_h(\boldsymbol{\rho}_h) - L_p(t))\boldsymbol{\rho}_h, \quad (x,t) \in \Omega \times [t_n, t_{n+1}], \tag{3.1}$$

where $L_p(t)$ is a pth-order approximation of $\mathcal{L}_h(\boldsymbol{\rho}_h(t))$ satisfying

$$\mathcal{L}_h(\boldsymbol{\rho}_h(t_n + s)) = L_p(t_n + s) + O(s^p), \qquad s \in [0, \tau]. \tag{3.2}$$

Define the integrating factor

$$\mathcal{T}(t) = \int_0^t L_p(s)\, ds. \tag{3.3}$$

Then, the equation (3.1) can be written as

$$\left(e^{-\mathcal{T}(t)}\boldsymbol{\rho}_h(t)\right)_t = e^{-\mathcal{T}(t)}\left(\mathcal{N}_h(\boldsymbol{\rho}_h(t)) + (\mathcal{L}_h(\boldsymbol{\rho}_h(t)) - L_p(t))\boldsymbol{\rho}_h(t)\right) \tag{3.4}$$

Define $\boldsymbol{w}(t) = e^{-\mathcal{T}(t)}\boldsymbol{\rho}_h(t)$. We get

$$(\boldsymbol{w}(t))_t = e^{-\mathcal{T}(t)}\left(\mathcal{N}_h(\boldsymbol{\rho}_h(t)) + (\mathcal{L}_h(\boldsymbol{\rho}_h(t)) - L_p(t))\boldsymbol{\rho}_h(t)\right) = H(\boldsymbol{w}, t). \tag{3.5}$$

Next, we introduce first- and second-order bound-preserving schemes for (3.5) using SSP-RK methods [18]. It is noteworthy that our proposed schemes are simplified to the exponential SSP-RK methods [24] when $\mathcal{L}_h(\boldsymbol{\rho}_h)$ is a constant matrix independent of $\boldsymbol{\rho}_h$.

3.1. First-order scheme. Consider the approximation $L_0(t) = \mathcal{L}_h(\boldsymbol{\rho}_h(t_n))$. Then the integrating factor (3.3) becomes $\mathcal{T}(t) = t\mathcal{L}_h(\boldsymbol{\rho}_h(t_n))$. By applying the first-order forward Euler scheme to (3.5), we derive the scheme

$$\boldsymbol{\rho}_h^{n+1} = e^{\tau\mathcal{L}_h(\boldsymbol{\rho}_h^n)}\left(\boldsymbol{\rho}_h^n + \tau\mathcal{N}_h(\boldsymbol{\rho}_h^n)\right), \tag{3.6}$$

which is equivalent to

$$\boldsymbol{\rho}_h^{n+1,1} = \boldsymbol{\rho}_h^n + \tau\mathcal{N}_h(\boldsymbol{\rho}_h^n) \tag{3.7a}$$

$$\boldsymbol{\rho}_h^{n+1} = e^{\tau\mathcal{L}_h(\boldsymbol{\rho}_h^n)}\boldsymbol{\rho}_h^{n+1,1}. \tag{3.7b}$$

Let $\boldsymbol{\rho}_h(t)$ be an exact solution of (2.5) and denote $\boldsymbol{\rho} = \boldsymbol{\rho}_h(t_n)$. Using Taylor's expansion, we have

$$\begin{aligned}\boldsymbol{\rho}_h(t_{n+1}) - \boldsymbol{\rho}_h^{n+1} &= \boldsymbol{\rho}_h(t_{n+1}) - e^{\tau\mathcal{L}_h(\boldsymbol{\rho})}\left(\boldsymbol{\rho} + \tau\mathcal{N}_h(\boldsymbol{\rho})\right)\\ &= \boldsymbol{\rho}_h(t_{n+1}) - \left(I + \tau\mathcal{L}_h(\boldsymbol{\rho}) + O(\tau^2)\right)\left(\boldsymbol{\rho} + \tau\mathcal{N}_h(\boldsymbol{\rho})\right)\\ &= \boldsymbol{\rho}_h(t_{n+1}) - \left(\boldsymbol{\rho} + \tau(\mathcal{L}_h(\boldsymbol{\rho})\boldsymbol{\rho} + \mathcal{N}_h(\boldsymbol{\rho}))\right) + O(\tau^2)\\ &= O(\tau^2).\end{aligned}$$

Hence. the scheme (3.7) is a first-order temporal discretization for (2.5).

Alternatively, we can express the scheme (3.7) in a weak form: Find $\rho_h^{n+1,1}, \rho_h^{n+1} \in V_h$ such that for all $\phi_h, \psi_h \in V_h$,

(3.8a)
$$\widetilde{\int}_{I_i} \frac{\rho_h^{n+1,1} - \rho_h^n}{\tau}\phi_h \, dx = -\widetilde{\int}_{I_i} (f_h^n u_h^n)\partial_x\phi_h \, dx + (\widehat{f_h^n u_h^n})_{i+\frac{1}{2}}(\phi_h)^-_{i+\frac{1}{2}} - (\widehat{f_h^n u_h^n})_{i-\frac{1}{2}}(\phi_h)^+_{i-\frac{1}{2}},$$

(3.8b)
$$\widetilde{\int}_{I_i} u_h^n\varphi_h \, dx = -\widetilde{\int}_{I_i} \xi_h^n\partial_x\varphi_h \, dx + (\widehat{\xi_h^n})_{i+\frac{1}{2}}(\varphi_h)^-_{i+\frac{1}{2}} - (\widehat{\xi_h^n})_{i-\frac{1}{2}}(\varphi_h)^+_{i-\frac{1}{2}},$$

(3.8c)
$$(\xi_h^n)_i = V(x_i) + (W * \rho_h^n)(x_i),$$

and $\rho_h^{n+1} = (\tilde{\rho}_h^L(\tau) + \tilde{\rho}_h^R(\tau))/2$ with $\tilde{\rho}_h : [0,\tau] \to V_h$ solving the system

(3.9a)
$$\widetilde{\int}_{I_i} (\tilde{\rho}_h)_t\phi_h \, dx = -\widetilde{\int}_{I_i} (F_h^n\eta_h)\,\partial_x\phi \, dx + (\widehat{F_h^n\eta_h})_{i+\frac{1}{2}}\phi^-_{i+\frac{1}{2}} - (\widehat{F_h^n\eta_h})_{i-\frac{1}{2}}\phi^+_{i-\frac{1}{2}},$$

(3.9b)
$$\widetilde{\int}_{I_i} \eta_h\psi_h \, dx = -\widetilde{\int}_{I_i} \tilde{\rho}_h\partial_x\psi_h \, dx + (\widehat{\tilde{\rho}_h})_{i+\frac{1}{2}}(\psi_h)^-_{i+\frac{1}{2}} - (\widehat{\tilde{\rho}_h})_{i-\frac{1}{2}}(\psi_h)^+_{i-\frac{1}{2}},$$

(3.9c)
$$\tilde{\rho}_h(0) = \rho_h^{n+1,1}.$$

and $\tilde{\rho}_h^L$, $\tilde{\rho}_h^R$ correspond to the solutions using two different fluxes according to (2.3a).

LEMMA 3.1. *Given ρ_h^n, the $\rho_h^{n+1,1}$ solved in (3.8) satisfies*

- *Mass conservation: $\widetilde{\int}_\Omega \rho_h^{n+1,1} \, dx = \widetilde{\int}_\Omega \rho_h^n \, dx$;*
- *Bound-preservation for the cell average: Assuming that ρ_h^n is within the range $[0,1]$ at the Gauss-Lobatto quadrature, i.e., $\boldsymbol{\rho}_h^n \in [0,1]$, the cell average $(\bar{\rho}_h)_i^{n+1,1} = \frac{1}{h}\widetilde{\int}_{I_i}\rho_h^{n+1,1} \, dx \in [0,1]$ if*

$$\alpha\lambda g \le \rho \text{ and } \alpha\lambda(1-g) \le 1-\rho, \tag{3.10}$$

where $\lambda = \tau/h$. Specifically, when $g = \rho$, it reduces to the CFL condition

$$\tau \le \frac{h}{\max\limits_{i=1,\cdots,N}\alpha_{i\pm\frac{1}{2}}}. \tag{3.11}$$

Proof. Let $\phi_h \equiv 1$ in (3.8a), we obtain that

$$\widetilde{\int}_{I_i} \rho_h^{n+1,1} \, dx = \widetilde{\int}_{I_i} \rho_h^n \, dx + \tau\left((\widehat{f_h^n u_h^n})_{i+\frac{1}{2}} - (\widehat{f_h^n u_h^n})_{i-\frac{1}{2}}\right), \quad i = 1, 2, \cdots, N. \tag{3.12}$$

Summing i from 1 to N, one get

$$\widetilde{\int}_{\Omega} \rho_h^{n+1,1}\, dx = \widetilde{\int}_{\Omega} \rho_h^{n}\, dx + \tau \left((\widehat{f_h^n u_h^n})_{N+\frac{1}{2}} - (\widehat{f_h^n u_h^n})_{-\frac{1}{2}} \right).$$

The numerical flux at the right boundary is equal to the numerical flux at the left boundary, as required by the periodic boundary condition and the definition of the numerical flux. Therefore, we can conclude that $\widetilde{\int}_I \rho_h^{n+1,1}\, dx = \widetilde{\int}_I \rho_h^{n}\, dx$.

Multiplying both side of (3.12) with $1/h$, it becomes

$$(\bar{\rho}_h)_i^{n+1,1} = (\bar{\rho}_h)_i^n + \lambda \left((\widehat{f_h^n u_h^n})_{i+\frac{1}{2}} - (\widehat{f_h^n u_h^n})_{i-\frac{1}{2}} \right), \quad i = 1, 2, \cdots, N.$$

Note that the Gauss–Lobatto quadrature is exact for evaluating the cell average $(\bar{\rho}_h)_i^n$ since $(\rho_h)_i^n$ is a first-order polynomial. For simplicity, we omit the indices n and h for the notation ρ_h^n in the subsequent proof. Then

$$\begin{aligned}
(\bar{\rho})_i^{n+1,1} =& \frac{1}{2}\left(\rho^-_{i+\frac{1}{2}} + \rho^+_{i-\frac{1}{2}}\right) + \frac{\lambda}{2}\left((fu)^+_{i+\frac{1}{2}} + (fu)^-_{i+\frac{1}{2}} + \alpha_{i+\frac{1}{2}}(g^+_{i+\frac{1}{2}} - g^-_{i+\frac{1}{2}})\right) \\
& - \frac{\lambda}{2}\left((fu)^+_{i-\frac{1}{2}} + (fu)^-_{i-\frac{1}{2}} + \alpha_{i-\frac{1}{2}}(g^+_{i-\frac{1}{2}} - g^-_{i-\frac{1}{2}})\right) \\
=& \frac{1}{2}\left[\rho^-_{i+\frac{1}{2}} + \lambda\left((fu)^+_{i+\frac{1}{2}} + (fu)^-_{i+\frac{1}{2}} + \alpha_{i+\frac{1}{2}}(g^+_{i+\frac{1}{2}} - g^-_{i+\frac{1}{2}})\right)\right. \\
& \left. -\lambda\left((fu)^-_{i+\frac{1}{2}} + (fu)^+_{i-\frac{1}{2}} + \alpha_{i+\frac{1}{2}}(g^-_{i+\frac{1}{2}} - g^+_{i-\frac{1}{2}})\right)\right] + \\
& \frac{1}{2}\left[\rho^+_{i-\frac{1}{2}} + \lambda\left((fu)^-_{i+\frac{1}{2}} + (fu)^+_{i-\frac{1}{2}} + \alpha_{i+\frac{1}{2}}(g^-_{i+\frac{1}{2}} - g^+_{i-\frac{1}{2}})\right)\right. \\
& \left. -\lambda\left((fu)^+_{i-\frac{1}{2}} + (fu)^-_{i-\frac{1}{2}} + \alpha_{i-\frac{1}{2}}(g^+_{i-\frac{1}{2}} - g^-_{i-\frac{1}{2}})\right)\right] \\
=& \frac{1}{2}\left[\lambda\alpha_{i+\frac{1}{2}}K^+(\rho^+_{i+\frac{1}{2}}, \alpha_{i+\frac{1}{2}}) + \lambda\alpha_{i+\frac{1}{2}}K^-(\rho^+_{i-\frac{1}{2}}, \alpha_{i+\frac{1}{2}})\right. \\
& \left. + \left(\rho^-_{i+\frac{1}{2}} - 2\lambda\alpha_{i+\frac{1}{2}}g^-_{i+\frac{1}{2}}\right)\right] \\
& \frac{1}{2}\left[\lambda\alpha_{i-\frac{1}{2}}K^-(\rho^-_{i-\frac{1}{2}}, \alpha_{i-\frac{1}{2}}) + \lambda\alpha_{i+\frac{1}{2}}K^+(\rho^-_{i+\frac{1}{2}}, \alpha_{i+\frac{1}{2}})\right. \\
& \left. + \left(\rho^+_{i-\frac{1}{2}} - \lambda(\alpha_{i+\frac{1}{2}} + \alpha_{i-\frac{1}{2}})g^+_{i-\frac{1}{2}}\right)\right],
\end{aligned}$$

where $K^{\pm} = g \pm fu/\alpha$. Note that $K^{\pm} \in [0, 1]$ by the choice of g. To make $(\bar{\rho}_h)_i^{n+1,1} \in [0, 1]$, it is sufficient to ensure $\rho - \lambda\alpha g \in [0, 1 - \lambda\alpha]$, which reduces to the restriction (3.10). ∎

Remark 3.2. In the definition of ξ_h, it is continuous and (3.8b) gives $u_h = \partial_x \mathcal{I}(\xi_h)$ on I_i after integration by parts. Hence the CFL condition (3.11) becomes

$$\tau \le \frac{h}{\max_i \|\partial_x \mathcal{I}(\xi_h)\|_{L^\infty(I_i)}} \sim h,$$

which is a favorable CFL condition for parabolic equations.

Remark 3.3. For efficient implementation, we can approximate the exponential matrix in (3.7b) as

$$\boldsymbol{\rho}_h^{n+1} = \left(I - \tau \mathcal{L}_h(\boldsymbol{\rho}_h^n)\right)^{-1} \boldsymbol{\rho}_h^{n+1,1}, \tag{3.13}$$

which is also a first-order scheme [5] and the bound preservation still holds using the similar proving process.

As stated in Lemma 3.1, the Euler forward stage (3.7a) only maintains bound-preservation on the cell average, not across all Gauss-Lobatto quadrature points. Following the methodology developed by [32], we can apply a bound-preserving limiter to enforce the boundedness of nodal values on all Gauss–Lobatto quadrature points without violating the mass conservation and accuracy. Precisely, let

$$\rho_h^{n+1,2}(x_{i\pm\frac{1}{2}}^{\mp}) = (\bar{\rho}_h)_i^{n+1,1} + \theta_i \left(\rho_h^{n+1,1}(x_{i\pm\frac{1}{2}}^{\mp}) - (\bar{\rho}_h)_i^{n+1,1}\right),$$

with

$$\theta_i = \min\left\{\frac{(\bar{\rho}_h)_i^{n+1,1}}{(\bar{\rho}_h)_i^{n+1,1} - m_i}, \frac{1-(\bar{\rho}_h)_i^{n+1,1}}{M_i - (\bar{\rho}_h)_i^{n+1,1}}, 1\right\},$$
$$m_i = \min \rho_h^{n+1,1}(x_{i\pm\frac{1}{2}}^{\mp}),\ M_i = \max \rho_h^{n+1,1}(x_{i\pm\frac{1}{2}}^{\mp}).$$

Then we get $\rho_h^{n+1,2}(x_{i\pm\frac{1}{2}}^{\mp}) \in [0,1]$ and $(\bar{\rho}_h)_i^{n+1,2} = (\bar{\rho}_h)_i^{n+1,1}$ (see [32]). Furthermore, the interpolation polynomial of $\{\rho_h^{n+1,2}(x_{i\pm\frac{1}{2}}^{\mp})\}$ on I_i satisfies mnloodsxwe

$$\left|\rho_h^{n+1,2}(x) - \rho_h^{n+1,1}(x)\right| \le C_k \max_{x\in\{x_{i\pm\frac{1}{2}}^{\mp}\}} \left|\rho\left(x, t_{n+1}\right) - \rho_h^{n+1,1}(x)\right|,$$

where $\rho(x, t_{n+1})$ is the exact solution at time t_{n+1} and C_k is a constant depending only on the polynomial degree k. Consequently, we can update

$$\boldsymbol{\rho}_h^{n+1} = e^{\tau\mathcal{L}(\boldsymbol{\rho}_h^n)}\boldsymbol{\rho}_h^{n+1,2}.$$

and conclude that $\boldsymbol{\rho}_h^{n+1} \in [0,1]$ by Lemma 2.2. Denote the bound limiter as $\mathcal{P}$. The first-order bound-preserving scheme can be summarized as

$$\boldsymbol{\rho}_h^{n+1} = e^{\tau\mathcal{L}(\boldsymbol{\rho}_h^n)}\mathcal{P}\left(\boldsymbol{\rho}_h^n + \tau\mathcal{N}_h(\boldsymbol{\rho}_h^n)\right). \tag{3.14}$$

Using the approximation (3.13), a more efficient first-order scheme is given by

$$\boldsymbol{\rho}_h^{n+1} = \left(I - \tau\mathcal{L}(\boldsymbol{\rho}_h^n)\right)^{-1}\mathcal{P}\left(\boldsymbol{\rho}_h^n + \tau\mathcal{N}_h(\boldsymbol{\rho}_h^n)\right). \tag{3.15}$$

THEOREM 3.4. *The updates* (3.14) *and* (3.15) *are bound preserving and mass conservative, provided the time step restriction specified in Lemma* 3.1 *is satisfied.*

3.2. Second-order scheme. Based on the first-order scheme (3.14) or (3.15), we can construct a first-order approximation for $\mathcal{L}_h(\boldsymbol{\rho}_h)$. Specially, let

$$\boldsymbol{\rho}_h^{n+1,1} = \boldsymbol{\rho}_h^n + \tau\mathcal{N}_h(\boldsymbol{\rho}_h^n), \tag{3.16a}$$
$$\widetilde{\boldsymbol{\rho}}_h^{n+1} = \left(I - \tau\mathcal{L}_h(\boldsymbol{\rho}_h^n)\right)^{-1}\boldsymbol{\rho}_h^{n+1,1}, \tag{3.16b}$$

and introduce the linear interpolation

$$L_1(t) = \frac{t-t_n}{\tau}\mathcal{L}_h(\widetilde{\boldsymbol{\rho}}_h^{n+1}) + \left(1 - \frac{t-t_n}{\tau}\right)\mathcal{L}_h(\boldsymbol{\rho}_h^n), \tag{3.17}$$

which provides a first approximation for $\mathcal{L}_h(\boldsymbol{\rho}_h)$ on the interval $[t_n, t_{n+1}]$. Consequently, the integrating factor (3.3) becomes

$$\mathcal{T}(t) = \frac{(t-t_n)^2}{2\tau}\mathcal{L}_h(\widetilde{\boldsymbol{\rho}}_h^{n+1}) - \frac{\tau}{2}\left(1 - \frac{t-t_n}{\tau}\right)^2 \mathcal{L}_h(\boldsymbol{\rho}_h^n). \tag{3.18}$$

Applying the second-order SSP Runge-Kutta [18] for (3.4), we obtain the scheme

$$\boldsymbol{\rho}_h^{n+1,2} = e^{\tau\tilde{\mathcal{L}}_h^{n+1}}\boldsymbol{\rho}_h^{n+1,1}, \tag{3.19a}$$

$$\boldsymbol{\rho}_h^{n+1} = \frac{1}{2}e^{\tau\tilde{\mathcal{L}}_h^{n+1}}\boldsymbol{\rho}_h^{n} + \frac{1}{2}\left(\boldsymbol{\rho}_h^{n+1,2} + \tau\mathcal{N}_h(\boldsymbol{\rho}_h^{n+1,2})\right), \tag{3.19b}$$

where

$$\tilde{\mathcal{L}}_h^{n+1} = \frac{1}{2}(\mathcal{L}(\widetilde{\boldsymbol{\rho}}_h^{n+1}) + \mathcal{L}_h(\boldsymbol{\rho}_h^n)).$$

Remark 3.5. Due to the properties $L_1(t_n) = \mathcal{L}_h(\boldsymbol{\rho}_h^n)$ and $L_1(t_{n+1}) = \mathcal{L}_h(\widetilde{\boldsymbol{\rho}}_h^{n+1})$, the stiff term $\mathcal{L}_h(\boldsymbol{\rho}_h)\boldsymbol{\rho}_h$ does not appear in the explicit Euler steps (3.16a) and (3.19b), even though it exists in the ODE (3.5). Hence, the stability of the explicit Euler step only depends on the nonlinear term $\mathcal{N}_h$.

Let $\boldsymbol{\rho}_h(t)$ be an exact solution for (2.5), we next show that (3.19) is second-order in time. Denote $\mathcal{G}(\boldsymbol{\rho}_h) = \mathcal{L}_h(\boldsymbol{\rho}_h)\boldsymbol{\rho}_h + \mathcal{N}_h(\boldsymbol{\rho}_h)$ and $\boldsymbol{\rho} = \boldsymbol{\rho}_h(t_n)$ for simplicity. The Taylor's expansion gives that

(3.20)

$$\begin{aligned}\boldsymbol{\rho}_h(t_{n+1}) &= \boldsymbol{\rho} + \tau\mathcal{G}(\boldsymbol{\rho}) + \frac{\tau^2}{2}\mathcal{G}_t(\boldsymbol{\rho}) + O(\tau^3)\\ &= \boldsymbol{\rho} + \tau\mathcal{G}(\boldsymbol{\rho}) + \frac{\tau^2}{2}\left(\left(\mathcal{L}_h'(\boldsymbol{\rho})\mathcal{G}(\boldsymbol{\rho})\right)\boldsymbol{\rho} + \mathcal{L}_h(\boldsymbol{\rho})\mathcal{G}(\boldsymbol{\rho})\right) + \tau^2\mathcal{N}_h'(\boldsymbol{\rho})\mathcal{G}(\boldsymbol{\rho}) + O(\tau^3)\end{aligned}$$

Substitute the exact solution into (3.16) and (3.19), we get

$$\widetilde{\boldsymbol{\rho}}_h^{n+1} = \boldsymbol{\rho} + \tau\mathcal{G}(\boldsymbol{\rho}) + O(\tau^2)$$

$$\mathcal{L}_h(\widetilde{\boldsymbol{\rho}}_h^{n+1}) = \mathcal{L}_h(\boldsymbol{\rho}) + \tau\mathcal{L}_h'(\boldsymbol{\rho})\mathcal{G}(\boldsymbol{\rho}) + O(\tau^2)$$

$$\begin{aligned}\boldsymbol{\rho}_h^{n+1,2} &= \boldsymbol{\rho} + \tau\left(\tilde{\mathcal{L}}_h^{\,n+1}\boldsymbol{\rho} + \mathcal{N}_h(\boldsymbol{\rho})\right) + \frac{\tau^2}{2}\tilde{\mathcal{L}}_h^{\,n+1}\left(\tilde{\mathcal{L}}_h^{\,n+1}\boldsymbol{\rho} + 2\mathcal{N}_h(\boldsymbol{\rho})\right) + O(\tau^3)\\ &= \boldsymbol{\rho} + \tau\mathcal{G}(\boldsymbol{\rho}) + \frac{\tau^2}{2}\mathcal{L}_h(\boldsymbol{\rho})\left(\mathcal{G}(\boldsymbol{\rho}) + \mathcal{N}_h(\boldsymbol{\rho})\right) + \frac{\tau}{2}(\mathcal{L}_h(\widetilde{\boldsymbol{\rho}}_h^{n+1}) - \mathcal{L}_h(\boldsymbol{\rho}))\boldsymbol{\rho} + O(\tau^3)\\ &= \boldsymbol{\rho} + \tau\mathcal{G}(\boldsymbol{\rho}) + \frac{\tau^2}{2}\mathcal{L}_h(\boldsymbol{\rho})\left(\mathcal{G}(\boldsymbol{\rho}) + \mathcal{N}_h(\boldsymbol{\rho})\right) + \frac{\tau^2}{2}\left(\mathcal{L}_h'(\boldsymbol{\rho})\mathcal{G}(\boldsymbol{\rho})\right)\boldsymbol{\rho} + O(\tau^3)\end{aligned}$$

$$\mathcal{N}_h(\boldsymbol{\rho}_h^{n+1,2}) = \mathcal{N}_h(\boldsymbol{\rho}) + \tau\mathcal{N}_h'(\boldsymbol{\rho})\mathcal{G}(\boldsymbol{\rho}) + O(\tau^2)$$

$$\begin{aligned}\boldsymbol{\rho}_h^{n+1} &= \frac{1}{2}\left(I + \tau\tilde{\mathcal{L}}_h^{\,n+1} + \frac{\tau^2}{2}(\tilde{\mathcal{L}}_h^{\,n+1})^2\right)\boldsymbol{\rho} + \frac{1}{2}\left(\boldsymbol{\rho}_h^{n+1,2} + \tau\mathcal{N}_h(\boldsymbol{\rho}_h^{n+1,2})\right) + O(\tau^3)\\ &= \boldsymbol{\rho} + \frac{\tau}{2}\left(\mathcal{L}_h(\boldsymbol{\rho})\boldsymbol{\rho} + \mathcal{N}_h(\boldsymbol{\rho})\right) + \frac{\tau^2}{4}\left(\mathcal{L}_h(\boldsymbol{\rho})\right)^2\boldsymbol{\rho} + \frac{1}{2}(\boldsymbol{\rho}_h^{n+1,2} - \boldsymbol{\rho})\\ &\quad + \frac{\tau}{4}(\mathcal{L}_h(\widetilde{\boldsymbol{\rho}}_h^{n+1}) - \mathcal{L}_h(\boldsymbol{\rho}))\boldsymbol{\rho} + \frac{\tau}{2}(\mathcal{N}_h(\boldsymbol{\rho}_h^{n+1,2}) - \mathcal{N}_h(\boldsymbol{\rho})) + O(\tau^3)\\ &= \boldsymbol{\rho} + \tau\mathcal{G}(\boldsymbol{\rho}) + \frac{\tau^2}{4}\left(\mathcal{L}_h(\boldsymbol{\rho})\right)^2\boldsymbol{\rho} + \frac{\tau^2}{4}\mathcal{L}_h(\boldsymbol{\rho})\left(\mathcal{G}(\boldsymbol{\rho}) + \mathcal{N}_h(\boldsymbol{\rho})\right)\end{aligned}$$

$$+\frac{\tau^2}{2}\left(\mathcal{L}_h'(\boldsymbol{\rho})\mathcal{G}(\boldsymbol{\rho})\right)\boldsymbol{\rho}+\frac{\tau^2}{2}\mathcal{N}_h'(\boldsymbol{\rho})\mathcal{G}(\boldsymbol{\rho})+O(\tau^3)$$

$$=\boldsymbol{\rho}+\tau\mathcal{G}(\boldsymbol{\rho})+\frac{\tau^2}{2}\left(\left(\mathcal{L}_h'(\boldsymbol{\rho})\mathcal{G}(\boldsymbol{\rho})\right)\boldsymbol{\rho}+\mathcal{L}_h(\boldsymbol{\rho})\mathcal{G}(\boldsymbol{\rho})\right)+\frac{\tau^2}{2}\mathcal{N}_h'(\boldsymbol{\rho})\mathcal{G}(\boldsymbol{\rho})+O(\tau^3).$$

Together with (3.20), one has

$$\boldsymbol{\rho}_h(t_{n+1})-\boldsymbol{\rho}_h^{n+1}=O(\tau^3),$$

which implies that the scheme (3.19) is a second-order temporal discretization for ODE (2.5). When the bound-preserving limiter is applied immediately after each Euler forward stage, we obtain a second-order bound-preserving scheme

$$\boldsymbol{\rho}_h^{n+1,1}=\left(I-\tau\mathcal{L}_h(\boldsymbol{\rho}_h^n)\right)^{-1}\mathcal{P}\left(\boldsymbol{\rho}_h^n+\tau\mathcal{N}_h(\boldsymbol{\rho}_h^n)\right), \tag{3.21a}$$

$$\boldsymbol{\rho}_h^{n+1,2}=e^{\frac{\tau}{2}\left(\mathcal{L}_h(\boldsymbol{\rho}_h^n)+\mathcal{L}_h(\boldsymbol{\rho}_h^{n+1,1})\right)}\mathcal{P}\left(\boldsymbol{\rho}_h^n+\tau\mathcal{N}_h(\boldsymbol{\rho}_h^n)\right), \tag{3.21b}$$

$$\boldsymbol{\rho}_h^{n+1}=\frac{1}{2}e^{\frac{\tau}{2}\left(\mathcal{L}_h(\boldsymbol{\rho}_h^n)+\mathcal{L}_h(\boldsymbol{\rho}_h^{n+1,1})\right)}\boldsymbol{\rho}_h^n+\frac{1}{2}\mathcal{P}\left(\boldsymbol{\rho}_h^{n+1,2}+\tau\mathcal{N}_h(\boldsymbol{\rho}_h^{n+1,2})\right). \tag{3.21c}$$

Similar to the first-order scheme (3.14), we have the following theorem for (3.21).

THEOREM 3.6. *The time discretization* (3.21) *of the semi-discrete scheme* (2.5) *is bound preserving and mass conservative as long as the time step restriction in Lemma* 3.1 *is satisfied.*

Remark 3.7. For practical implementation, one can use the second-order approximation

$$e^{\tau\mathcal{L}}\approx\left(I-\tau\mathcal{L}+\frac{\tau^2}{2}\mathcal{L}^2\right)^{-1} \tag{3.22}$$

in (3.21) to optimize computational efficiency. However, the guarantee of bound-preservation is compromised due to the inclusion of $\mathcal{L}^2$. A pragmatic approach would be to initially utilize the approximation from (3.22). If a value surpasses the expected density bounds, the respective step should be discarded. Subsequently, one should revert to computing using the exponential matrix.

4. Numerical results. In this section, we examine the performance and accuracy of our proposed numerical schemes (3.21) for computing several examples on domain $\Omega=[-L,L]^d, d=1,2$. The error is measured in the discrete norms

$$\begin{aligned}
&err_{L^1}=\widetilde{\int_\Omega}\left|\rho_h(\boldsymbol{x},t)-\rho(\boldsymbol{x},t)\right|d\boldsymbol{x}=\left(\frac{h}{2}\right)^d\left\|\boldsymbol{\rho}_h-\boldsymbol{\rho}\right\|_{\ell^1},\\
&err_{L^2}=\sqrt{\widetilde{\int_\Omega}\left|\rho_h(\boldsymbol{x},t)-\rho(\boldsymbol{x},t)\right|^2d\boldsymbol{x}}=\left(\frac{h}{2}\right)^{\frac{d}{2}}\left\|\boldsymbol{\rho}_h-\boldsymbol{\rho}\right\|_{\ell^2},\\
&err_{L^\infty}=\left\|\boldsymbol{\rho}_h-\boldsymbol{\rho}\right\|_\infty.
\end{aligned} \tag{4.1}$$

Here ρ_h is the numerical solution obtained by the scheme (3.21), and ρ is the exact solution for (1.1) or a reference solution computed by our approach in a finer mesh. Vectors $\boldsymbol{\rho}_h$ and $\boldsymbol{\rho}$ are the values of ρ_h and ρ on all Gauss–Lobatto quadrature points, respectively. We chose $g=\rho$ in the flux (2.3) and use CFL condition $\tau=\frac{1}{4L}h$ for all following numerical tests. The exponential matrix is efficiently calculated by the techniques introduced in [20].

4.1. Accuracy test. We first examine the accuracy of (3.21) on an initial value problem with a source term S:

$$\begin{cases} \rho_t = \nabla\rho(1-\rho)\nabla(\rho + \sin(\mathbf{1}^\top \boldsymbol{x}) + W * \rho) + S(\boldsymbol{x}, t), \quad \boldsymbol{x} \in [-\pi, \pi]^d, t > 0, \\ \rho(\boldsymbol{x}, 0) = \dfrac{1}{4}\left(\sin(\mathbf{1}^\top \boldsymbol{x}) + 1\right), \end{cases} \tag{4.2}$$

where $\mathbf{1} = (1, \cdots, 1)^\top \in \mathbb{R}^d$ and $W(\boldsymbol{x}) = \cos(\mathbf{1}^\top \boldsymbol{x})/(2\pi)$. Here periodic boundary conditions are applied and the source term S is used to ensure that the exact solution is

$$\rho(\boldsymbol{x}, t) = \frac{1}{4}\left(\sin(\mathbf{1}^\top \boldsymbol{x} + t) + 1\right). \tag{4.3}$$

In this test, $F = f = \rho(1-\rho)$, $H' = \rho$ and $V = \sin(\mathbf{1}^\top \boldsymbol{x})$. We calculate the error at $T = 1$. Table 1 and Table 2 show the corresponding second-order convergence in one and two dimensions.

TABLE 1
Accuracy test in one dimension for computing a solution to the equation (4.2). *The error is calculated at* $T = 1$.

N	L^1 error	Order	L^2 error	Order	L^∞ error	Order
20	2.074e-02		9.966e-03		8.508e-03	
40	5.032e-03	2.043	2.442e-03	2.029	2.639e-03	1.689
80	1.232e-03	2.030	6.006e-04	2.024	6.788e-04	1.959
160	3.029e-04	2.024	1.463e-04	2.038	1.752e-04	1.954

TABLE 2
Accuracy test in two dimensions for computing a solution to the equation (4.2). *The error is calculated at* $T = 1$.

N	L^1 error	Order	L^2 error	Order	L^∞ error	Order
10×10	9.177e-01		1.884e-01		1.051e-01	
20×20	2.182e-01	2.073	4.636e-02	2.023	3.046e-02	1.787
40×40	5.095e-02	2.098	1.038e-02	2.159	6.019e-03	2.339

4.2. Saturation experiment. To demonstrate the bound-preserving property of our numerical scheme, we consider the saturation experiment given by

$$\begin{cases} \rho_t = \nabla\left(\rho(1-\rho)\nabla\left(D\ln(\rho) + \dfrac{C}{2}|\boldsymbol{x}|^2\right)\right), \\ \rho(\boldsymbol{x}, 0) = \rho_0. \end{cases} \tag{4.4}$$

Here D and C are positive numbers. In this case, $f = \rho(1-\rho)$, $H = D\left(\rho\ln(\rho) - \rho\right)$, $F = fH'' = D(1-\rho)$, $V(x) = \frac{C}{2}|\boldsymbol{x}|^2$ and $W(\boldsymbol{x}) = 0$. The exact solution of (4.4) is bounded on $[0, 1]$ and the steady state depends on the initial mass $m = \|\rho_0\|_{L^1}$. There exists a threshold $m_c = (\frac{2\pi D}{C})^{d/2}$ such that the steady state can be written as

$$\rho_\infty(\boldsymbol{x}) = \begin{cases} A\exp\left(-\dfrac{C}{2D}|\boldsymbol{x}|^2\right), & m \le m_c \\ B\exp\left(-\dfrac{C}{2D}\max\left\{|\boldsymbol{x}|^2 - \ell^2, 0\right\}\right), & m > m_c \end{cases} \tag{4.5}$$

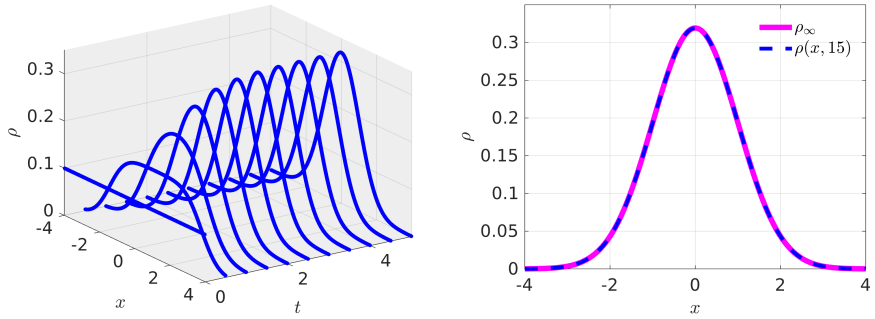

FIG. 1. *Computation of a smooth solution to the one-dimensional saturation experiment* (4.4) *with* $\rho_0 \equiv 0.1$. ***Left:*** *Evolution of* $\rho_h(x,t)$; ***Right:*** *Comparison between* ρ_∞ *and the numerical solution calculated at* $T = 15$;

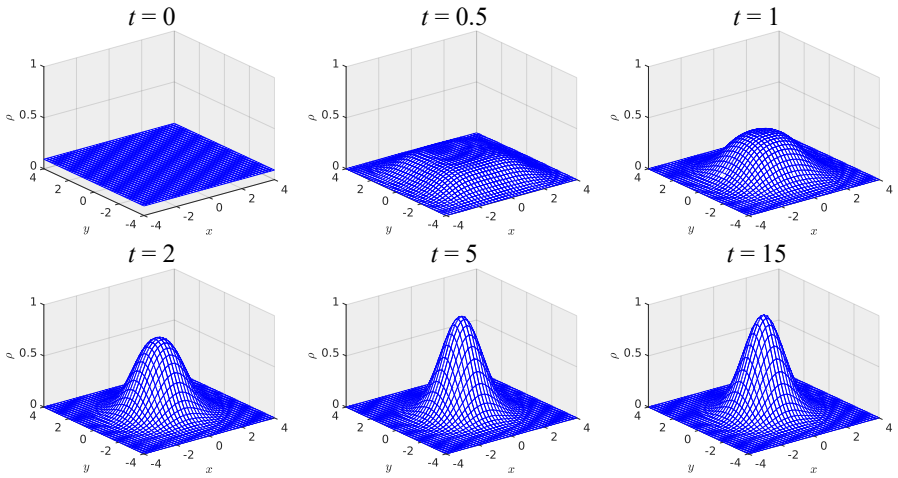

FIG. 2. *Evolution of a solution* $\rho(\boldsymbol{x}, t)$ *with* $\rho_0 \equiv 0.09375$ *in a two-dimensional saturation experiment* (4.4).

where A, B are positive constants such that $\|\rho_\infty\|_{L^1} = \|\rho_0\|_{L^1} = m$, and ℓ can be determined from the initial datum (see reference [1]). Numerically, we solve the equation (4.4) over the domain $\Omega = [-4, 4]^d$ with parameters $C = 1, D = 1$. Consequently, the threshold $m_c = (2\pi)^{d/2}$ in (4.5).

To verify the convergence order, we begin with a uniform initial data $\rho_0 \equiv 0.1$ in one dimension and $\rho_0 \equiv 0.09375$ in two dimensions. This ensures that $\|\rho_0\|_{L^1} \leq m_c$ and leads to a smooth steady state according to (4.5). We then calculate the numerical solution at $T = 15$ as the numerical steady state. Figure 1 and Figure 2 illustrate the numerical evolution to the steady state in one and two dimensions, respectively. Furthermore, Table 5 and Table 4 demonstrate the anticipated second-order accuracy in both one and two dimensions. In Figure 3, we present the behavior of the relative entropy $E(t|\infty) = E(\rho_h(t)) - E(\rho_\infty)$ and the associated bound of our solution. The results indicate that our scheme preserves the bound, aligning with our theoretical analysis, and also exhibits energy dissipation during this test.

To observe saturation, we begin with a uniform initial density of $\rho_0 \equiv 0.415$ in

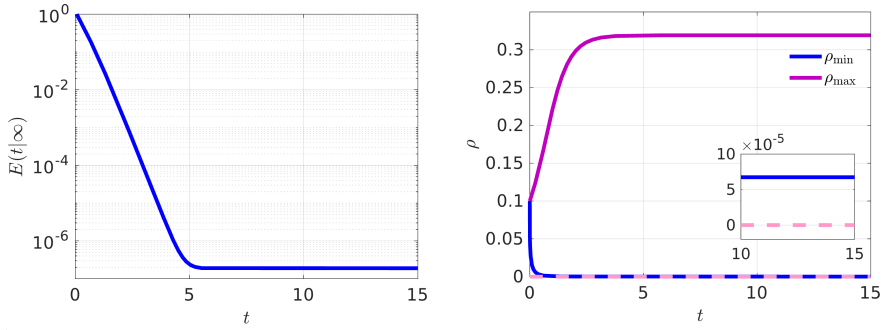

FIG. 3. *Computation of a smooth solution to the one-dimensional saturation experiment* (4.4) *with* $\rho_0 = 0.1$. ***Left:*** *Behaviour of the relative energy* $E(t;\infty)$; ***Right:*** *Behaviour of bound.*

TABLE 3
Accuracy test in one dimension for computing the steady state to the equation (4.4) *with a uniform initial density* $\rho_0 \equiv 0.1$. *The error is calculated at* $T = 15$.

N	L^1 error	Order	L^2 error	Order	L^∞ error	Order
20	2.091e-02		9.166e-03		6.682e-03	
40	2.055e-03	3.347	9.324e-04	3.297	7.039e-04	3.247
80	5.159e-04	1.994	2.234e-04	2.062	1.557e-04	2.176
160	1.299e-04	1.989	5.502e-05	2.021	3.624e-05	2.103

TABLE 4
Accuracy test in two dimensions for computing a smooth steady state to the equation (4.4) *with uniform initial density* $\rho_0 \equiv 0.09375$. *The error is calculated at* $T = 15$

N	L^1 error	Order	L^2 error	Order	L^∞ error	Order
10×10	5.152e-01		8.907e-02		2.554e-02	
20×20	1.436e-01	1.843	2.486e-02	1.841	9.420e-03	1.439
40×40	3.032e-02	2.243	5.063e-03	2.296	2.062e-03	2.192

one dimension and $\rho_0 \equiv 0.147$ in two dimensions such that $\|\rho_0\|_{L^1} > m_c$. According to (4.5), this initial density results in a non-smooth steady state that is bounded between 0 and 1. A numerical solution obtained at $T = 15$ is considered as the numerical steady state. The evolution in both one and two dimensions can be found in Figure 4 and Figure 5. As depicted in Figure 4, in contrast to the results from [9, Section 4.1], our DG discretization method yields a superior approximation devoid of oscillations near the upper bound $\rho_{\max} = 1$. Additionally, we present the behavior of the relative entropy and the bound in Figure 6, which shows that the numerical solution generated by our scheme demonstrates bound preservation within the interval $[0, 1]$ and also exhibits entropy dissipation. In order to test the accuracy of this example, we calculated the error at $T = 1$, employing reference solutions calculated at a finer mesh size with $N = 320$ for one-dimensional computations and $N = 80 \times 80$ for two-dimensional computations. Table 5 and Table 6 present the second-order accuracy of our approach in both one and two dimensions.

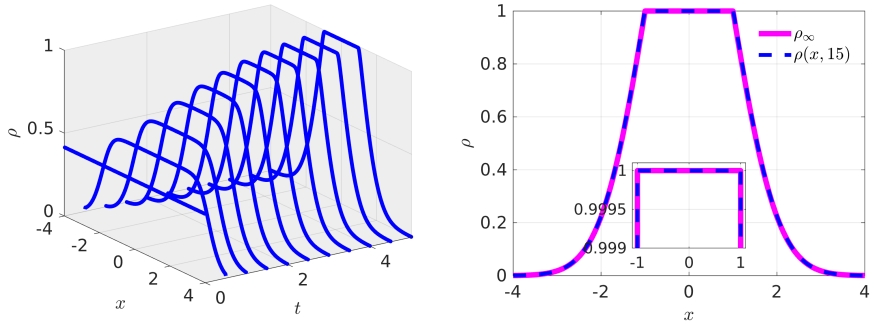

FIG. 4. *Computation of a non-smooth steady state to the one-dimensional saturation experiment* (4.4) *with* $\rho_0 = 0.415$. ***Left:*** *Evolution of* $\rho(x,t)$; ***Right:*** *Comparison between* ρ_∞ *and the numerical solution calculated at* $T = 15$.

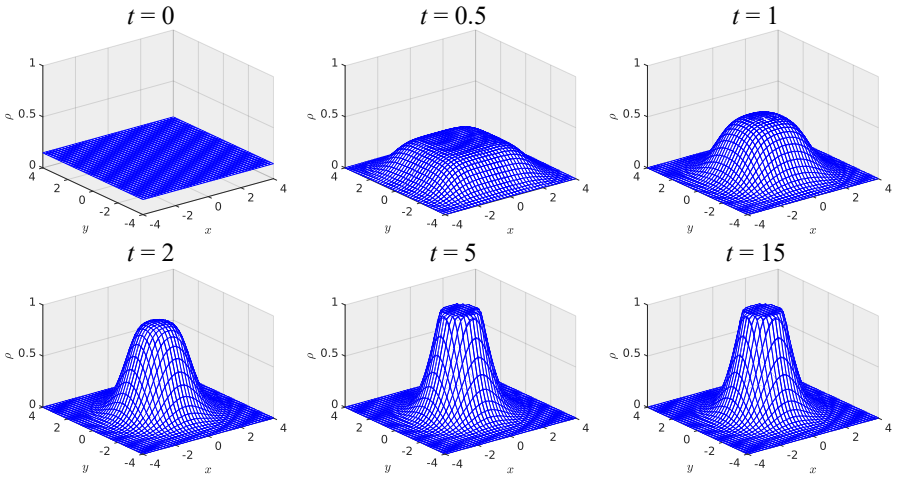

FIG. 5. *Evolution of a solution* $\rho(\boldsymbol{x},t)$ *with* $\rho_0 = 0.147$ *in a two-dimensional saturation experiment* (4.4).

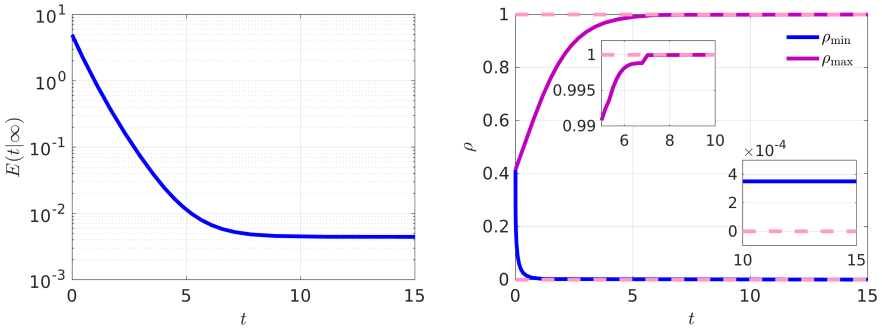

FIG. 6. *Computation of a non-smooth steady state to the one-dimensional saturation experiment* (4.4) *with* $\rho_0 = 0.415$. ***Left:*** *Behaviour of the relative entropy* $E(t;\infty)$; ***Right:*** *Behaviour of bound.*

TABLE 5
Accuracy test in one dimension for computing a solution to the equation (4.4) *with uniform initial density* $\rho_0 \equiv 0.415$. *The error is calculated at* $T = 1$.

N	L^1 error	Order	L^2 error	Order	L^∞ error	Order
20	3.022e-02		1.467e-02		1.158e-02	
40	7.603e-03	1.991	3.811e-03	1.944	4.007e-03	1.531
80	1.789e-03	2.088	8.774e-04	2.119	9.146e-04	2.131
160	3.669e-04	2.285	1.792e-04	2.291	1.775e-04	2.365

TABLE 6
Accuracy test in two dimensions for computing a solution to the equation (4.4) *with uniform initial density* $\rho_0 = 0.147$. *The error is calculated at* $T = 1$.

N	L^1 error	Order	L^2 error	Order	L^∞ error	Order
10×10	5.152e-01		8.897e-02		2.555e-02	
20×20	1.436e-01	1.843	2.483e-02	1.841	9.405e-03	1.442
40×40	3.032e-02	2.244	5.057e-03	2.296	2.059e-03	2.191

4.3. Aggregation–diffusion equation. We proceed with our study of the scheme (3.21) on the aggregation-diffusion equation

$$\rho_t = \nabla\rho\nabla\left(\frac{\nu m}{m-1}\rho^{m-1} + W * \rho\right) = \nu\Delta\rho^m + \nabla\rho\nabla(W * \rho) \tag{4.6}$$

with interaction kernel $W = e^{|\boldsymbol{x}|^2}/(2\pi)^{d/2}$. The parameters $\nu > 0$ and $m > 1$ are set to $\nu = 0.05$ and $m = 3$ in our computation. In this example, $f = \rho$, $F = \nu m\rho^{m-1}$, $V = 0$, and the density ρ remains positive. For our tests, we utilize periodic boundary conditions and evaluate the solution over the domain $[-6, 6]$ in one dimension and $[-4, 4]^2$ in two dimensions.

For the one-dimensional test, we compute the solution up to time $T = 200$, using a smooth initial datum $\rho_0 = \frac{1}{\sqrt{2\pi}}(e^{-(x-2)^2/2} + e^{-(x+2)^2/2})$. The evolution and behavior of entropy are plotted in Figure 7. As time progresses, the density starts at a smooth initial state with two peaks and then converges to a non-smooth steady state with only one peak and two discontinuity points. The entropy also shows dissipation throughout the computation. To ascertain the accuracy of this example, we compute the error at $T = 1$ and refer to a benchmark solution computed with a mesh of $N = 320$ points. Table 7 demonstrates the desired second-order accuracy of our scheme (3.21).

TABLE 7
Accuracy test in one dimension for computing a solution to the equation (4.6) *with* $\rho_0 = \frac{1}{\sqrt{2\pi}}(e^{-(x-2)^2/2} + e^{-(x+2)^2/2})$. *The error is calculated at* $T = 1$.

N	L^1 error	Order	L^2 error	Order	L^∞ error	Order
20	3.993e-02		1.489e-02		9.097e-03	
40	1.608e-02	1.312	5.739e-03	1.376	3.056e-03	1.574
80	4.900e-03	1.714	1.742e-03	1.720	9.306e-04	1.715
160	1.040e-03	2.236	3.772e-04	2.208	2.113e-04	2.139

In two dimensions, we consider a test with a discontinuous initial state given by $\rho_0(\boldsymbol{x}) = 1_{[-2,2]\times[-2,2]}(\boldsymbol{x})$. Figure 8 shows the dynamic evolution used to compute a

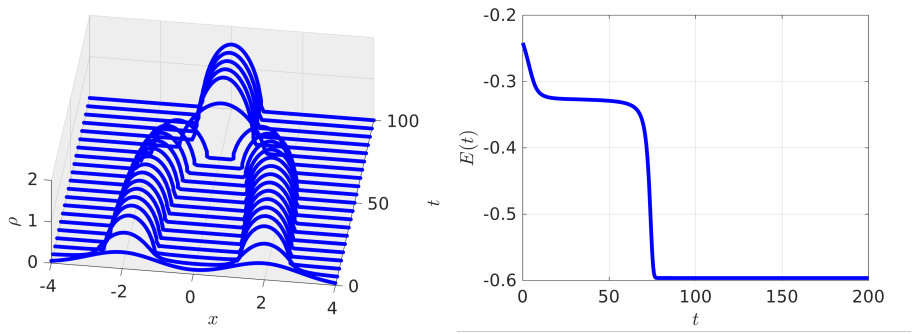

FIG. 7. *Computation of a steady state to the one-dimensional aggregation–diffusion equation* (4.6) *with* $\rho_0 = \frac{1}{\sqrt{2\pi}}(e^{-(x-2)^2/2} + e^{-(x+2)^2/2})$. ***Left:*** *Evolution of* $\rho(x,t)$; ***Right:*** *Behaviour of the discrete entropy.*

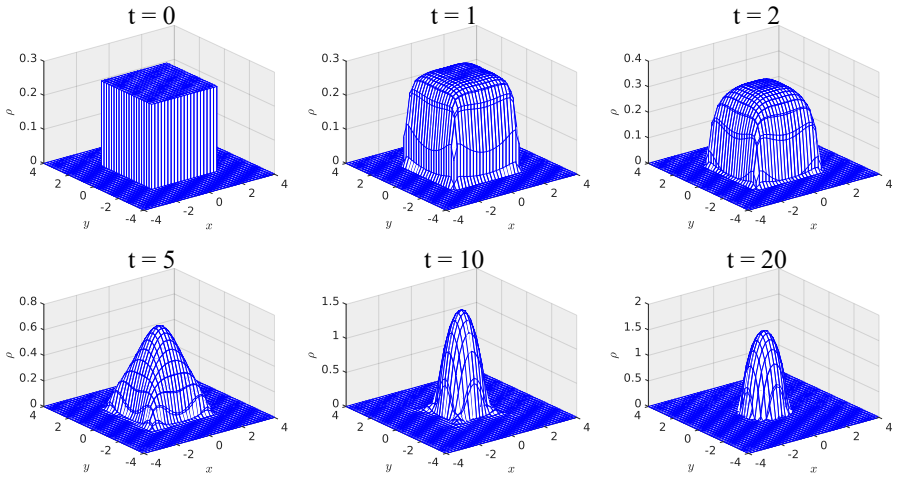

FIG. 8. *Evolution of a solution* $\rho(\boldsymbol{x},t)$ *to a two-dimensional aggregation–diffusion equation* (4.6) *with* $\rho_0 = 1_{[-2,2]\times[-2,2]}$.

solution at $T = 20$ with $N = 40 \times 40$ cells. In this setting, ρ exhibits a transition from a discontinuous distribution to a concentrated central peak. To test the accuracy in this two-dimensional case, we computed the error at $T = 1$ with a reference solution calculated using a mesh of $N = 80 \times 80$ cells. The error table and convergence rate are provided in Table 8. In this example, the L^1 error shows second-order accuracy, while the observed degeneracy in L^2 and L^∞ accuracy is due to the discontinuity in the initial state and the non-smoothness of the solution.

Conclusions. This paper presented a fully discrete scheme for solving a class of degenerate parabolic equations. Spatially, we applied a discontinuous Galerkin method to an appropriate reformulation of equations, resulting in an ODE with a splitting structure. Our numerical examples demonstrated that this spatial discretization induces fewer oscillations in the solution. Temporally, we proposed a second-order exponential scheme, which involves only linear solvers, by introducing integrating factors into the ODE and utilizing SSP-RK methods. Notably, our approach consistently

TABLE 8
Accuracy test in two dimensions for computing a solution to the equation (4.6) *with* $\rho_0 = 1_{[-2,2]\times[-2,2]}$. *The error is calculated at* $T = 1$.

N	L^1 error	Order	L^2 error	Order	L^∞ error	Order
10×10	1.674e+00		5.822e-01		2.587e-01	
20×20	4.250e-01	1.978	1.899e-01	1.616	1.188e-01	1.124
40×40	1.110e-01	1.937	6.151e-02	1.627	5.446e-02	1.125

evidences second-order accuracy, bound preservation and mass conservation with a favorable CFL condition $\tau \sim h$ both in theory and in practice. Nevertheless, the calculation of the exponential matrix partly limited our efficiency, and energy dissipation was only shown in our numerical tests but not proved theoretically. Indeed, devising an efficient linear scheme that simultaneously provides high-order accuracy in both space and time, while guaranteeing bound preservation, mass conservation, and energy dissipation, continues to be a major challenge.

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