

15TH WORKSHOP ON NON-PERTURBATIVE QUANTUM CHROMODYNAMICS

L'Institut d'Astrophysique de Paris
June 11-14, 2018
14eme arrondissement

Parton Distribution functions in Lattice QCD

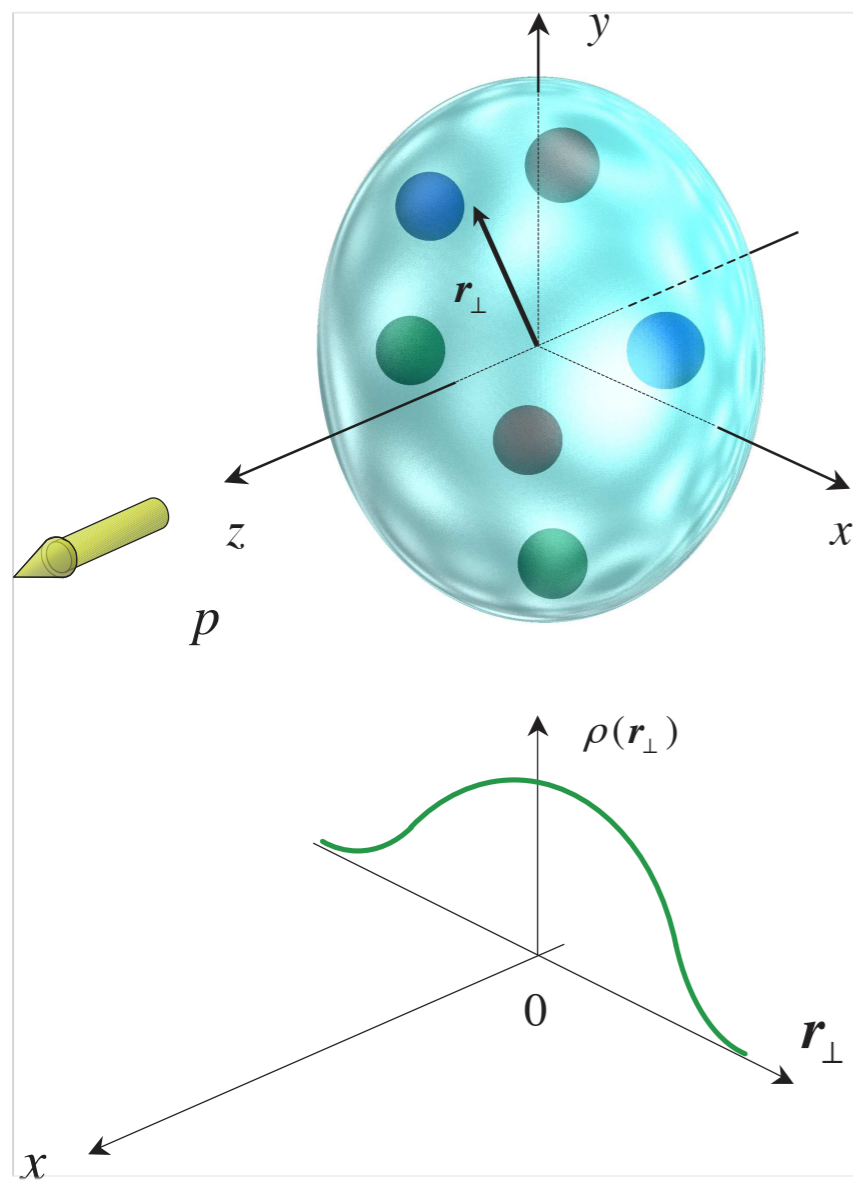
Kostas Orginos

Collaborators: *Joe Karpie, Anatoly Radyushkin, Savvas Zafeiropoulos*

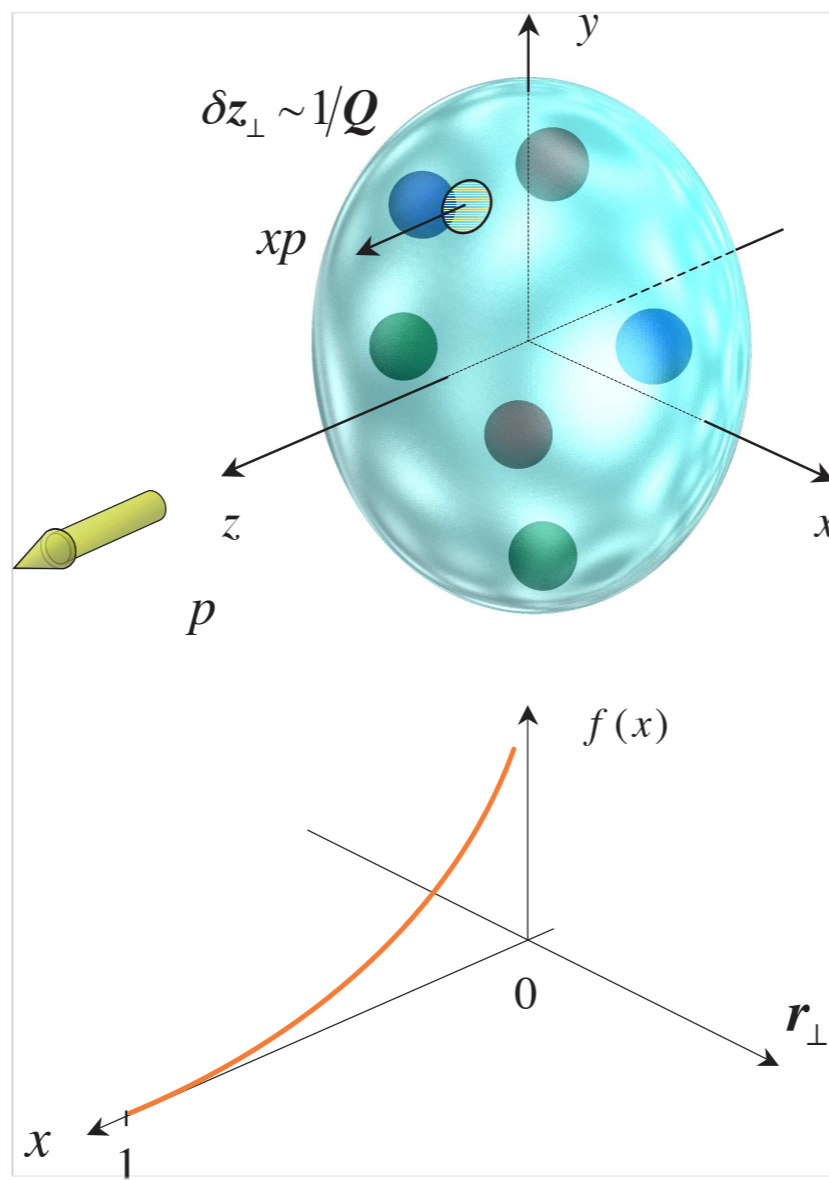
Introduction

- Quantum Chromodynamics: Theory of strong interactions
 - Describes the forces that bind together quarks to form hadrons such as the proton
- Non-linear and strongly coupled quantum field theory
- Proton is a relativistic many body system (partons)
 - It's structure is described in terms of parton densities
- Proton structure can be in principle accessed with theoretical computations
 - It requires numerical methods: Lattice QCD
- Proton structure is “universal”
 - Once determined it can be used to predict experimental results
 - It is currently determined experimentally and used as input to understand other experiments
 - Example: search for new physics at LHC

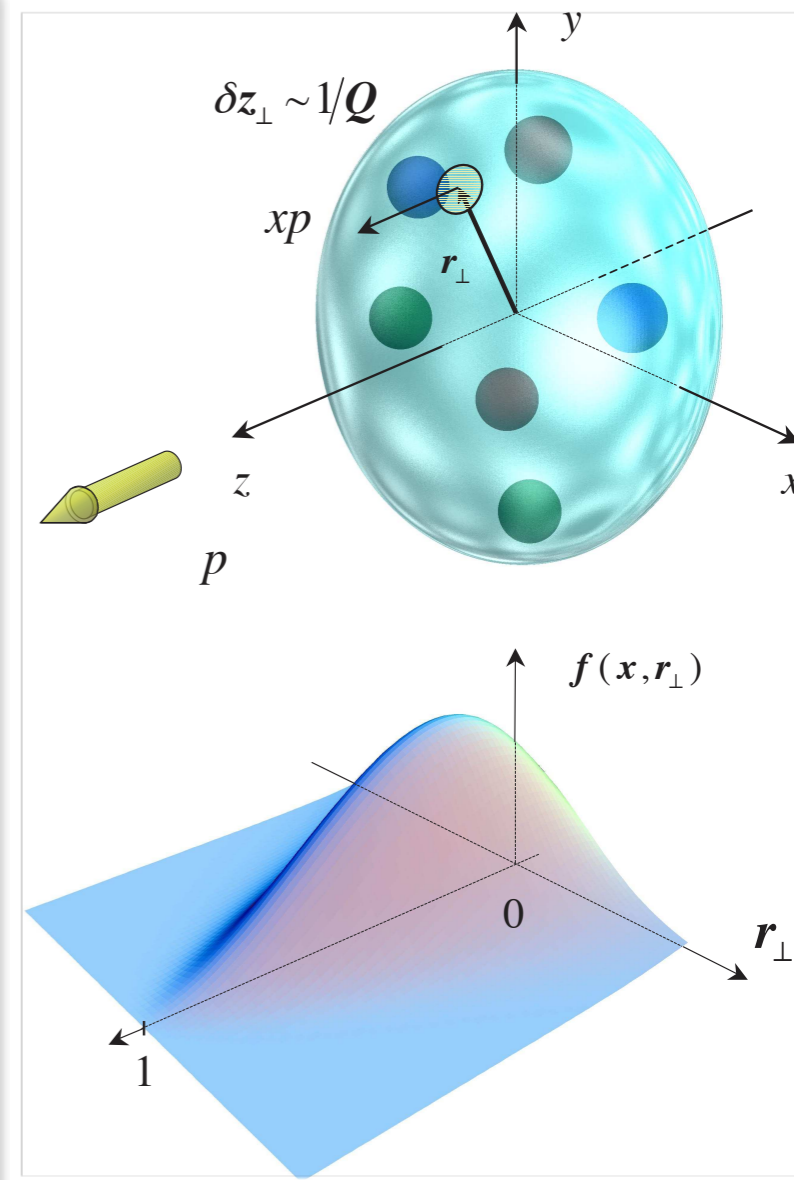
X. Ji, D. Muller, A. Radyushkin (1994-1997)



Form Factors

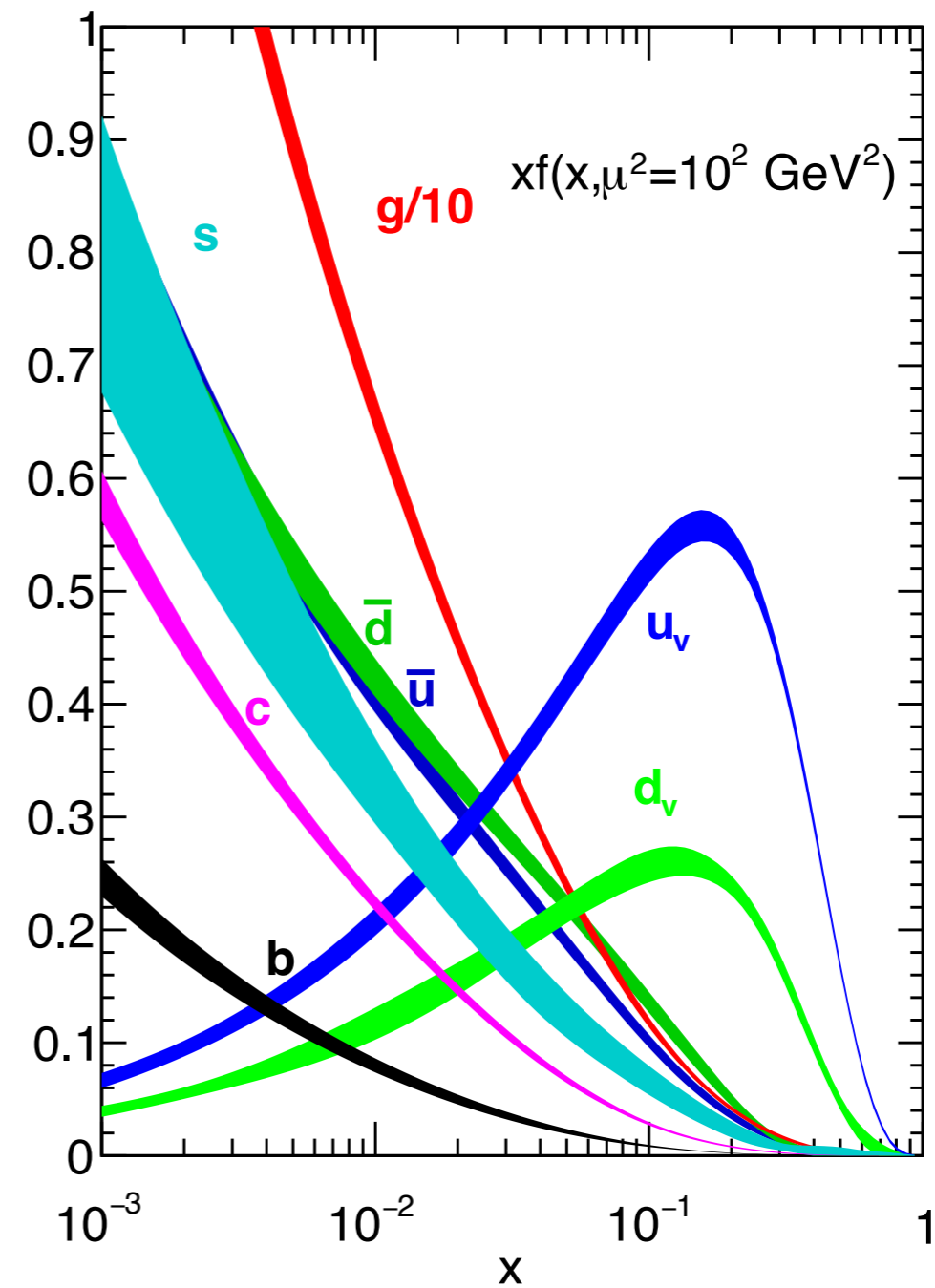
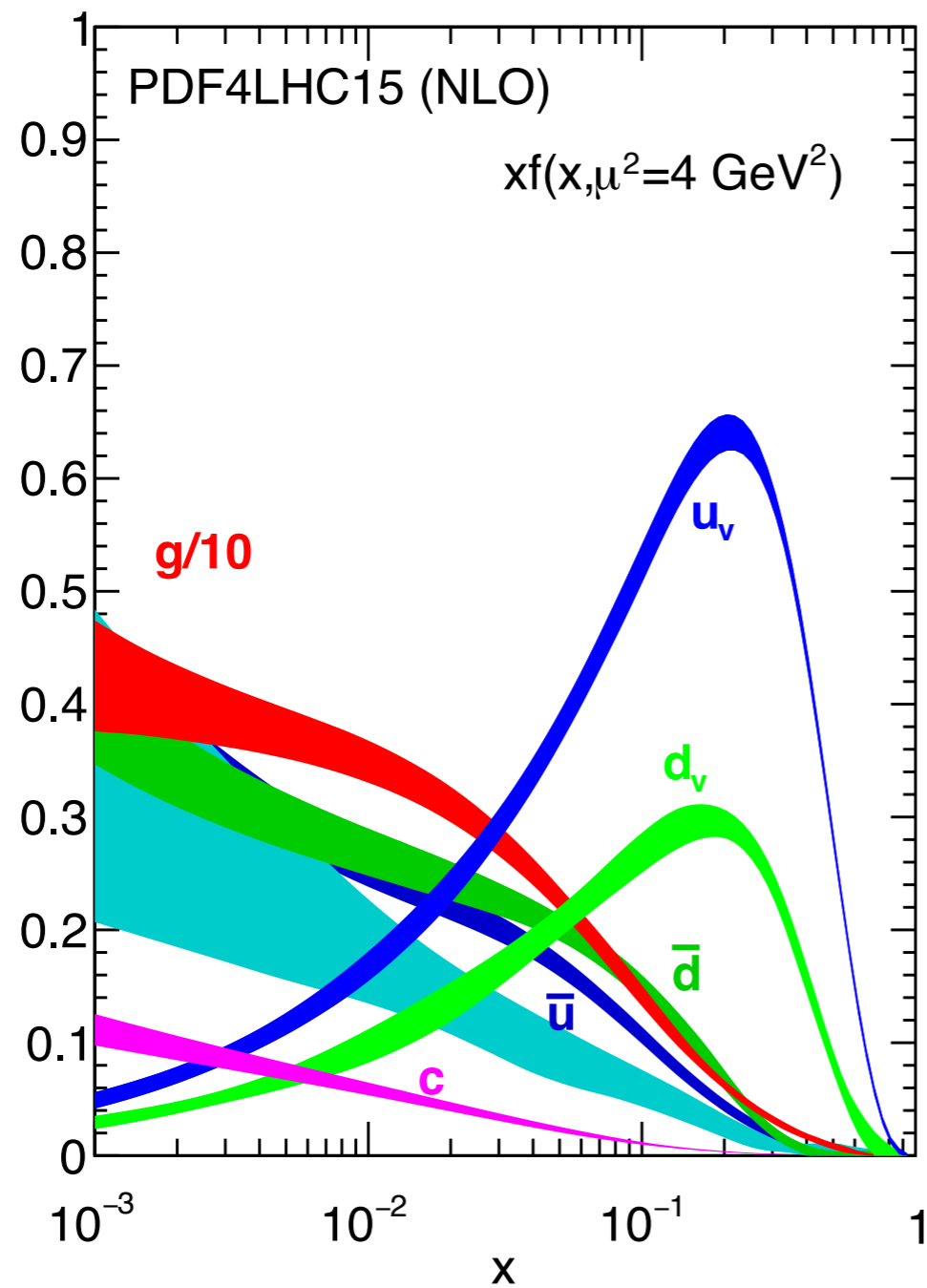


Parton Distribution functions



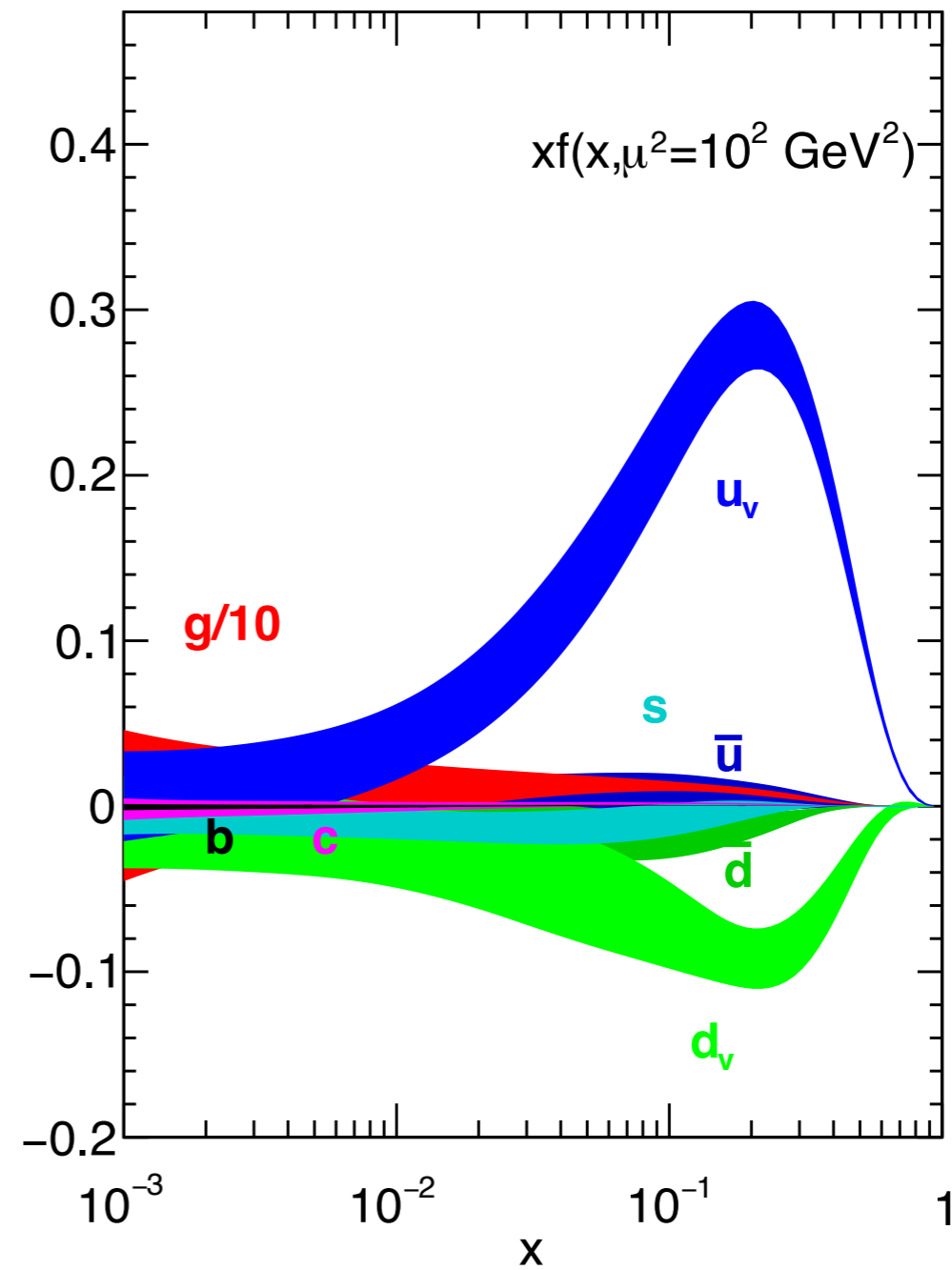
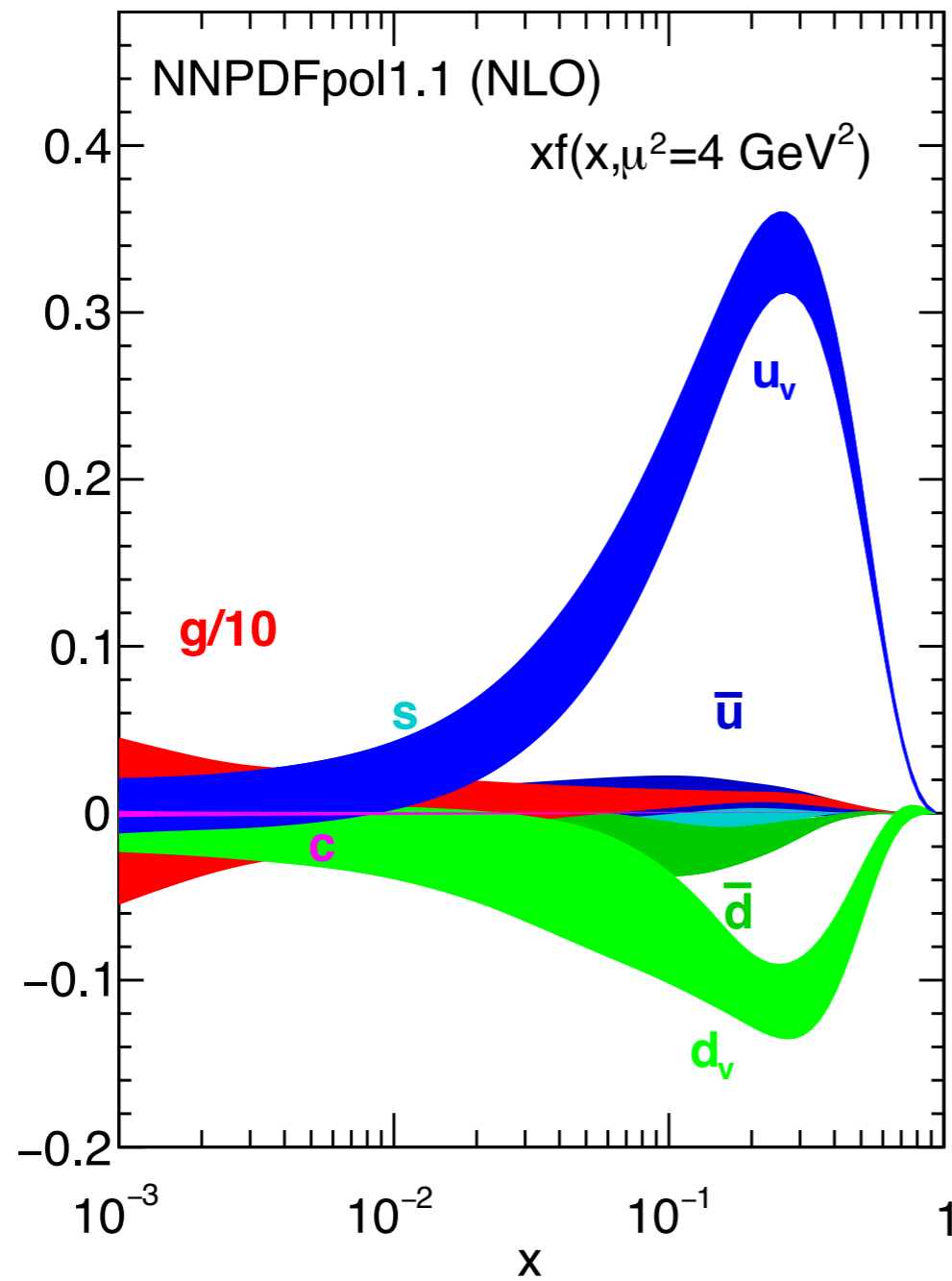
Generalized Parton Distribution functions

Determination of Parton distribution functions from Experiment



Fits to experimental data

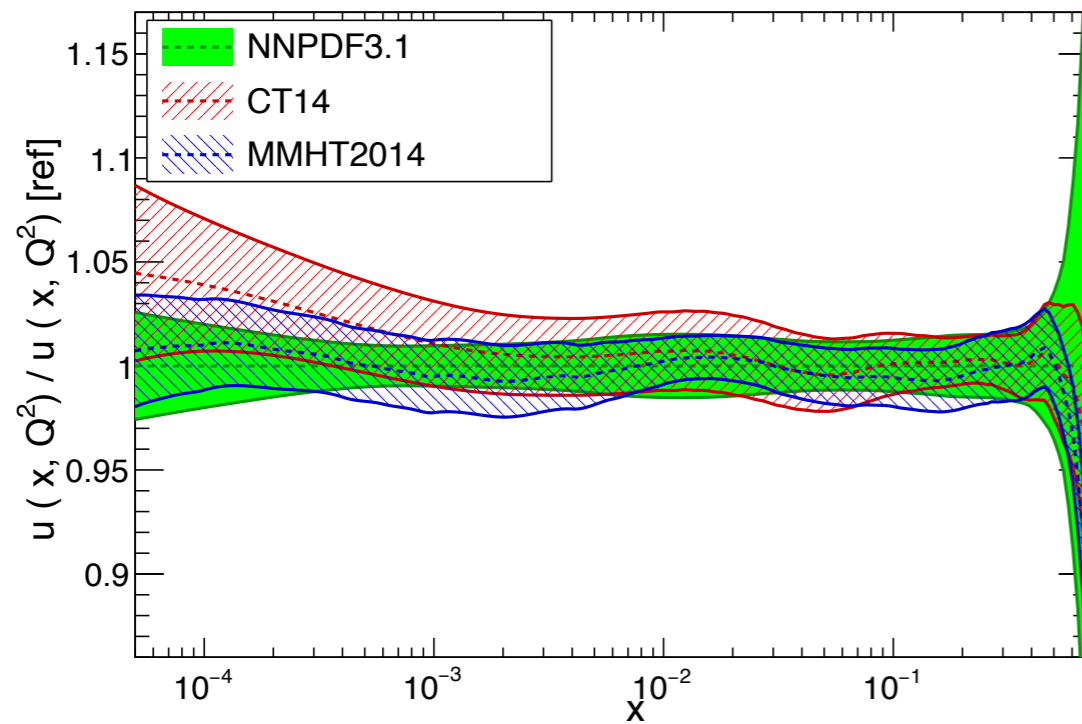
Determination of Parton distribution functions from Experiment



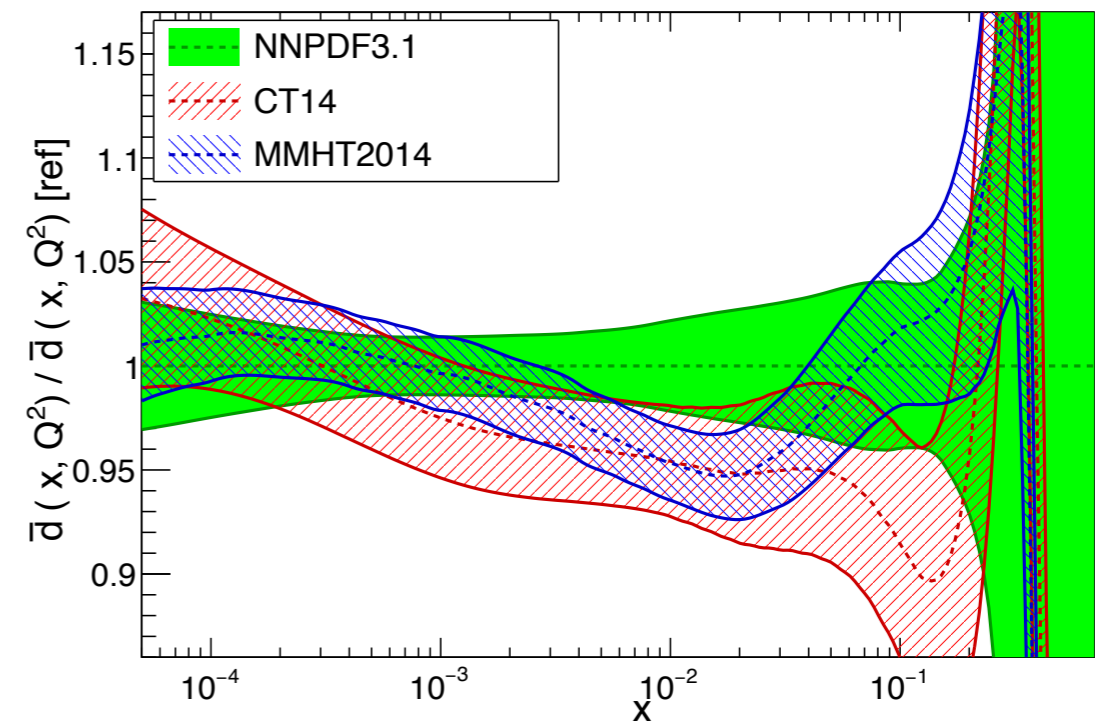
Fits to experimental data

Determination of Parton distribution functions from Experiment

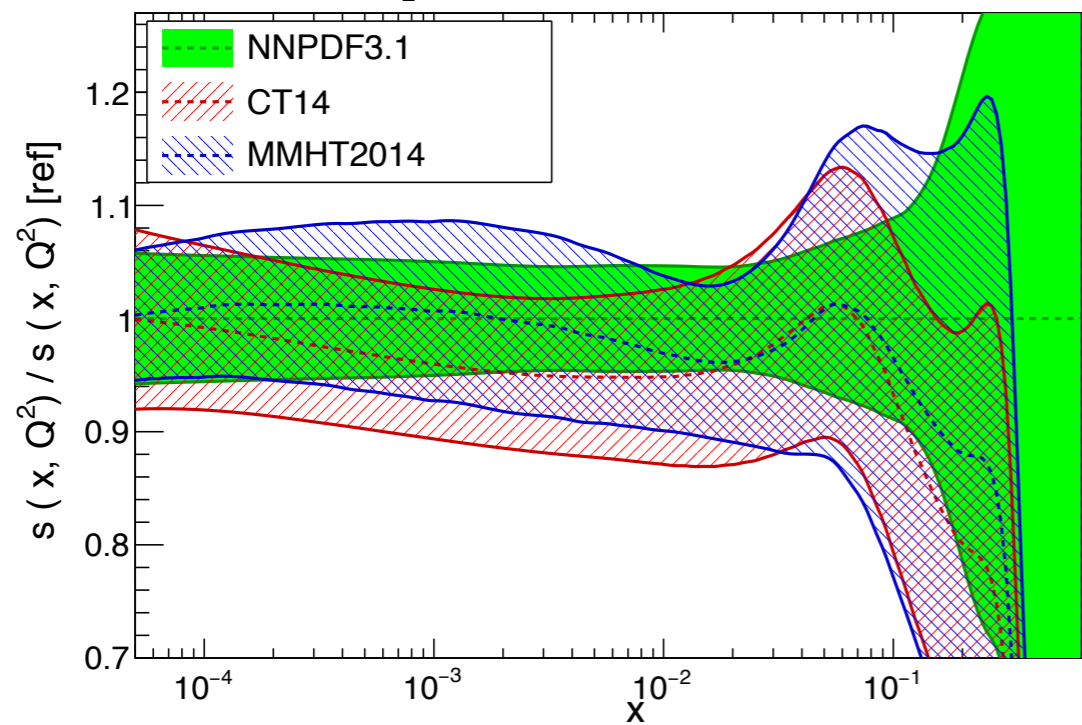
NNLO, $Q = 100$ GeV



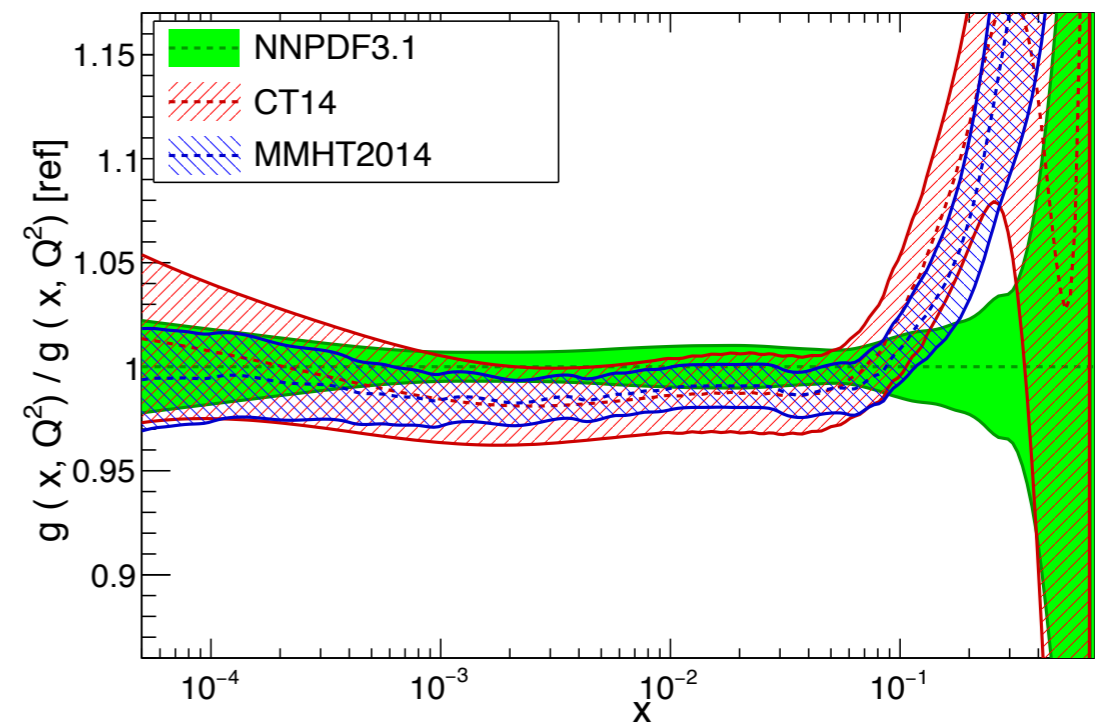
NNLO, $Q = 100$ GeV



NNLO, $Q = 100$ GeV



NNLO, $Q = 100$ GeV



Parton distributions and lattice QCD calculations: a community white paper

[arXiv:1711.07916](https://arxiv.org/abs/1711.07916)

Parton distributions and lattice QCD calculations: a community white paper

Huey-Wen Lin^{1,2}, Emanuele R. Nocera^{3,4}, Fred Olness⁵, Kostas Orginos^{6,7}, Juan Rojo^{8,9} (editors),
Alberto Accardi^{7,10}, Constantia Alexandrou^{11,12}, Alessandro Bacchetta¹³, Giuseppe Bozzi¹³,
Jiunn-Wei Chen¹⁴, Sara Collins¹⁵, Amanda Cooper-Sarkar¹⁶, Martha Constantinou¹⁷, Luigi Del Debbio⁴,
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Aleksander Kusina²⁴, Keh-Fei Liu²⁵, Simonetta Liuti^{26,27}, Christopher Monahan²⁸, Pavel Nadolsky⁵,
Jian-Wei Qiu⁷, Ingo Schienbein²³, Gerrit Schierholz²⁹, Robert S. Thorne²¹, Werner Vogelsang³⁰,
Hartmut Wittig³¹, C.-P. Yuan¹, and James Zanotti³²

[arXiv:1711.07916](https://arxiv.org/abs/1711.07916)

GPDs: Definition

GPDs:

$$\bar{u}(P') \left(\gamma^+ H(x, \xi, t) + i \frac{\sigma^{+k} \Delta_k}{2m} E(x, \xi, t) \right) = \int_{-\infty}^{\infty} \frac{d\omega^-}{4\pi} e^{-i\xi P^+ \omega^-} \left\langle P' \left| T \bar{\psi}(0, \omega^-, \mathbf{0}_T) W(\omega^-, 0) \gamma^+ \frac{\lambda^a}{2} \psi(0) \right| P \right\rangle_{\mathbf{C}}$$

$$W(\omega^-, 0) = \mathcal{P} \exp \left[-ig_0 \int_0^{\omega^-} dy^- A_{\alpha}^+(0, y^-, \mathbf{0}_T) T_{\alpha} \right]$$

$$\langle P' | P \rangle = (2\pi)^3 2P^+ \delta(P^+ - P'^+) \delta^{(2)}(\mathbf{P}_T - \mathbf{P}'_T)$$

$$\Delta = P' - P$$

$$t = \Delta^2$$

Moments:

$$\int_{-1}^1 dx x^{n-1} \begin{bmatrix} H(x, \xi, t) \\ E(x, \xi, t) \end{bmatrix} = \sum_{k=0}^{[(n-1)/2]} (2\xi)^{2k} \begin{bmatrix} A_{n,2k}(t) \\ B_{n,2k}(t) \end{bmatrix} \pm \delta_{n,\text{even}} (2\xi)^n C_n(t).$$

Matrix elements of twist-2 operators $\mathcal{O}_0^{\{\mu_1 \dots \mu_n\}} = i^{n-1} \bar{\psi}(0) \gamma^{\{\mu_1} D^{\mu_2} \dots D^{\mu_n\}} \frac{\lambda^a}{2} \psi(0) - \text{traces}$

PDFs: Definition

Light-cone PDFs:

$$f^{(0)}(\xi) = \int_{-\infty}^{\infty} \frac{d\omega^-}{4\pi} e^{-i\xi P^+ \omega^-} \left\langle P \left| T \bar{\psi}(0, \omega^-, \mathbf{0}_T) W(\omega^-, 0) \gamma^+ \frac{\lambda^a}{2} \psi(0) \right| P \right\rangle_{\text{C}}.$$

$$W(\omega^-, 0) = \mathcal{P} \exp \left[-ig_0 \int_0^{\omega^-} dy^- A_{\alpha}^+(0, y^-, \mathbf{0}_T) T_{\alpha} \right] \quad \langle P' | P \rangle = (2\pi)^3 2P^+ \delta(P^+ - P'^+) \delta^{(2)}(\mathbf{P}_T - \mathbf{P}'_T)$$

Moments:

$$a_0^{(n)} = \int_0^1 d\xi \xi^{n-1} \left[f^{(0)}(\xi) + (-1)^n \bar{f}^{(0)}(\xi) \right] = \int_{-1}^1 d\xi \xi^{n-1} f(\xi)$$

Local matrix elements:

$$\left\langle P \left| \mathcal{O}_0^{\{\mu_1 \dots \mu_n\}} \right| P \right\rangle = 2a_0^{(n)} (P^{\mu_1} \dots P^{\mu_n} - \text{traces}) \quad \mathcal{O}_0^{\{\mu_1 \dots \mu_n\}} = i^{n-1} \bar{\psi}(0) \gamma^{\{\mu_1} D^{\mu_2} \dots D^{\mu_n\}} \frac{\lambda^a}{2} \psi(0) - \text{traces}$$

Introduction (cont.)

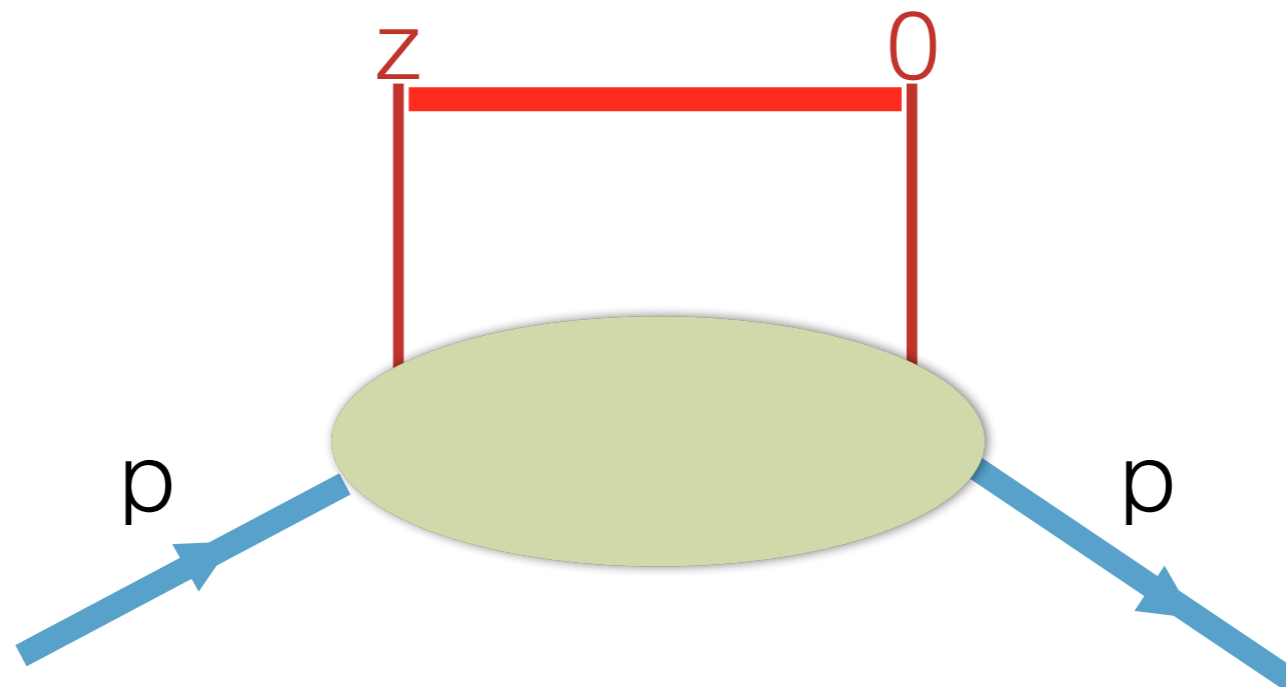
- Goal: Compute hadron structure properties from QCD
 - Parton distribution functions (PDFs)
- Operator product: Mellin moments are local matrix elements that can be computed in Lattice QCD
 - Power divergent mixing limits us to few moments
- Few years ago X. Ji suggested an approach for obtaining PDFs from Lattice QCD
- First calculations already available
 - X. Ji, Phys.Rev.Lett. 110, (2013)*
 - Y.-Q. Ma J.-W. Qiu (2014) 1404.6860*
 - H.-W. Lin, J.-W. Chen, S. D. Cohen, and X. Ji, Phys.Rev. D91, 054510 (2015)*
 - C. Alexandrou, et al, Phys. Rev. D92, 014502 (2015)*
- A new approach for obtaining PDFs from LQCD introduced by A. Radyushkin
 - A. Radyushkin Phys.Lett. B767 (2017)*
- Hadronic tensor methods
 - K-F Liu et al Phys. Rev. Lett. 72 (1994) , Phys. Rev. D62 (2000) 074501*
 - Detmold and Lin 2005*
 - M. T. Hansen et al arXiv:1704.08993.*
 - UKQCD-QCDSF-CSSM Phys. Lett. B714 (2012), arXiv:1703.01153*
 - Ma and Qiu : [arXiv:1709.03018](https://arxiv.org/abs/1709.03018)*

Pseudo-PDFs

Unpolarized PDFs proton:

$$\mathcal{M}^\alpha(z, p) \equiv \langle p | \bar{\psi}(0) \gamma^\alpha \hat{E}(0, z; A) \psi(z) | p \rangle$$

$$\hat{E}(0, z; A) = \mathcal{P} \exp \left[-ig \int_0^z dz'_\mu A_\alpha^\mu(z') T_\alpha \right]$$



Lorentz decomposition:

$$\mathcal{M}^\alpha(z, p) \equiv \langle p | \bar{\psi}(0) \gamma^\alpha \hat{E}(0, z; A) \psi(z) | p \rangle$$

$$\mathcal{M}^\alpha(z, p) = 2p^\alpha \mathcal{M}_p(-(zp), -z^2) + z^\alpha \mathcal{M}_z(-(zp), -z^2)$$

$$z = (0, z_-, 0)$$

Collinear PDFs: Choose

$$p = (p_+, 0, 0)$$

$$\gamma^+$$

$$\mathcal{M}^+(z, p) = 2p^+ \mathcal{M}_p(-p_+ z_-, 0)$$

Definition of PDF:

$$\mathcal{M}_p(-p_+ z_-, 0) = \int_{-1}^1 dx f(x) e^{-ixp_+ z_-}$$

$$\mathcal{M}_p(-pz, -z^2)$$

is a Lorentz invariant therefore
computable in any frame

$$\nu = -zp$$

ν is called Ioffe time

B. L. Ioffe, Phys. Lett. 30B, 123 (1969)

$$\mathcal{M}_p(\nu, -z^2) \equiv \int_{-1}^1 dx \mathcal{P}(x, -z^2) e^{ix\nu}$$

$$\mathcal{P}(x, 0) = f(x)$$

It can be shown that the domain of x is $[-1, 1]$

A. Radyushkin Phys. Lett. B767 (2017)

One can obtain PDFs in the limit of $z^2 \rightarrow 0$

This limit is singular but using OPE, PDFs are defined

$$\mathcal{M}_p(x, -z^2) = \int_0^1 du \mathcal{C}(u, z^2 \mu^2, \alpha_s(\mu)) \mathcal{Q}(u\nu, \mu) + \mathcal{O}(z^2 \Lambda_{qcd}^2)$$

$\mathcal{Q}(\nu, \mu)$ is called the Ioffe time PDF

V. Braun, et. al Phys. Rev. D 51, 6036 (1995)

$$\mathcal{Q}(\nu, \mu) = \int_{-1}^1 dx e^{-ix\nu} f(x, \mu)$$

Rossi & Testa argue that in lattice computations $1/a$ divergences may hide in the polynomial terms.

Rossi & Testa: PhysRev D 96, 014507 (2017), arXiv:1806.00808

Lattice QCD calculation:

$$\mathcal{M}^\alpha(z, p) \equiv \langle p | \bar{\psi}(0) \gamma^\alpha \hat{E}(0, z; A) \psi(z) | p \rangle$$

Choose

$$p = (p_0, 0, 0, p_3)$$

$$z = (0, 0, 0, z_3)$$

$$\gamma^0$$

On shell equal time matrix element
computable in Euclidean space

Briceno *et al* arXiv:1703.06072

Obtaining only the relevant

$$\mathcal{M}_p(\nu, z_3^2) = \frac{1}{2p_0} \mathcal{M}^0(z_3, p_3)$$

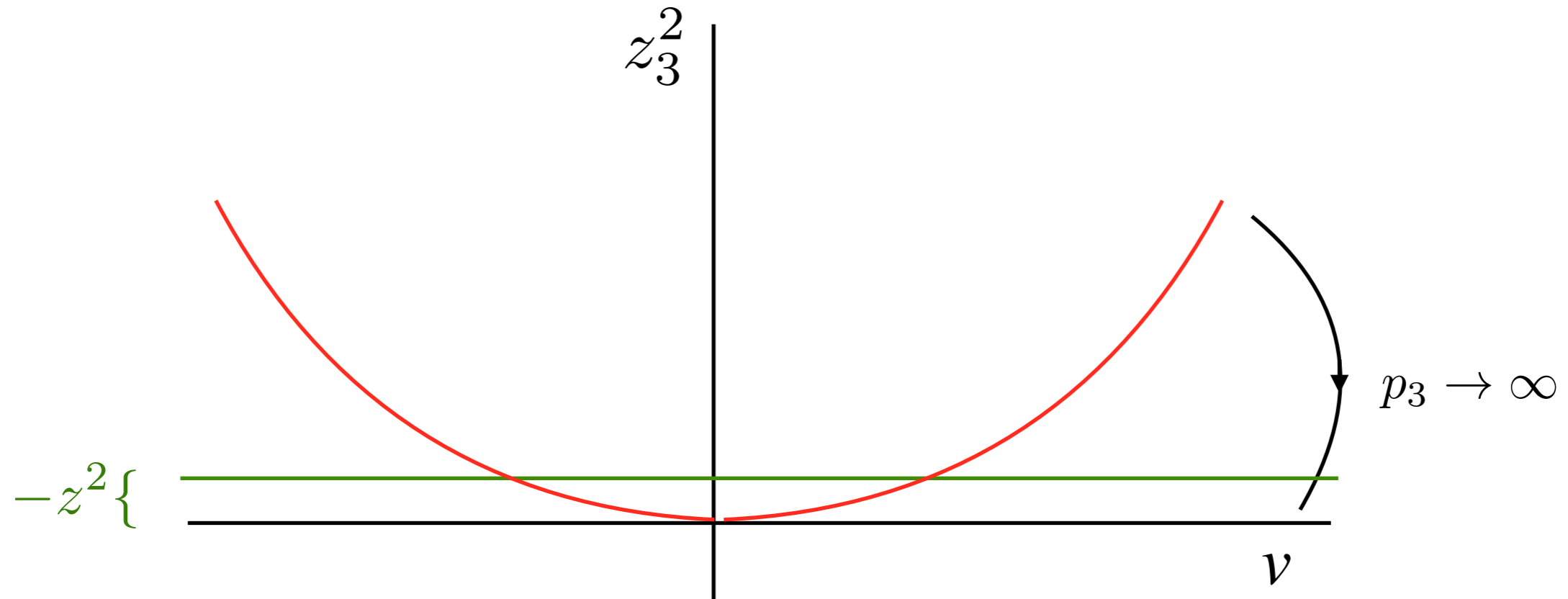
Choosing γ^0 was also suggested also by M. Constantinou at GHP2017 based on an operator mixing argument for the renormalized matrix element

Alexandrou *et al* arXiv:1706.00265

$$Q(y, p_3) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu \mathcal{M}_p(\nu, \nu^2/p_3^2) e^{-iy\nu} \quad \text{Ji's quasi-PDF}$$

Large values of $z_3 = \nu/p_3$ are problematic

Alternative approach to the light-cone:



$$\mathcal{P}(x, -z^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu \mathcal{M}_p(\nu, -z^2) e^{-ix\nu}$$

$-z^2 \rightarrow 0$ PDFs can be recovered

Lattice QCD requirements

$$aP_{max} = \frac{2\pi}{4} \sim \mathcal{O}(1)$$

$$a \sim 0.1 fm \rightarrow P_{max} = 10\Lambda$$

$$\Lambda \sim 300 MeV$$

$$a \sim 0.05 fm \rightarrow P_{max} = 20\Lambda$$

For practical calculations large momentum is needed

*Higher twist effect suppression (qpdfs)

*Wide coverage of Ioffe time ν

$P = 3$ GeV is already demanding due to statistical noise
achievable with easily accessible lattice spacings

$P = 6$ GeV exponentially harder
requires current state of the art lattice spacing

Statistical noise

Nucleon with momentum P two-point function:

$$C_{2p}(P, t) = \langle O_N(P, t) O_N^\dagger(P, 0) \rangle \sim \mathcal{Z} e^{-E(P)t}$$

Variance of nucleon two-point function:

$$\text{var} [C_{2p}(P, t)] = \langle O_N(P, t) O_N(P, t)^\dagger O_N(P, 0) O_N^\dagger(P, 0) \rangle \sim \mathcal{Z}_{3\pi} e^{-3m_\pi t}$$

Variance is independent of the momentum

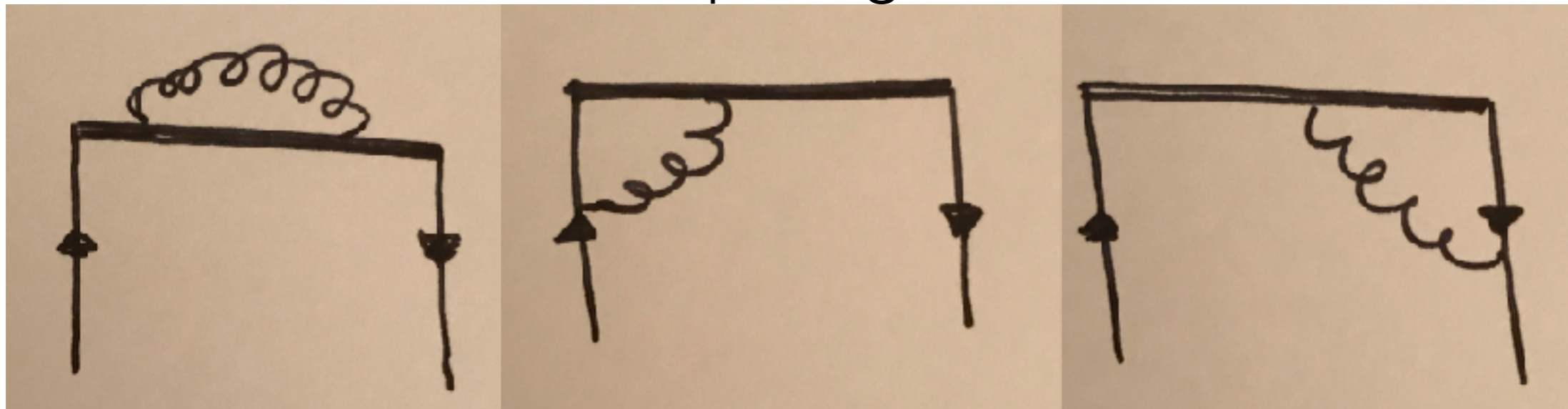
$$\frac{\text{var} [C_{2p}(P, t)]^{1/2}}{C_{2p}(P, t)} \sim \frac{\mathcal{Z}}{\mathcal{Z}_{3\pi}} e^{-[E(P) - 3/2m_\pi]t}$$

Statistical accuracy drops exponentially with the increasing momentum limiting the maximum achievable momentum.

Renormalization

$$\mathcal{M}_{ren}^0(z, p, \mu) = \lim_{a \rightarrow 0} Z_{\mathcal{O}}(z, \mu, a) \mathcal{M}^0(z, P, a)$$

One loop diagrams



Linear divergence

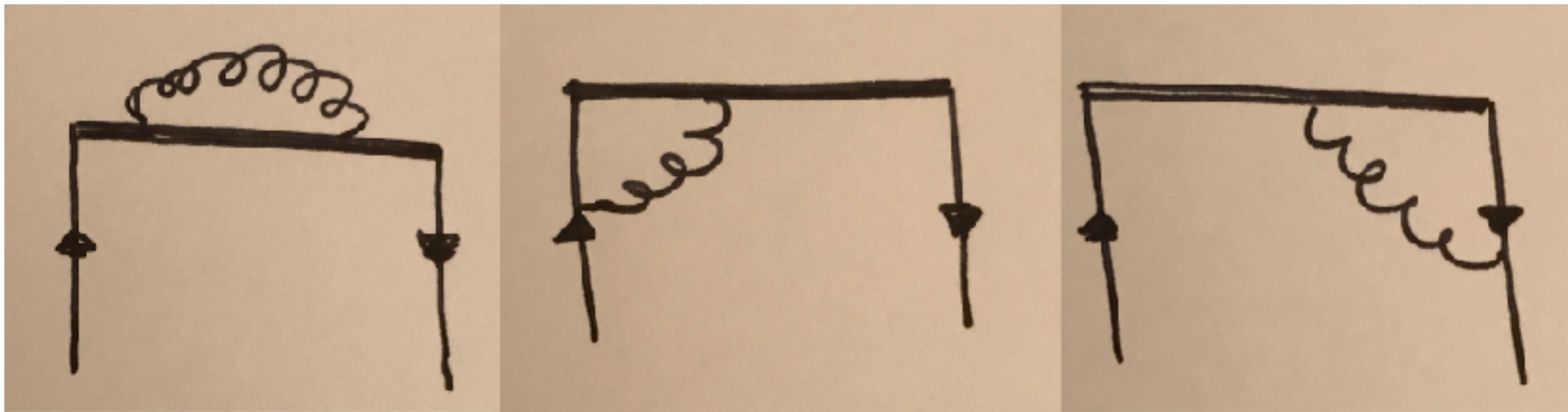
Logarithmic divergence

Dotsenko Nucl.Phys. B169 (1980) 527

Chen et al. Nucl.Phys. B915 (2017)

Ishikawa et al. arXiv:1707.03107, arXiv:1609.02018

Radyushkin arXiv:1710.08813



One loop calculation of the UV divergences results in

$$\mathcal{M}^0(z, P, a) \sim e^{-m|z|/a} \left(\frac{a^2}{z^2} \right)^{2\gamma_{end}}$$

after re-summation of one loop result resulting exponentiation

- J.G.M.Gatheral, Phys.Lett.133B,90(1983)
- J.Frenkel, J.C.Taylor, Nucl.Phys.B246,231(1984),
- G.P.Korchensky, A.V.Radyushkin, Nucl.Phys.B283,342(1987).

Multiplicatively renormalizable

Consider the ratio $\mathfrak{M}(\nu, z_3^2) \equiv \frac{\mathcal{M}_p(\nu, z_3^2)}{\mathcal{M}_p(0, z_3^2)}$

UV divergences will cancel in this ratio resulting a renormalization group invariant (RGI) function

The lattice regulator can now be removed

$\mathfrak{M}^{cont}(\nu, z_3^2)$ Universal independent of the lattice

$\mathcal{M}_p(0, 0) = 1$ Isovector matrix element

$$\mathfrak{M}(\nu, z^2) = \int_0^1 d\alpha \mathfrak{E}(\alpha, z^2 \mu^2, \alpha_s(\mu)) \mathcal{Q}(\alpha\nu, \mu) + \sum_{k=1}^{\infty} \mathcal{B}_k(\nu) (z^2)^k$$

$$\mathcal{B}_k(\nu) (z^2)^k \sim \mathcal{O}(\Lambda_{qcd}^{2k})$$

Polynomial corrections to the Ioffe time PDF may be suppressed

B. U. Musch, *et al* Phys. Rev. D 83, 094507 (2011)

M. Anselmino *et al.* 10.1007/JHEP04(2014)005

A. Radyushkin Phys.Lett. B767 (2017)

Rossi & Testa argument may not apply here if we use

$$\mathfrak{M}^{cont}(\nu, z_3^2)$$

Rossi & Testa: PhysRev D 96, 014507 (2017), arXiv:1806.00808

Possible mechanism for polynomial correction suppression

Approximate TMD factorization

A. Radyushkin Phys.Lett. B767 (2017)

M. Anselmino et al. 10.1007/JHEP04(2014)005

B. U. Musch, *et al* Phys. Rev. D 83, 094507 (2011)

$$\mathcal{M}_p(\nu, -z^2) \equiv \int_{-1}^1 dx \mathcal{P}(x, -z^2) e^{ix\nu}$$

Taking $z = (0, z_-, z_\perp)$ we can identify $\mathcal{P}(x, z_\perp^2) = \int d^2 k_\perp \mathcal{F}(x, k_\perp^2) e^{ik_\perp z_\perp}$

$\mathcal{F}(x, k_\perp^2)$ the primordial TMD

Assuming $\mathcal{F}(x, k_\perp^2) = f(x)g(k_\perp^2)$ we obtain $\mathcal{P}(x, z_\perp^2) = f(x)\tilde{g}(z_\perp^2)$

Implying that $\mathcal{M}_p(\nu, -z^2) = \mathcal{Q}(\nu, -z^2)\mathcal{M}_p(0, -z^2)$

where $\mathcal{M}_p(0, -z^2) = \tilde{g}(-z^2)$

$$\mu^2 \frac{d}{d\mu^2} \mathcal{Q}(\nu, \mu^2) = - \frac{2}{3} \frac{\alpha_s}{2\pi} \int_0^1 du B(u) \mathcal{Q}(u\nu, \mu^2)$$

$$B(u) = \left[\frac{1+u^2}{1-u} \right]_+$$

DGLAP kernel in position space

V. Braun, et. al Phys. Rev. D 51, 6036 (1995)

At 1-loop

$$\mathcal{Q}(\nu, \mu'^2) = \mathcal{Q}(\nu, \mu^2) - \frac{2}{3} \frac{\alpha_s}{2\pi} \ln(\mu'^2 / \mu^2) \int_0^1 du B(u) \mathcal{Q}(u\nu, \mu^2)$$

Matching to \overline{MS} computed at 1-loop

$$\mathfrak{M}(\nu, z^2) = \int_0^1 d\alpha \mathfrak{E}(\alpha, z^2 \mu^2, \alpha_s(\mu)) \mathcal{Q}(\alpha\nu, \mu) + \sum_{k=1}^{\infty} \mathcal{B}_k(\nu) (z^2)^k$$

Radyushkin 1801.02427
Zhang et al. arXiv 1801.03032

Numerical Tests

with

J. Karpie, A. Radyushkin, S. Zafeiropoulos

[Phys.Rev. D96 \(2017\) no.9, 094503](#)

Numerical Tests

- Quenched approximation $\beta=6.0$
 $32^3 \times 64 \quad m_\pi \sim 600 \text{MeV}$
- Need series of small z_3
- Need a range of momenta to scan ν
- Goals:
 - Check polynomial corrections
 - Understand the systematics of the approach

Matrix element calculation

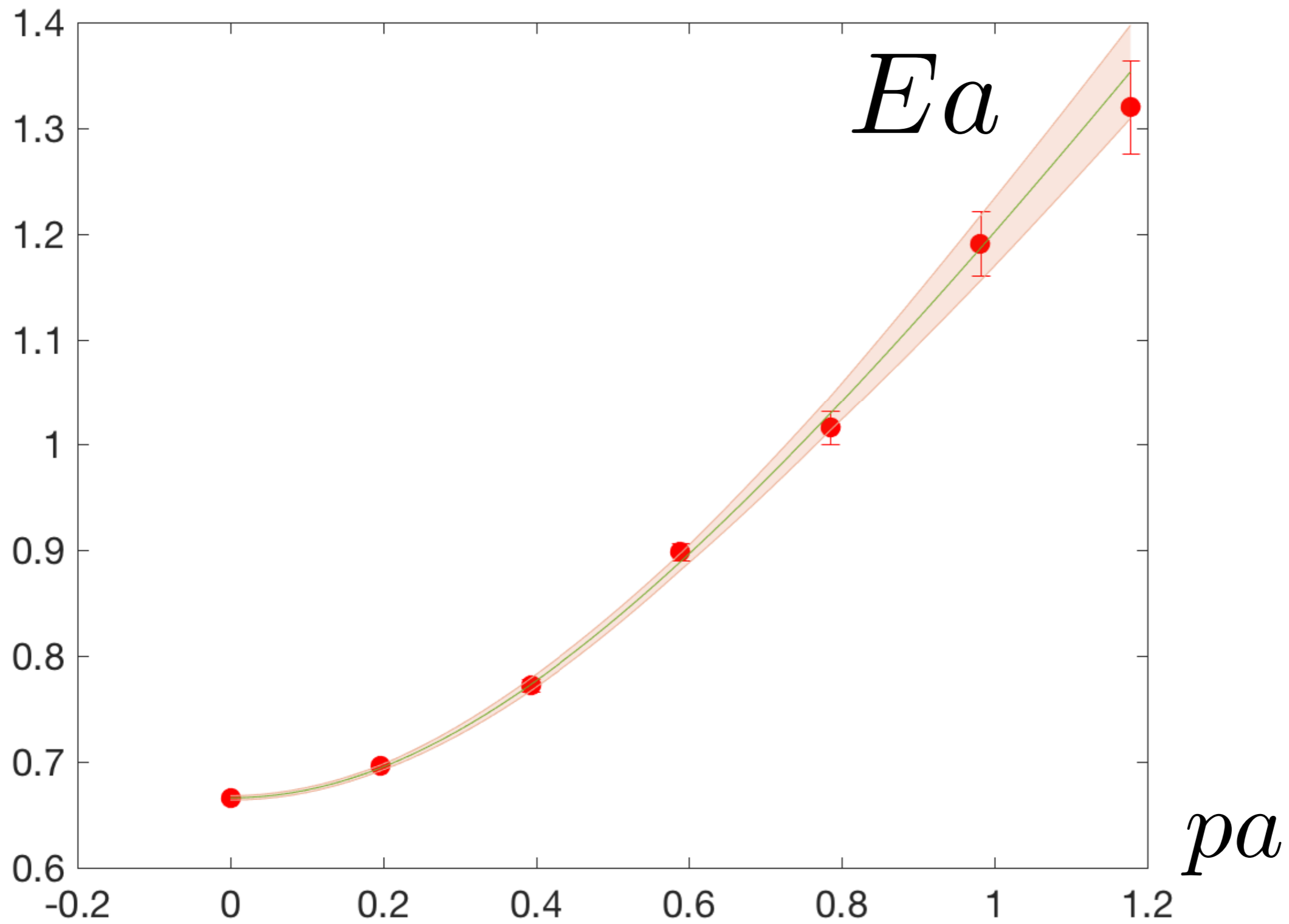
$$C_P(t) = \langle \mathcal{N}_P(t) \overline{\mathcal{N}}_P(0) \rangle \quad C_P^{\mathcal{O}^0(z)}(t) = \langle \mathcal{N}_P(t) \mathcal{O}^0(z) \overline{\mathcal{N}}_P(0) \rangle$$

$$\mathcal{M}_{\text{eff}}(z_3 P, z_3^2; t) = \frac{C_P^{\mathcal{O}^0(z)}(t+1)}{C_P(t+1)} - \frac{C_P^{\mathcal{O}^0(z)}(t)}{C_P(t)}$$

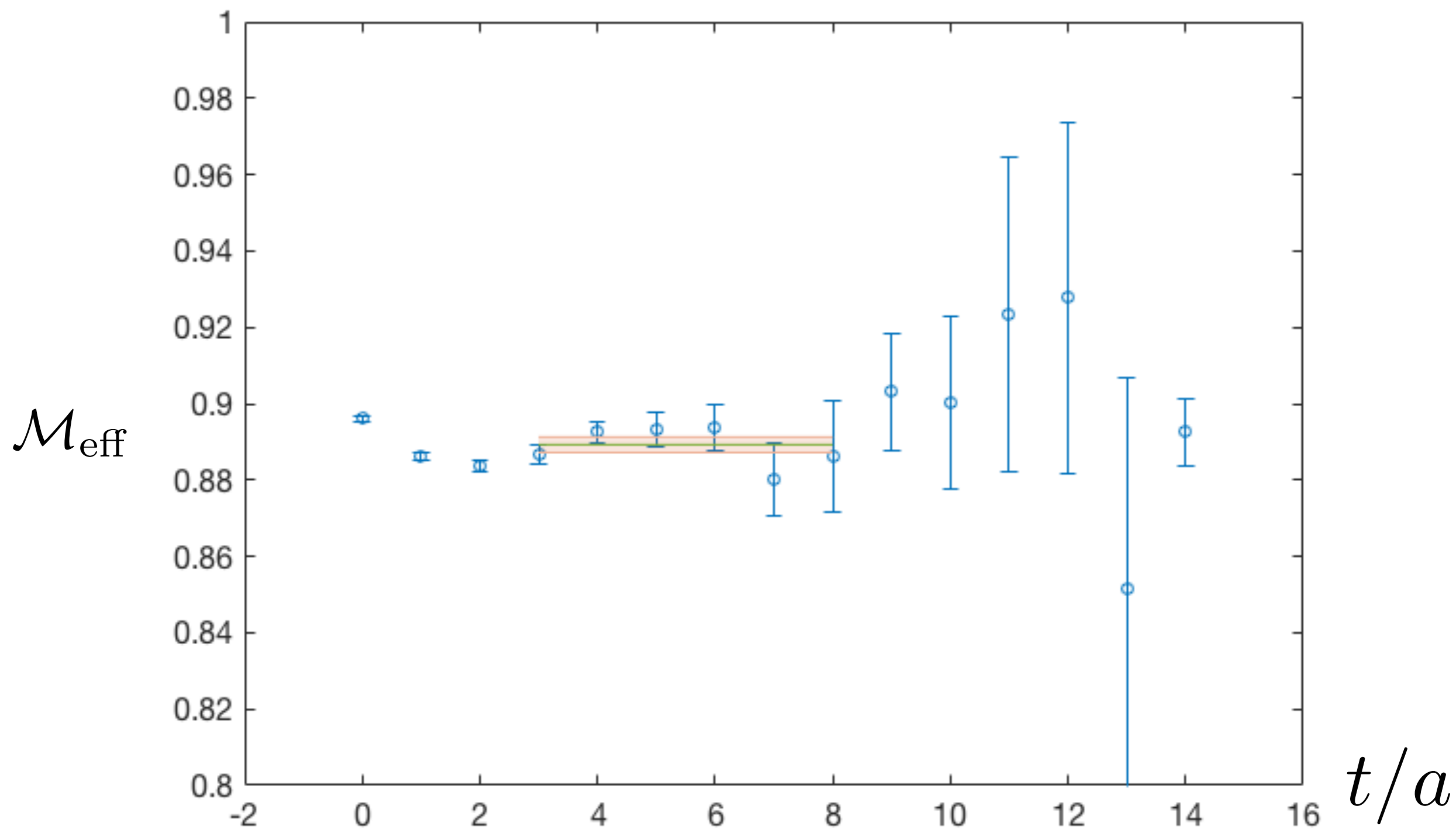
C. Bouchard, et al arXiv:1612.06963 [hep-lat]

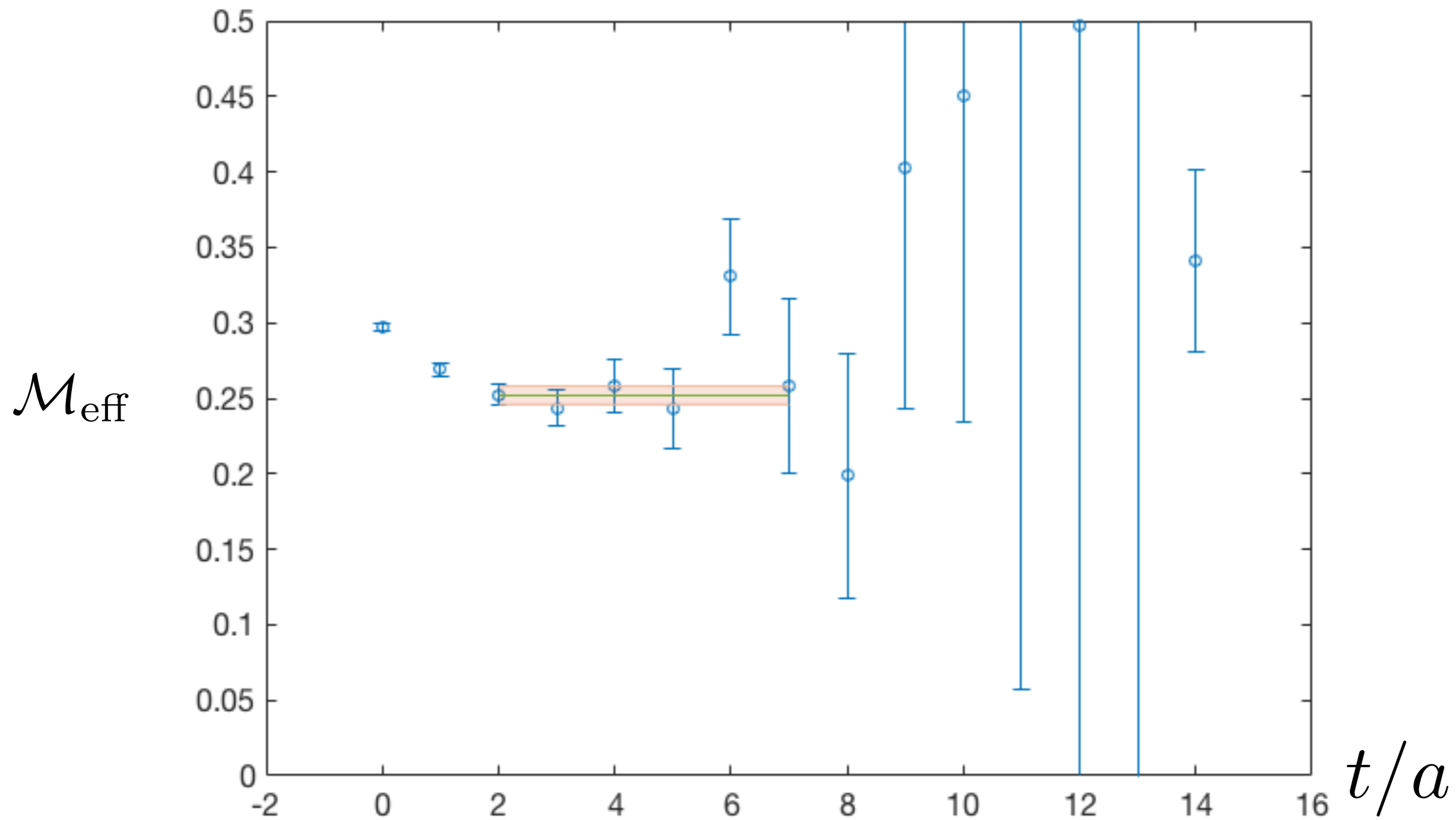
$$\mathfrak{M}(\nu, z_3^2) = \lim_{t \rightarrow \infty} \frac{\mathcal{M}_{\text{eff}}(z_3 P, z_3^2; t)}{\mathcal{M}_{\text{eff}}(z_3 P, z_3^2; t)|_{z_3=0}} \times \frac{\mathcal{M}_{\text{eff}}(z_3 P, z_3^2; t)|_{z_3=0}}{\mathcal{M}_{\text{eff}}(z_3 P, z_3^2; t)|_{P=0}}$$

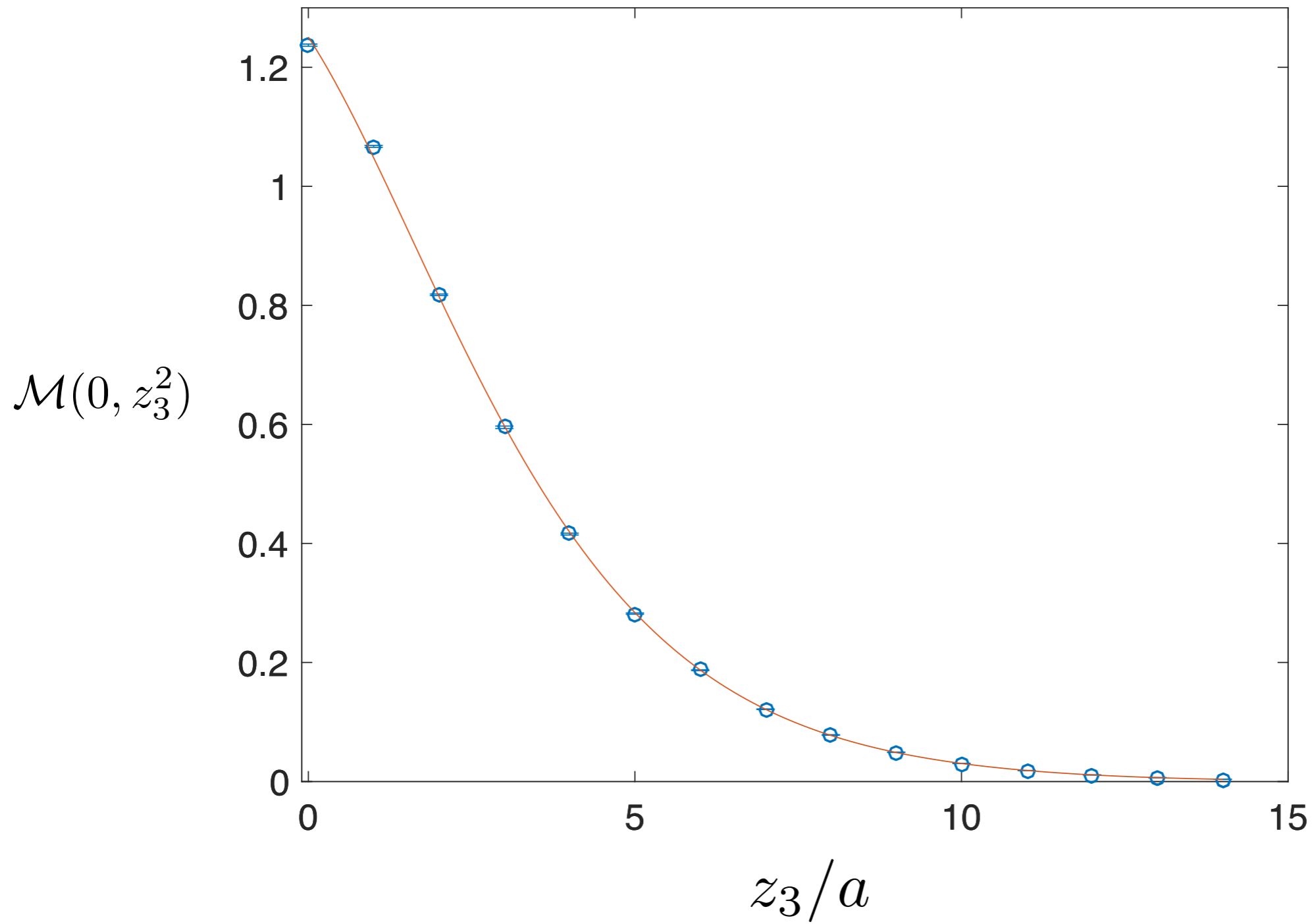
Constructed to remove lattice spacing errors



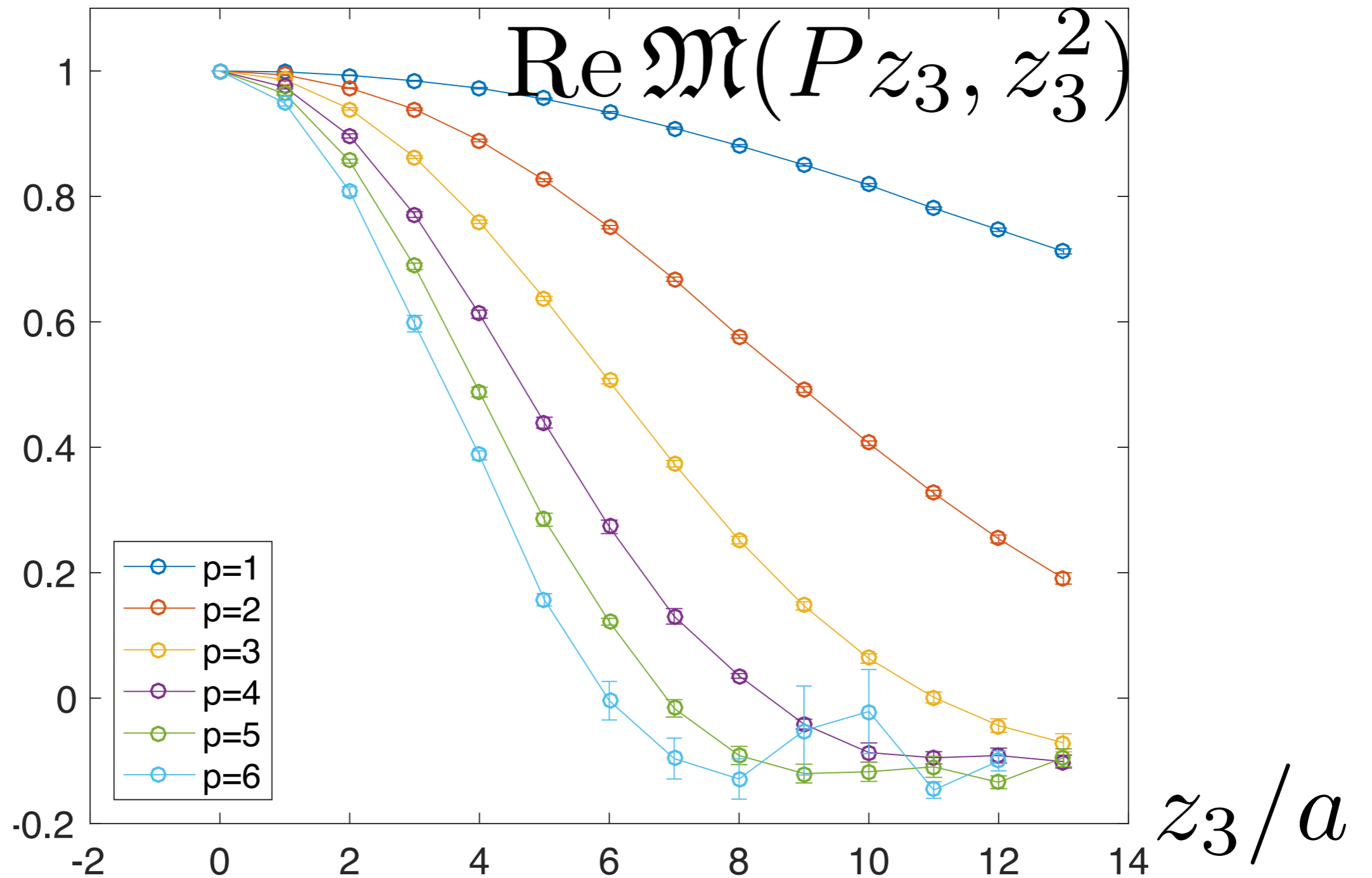
Gaussian smeared sources







Cusp indicates “linear” divergence of Wilson line



Ratio removes the linear" divergence of Wilson line

Real Part

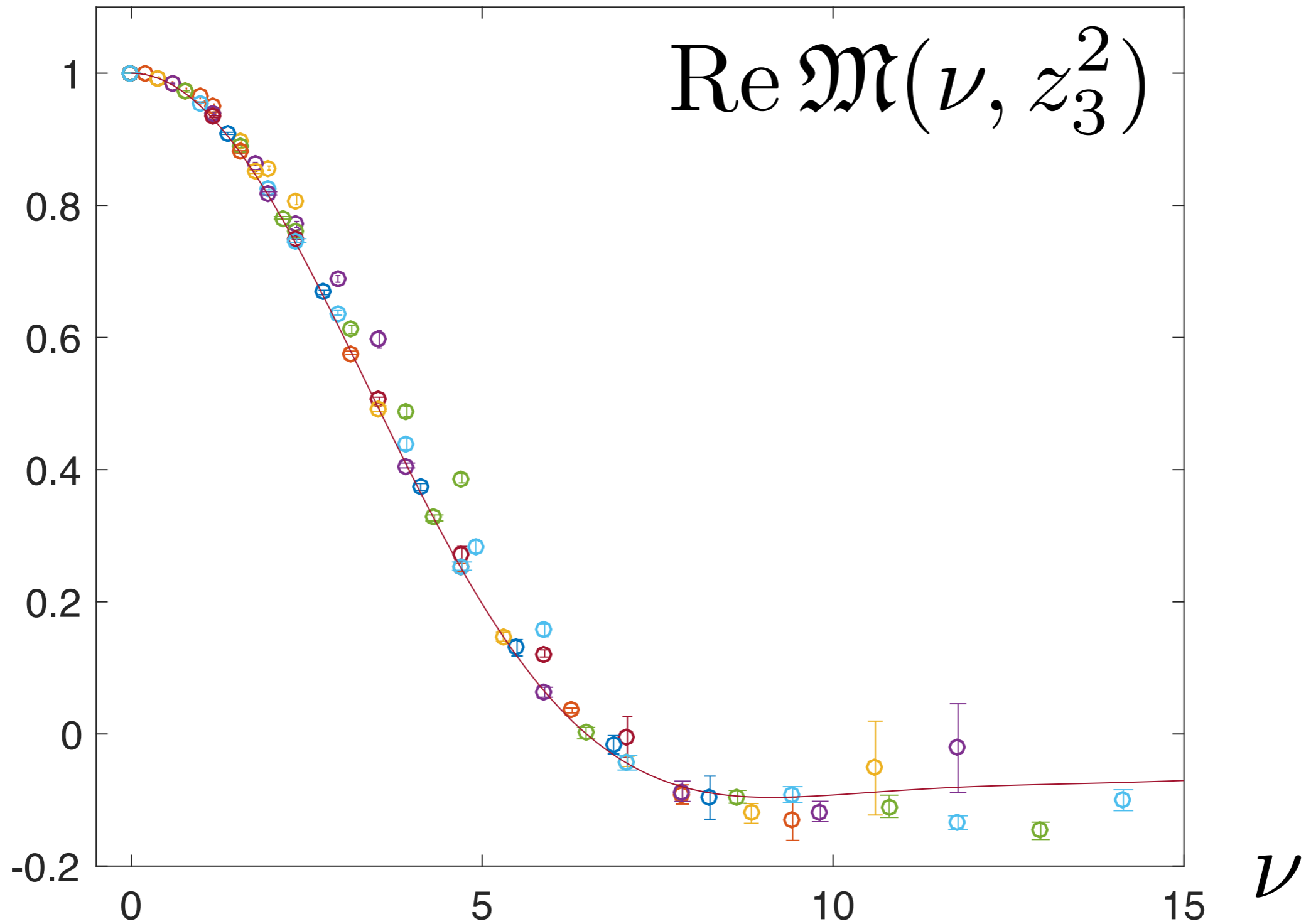
Isovector distribution

$$\mathfrak{M}_R(\nu, z^2 = 1/\mu^2) \equiv \int_0^1 dx \cos(\nu x) q_\nu(x, \mu^2)$$

$$q_\nu(x) = q(x) - \bar{q}(x)$$

$$q(x) = u(x) - d(x)$$

$$\overline{MS} \quad \mu^2 = (2e^{-\gamma_E} / z_3)^2$$



Points almost collapse on a universal curve

$$q_\nu(x) = \frac{315}{32} \sqrt{x} (1-x)^3$$

Imaginary Part

Isovector distribution

$$\mathfrak{M}_I(\nu, z^2 = 1/\mu^2) \equiv \int_0^1 dx \sin(\nu x) q_+(x, \mu^2).$$

$$q_+(x) = q(x) + \bar{q}(x)$$

$$q(x) = u(x) - d(x)$$

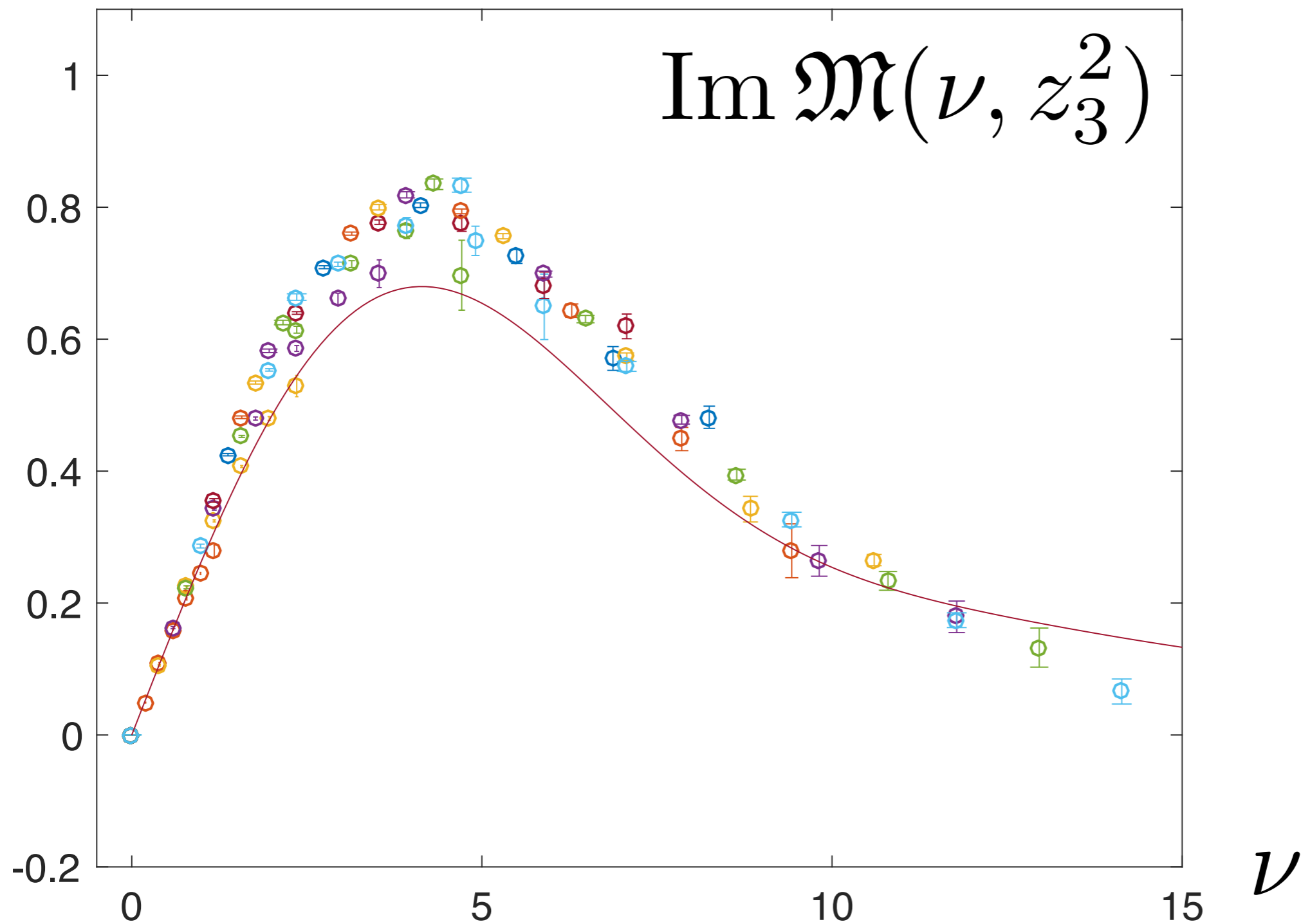
$$q_+(x) = q_v(x) + 2\bar{q}(x)$$

$$q_v(x) = q(x) - \bar{q}(x)$$

$$\overline{MS} \quad \mu^2 = (2e^{-\gamma_E} / z_3)^2$$

anti-quarks contribute to the imaginary part

$$q_+(x) = q_v(x) + 2\bar{q}(x)$$

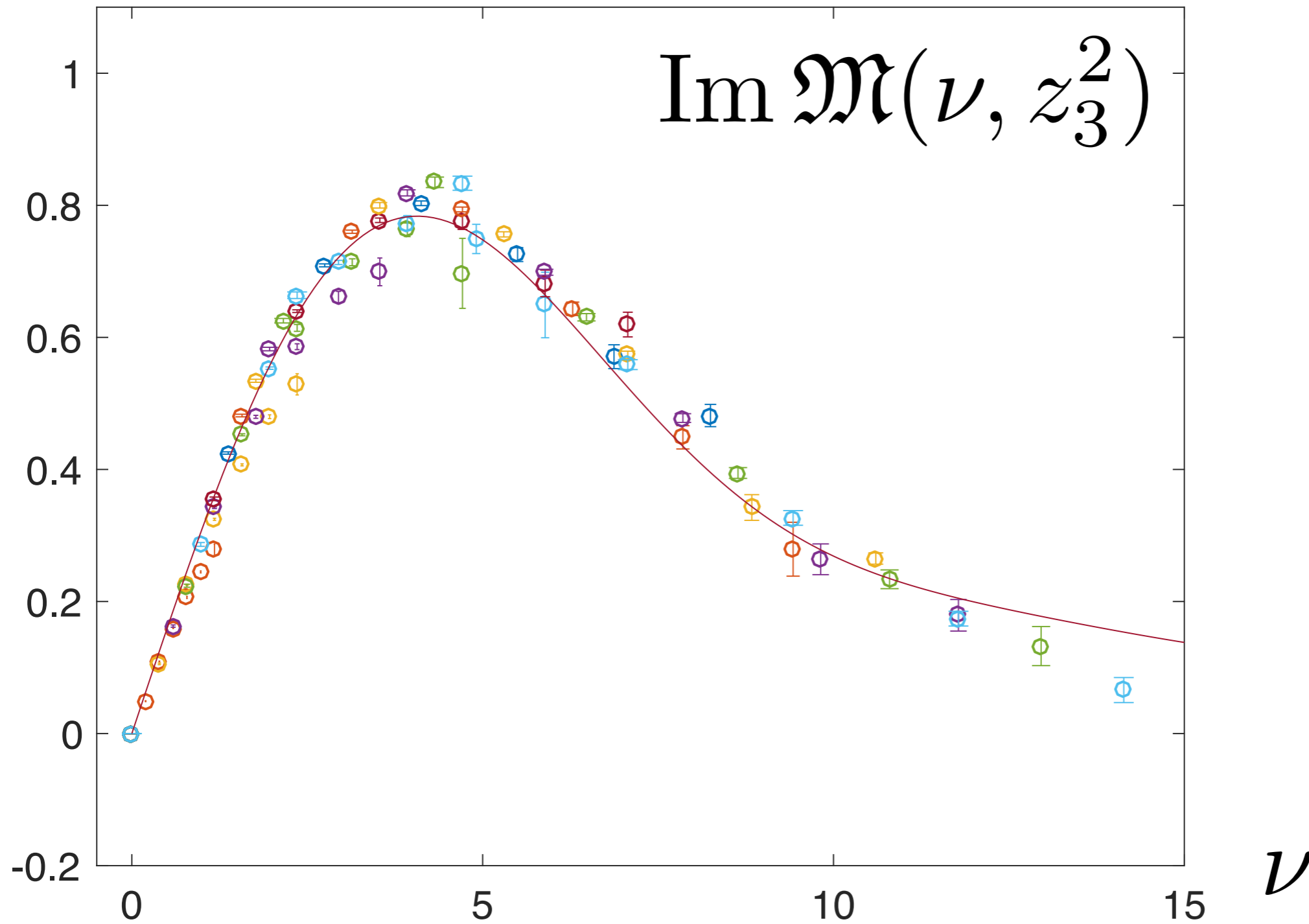


$$q_v(x) = \frac{315}{32} \sqrt{x} (1-x)^3$$

$$\bar{q}(x) = 0$$

anti-quarks contribute to the imaginary part

$$q_+(x) = q_v(x) + 2\bar{q}(x)$$



$$q_v(x) = \frac{315}{32} \sqrt{x} (1-x)^3$$

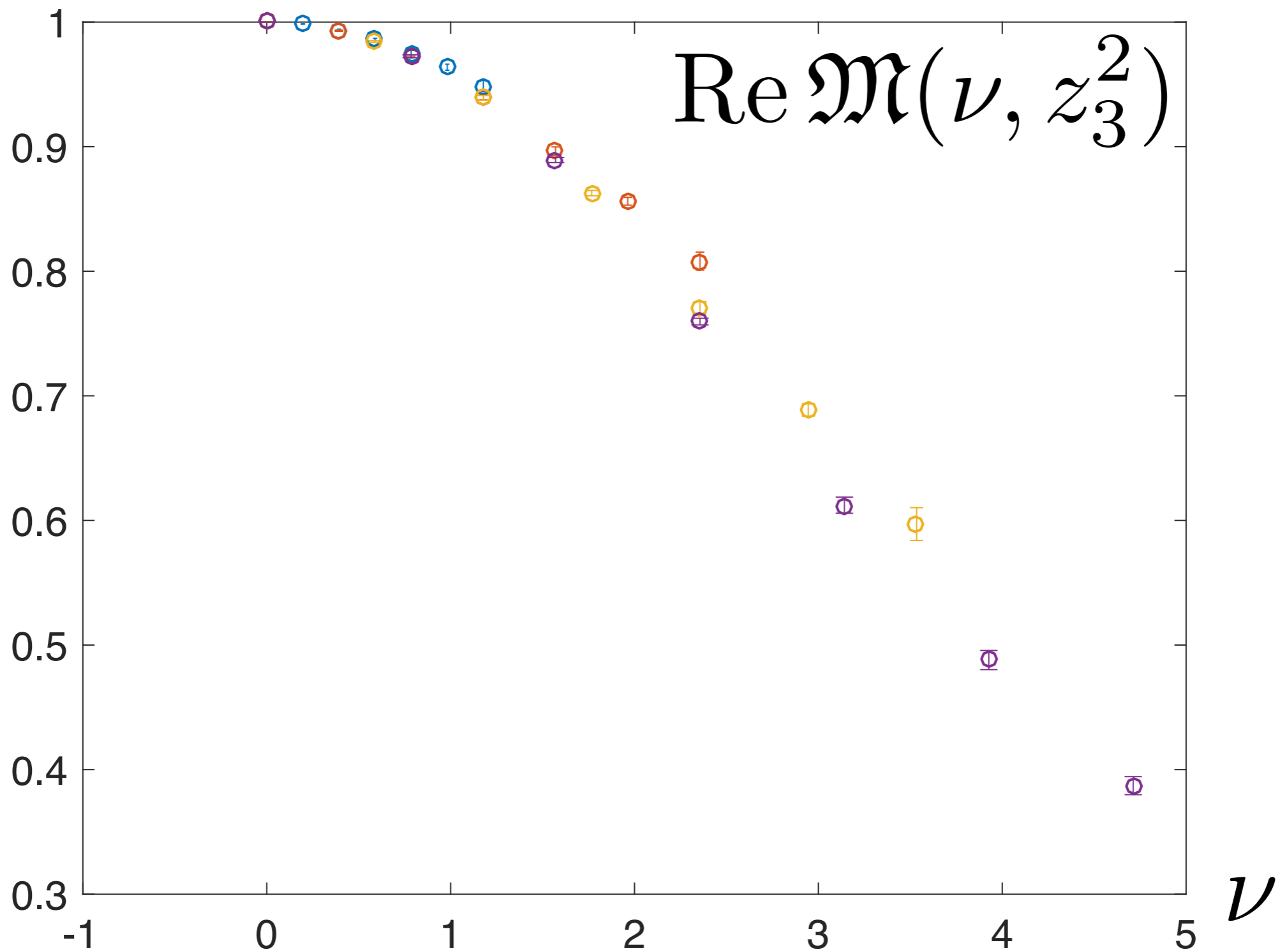
$$\bar{q}(x) = 1.4 x (1-x)^3$$

Points in previous plots obtained in with different z/a
i.e. correspond to the Ioffe time PDF at different scales!

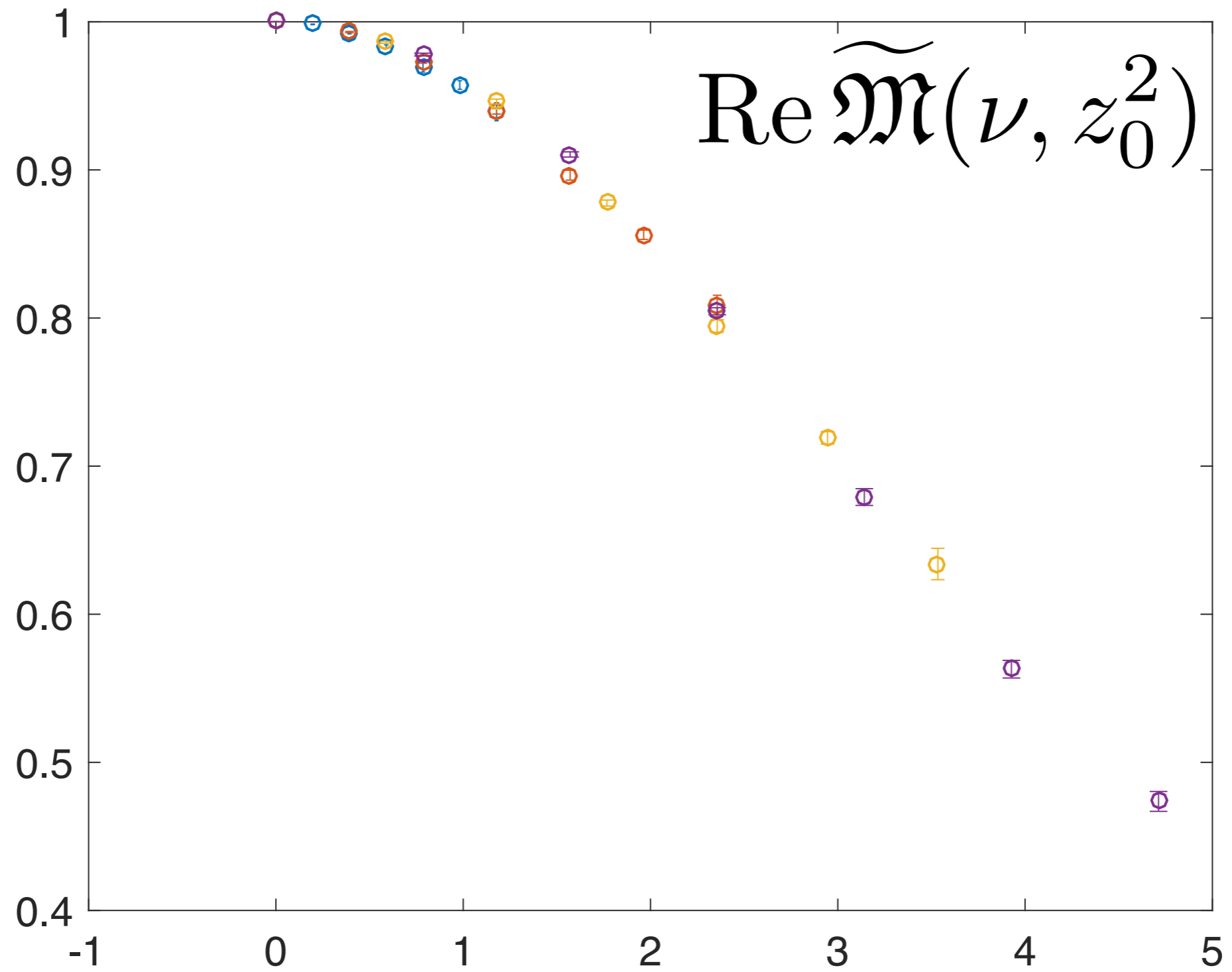
DGLAP evolution:

$$\mathfrak{M}(\nu, z'_3) = \mathfrak{M}(\nu, z_3^2) - \frac{2}{3} \frac{\alpha_s}{\pi} \ln(z_3'^2 / z_3^2) B \otimes \mathfrak{M}(\nu, z_3^2)$$

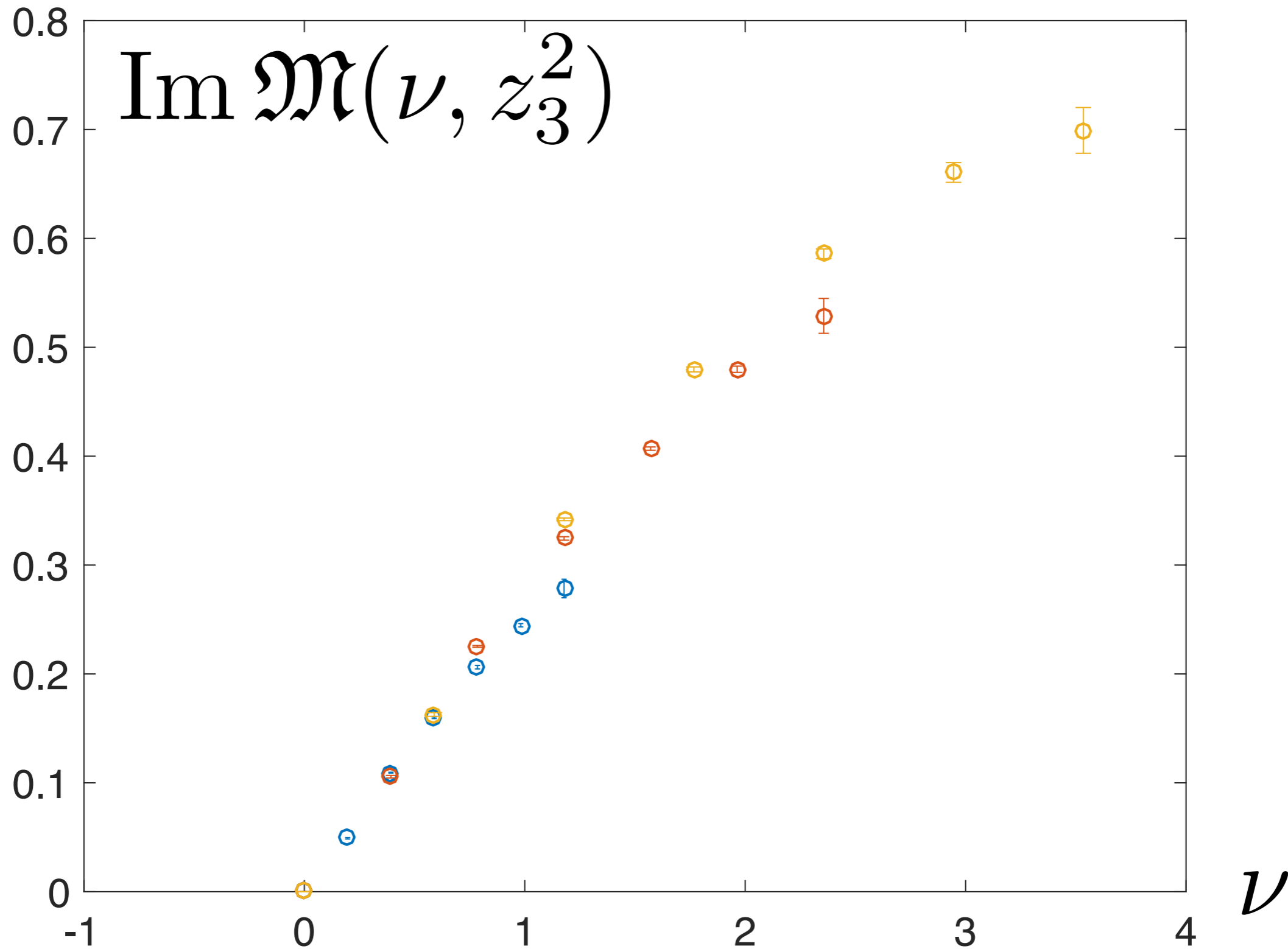
Apply evolution only at short distance points [~ 1 GeV]



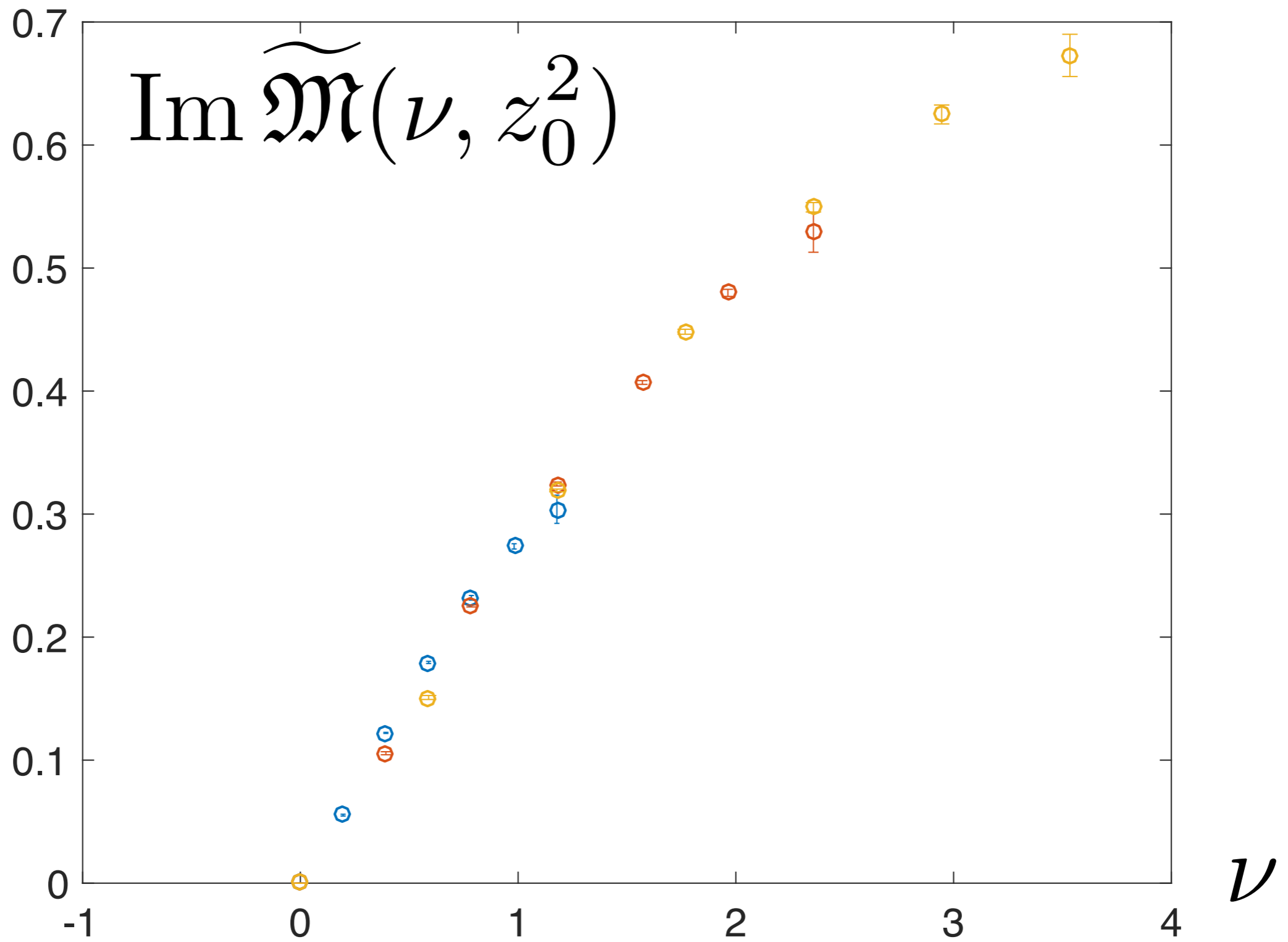
Data corresponding to $z/a = 1, 2, 3, 4$



Evolved to 1GeV



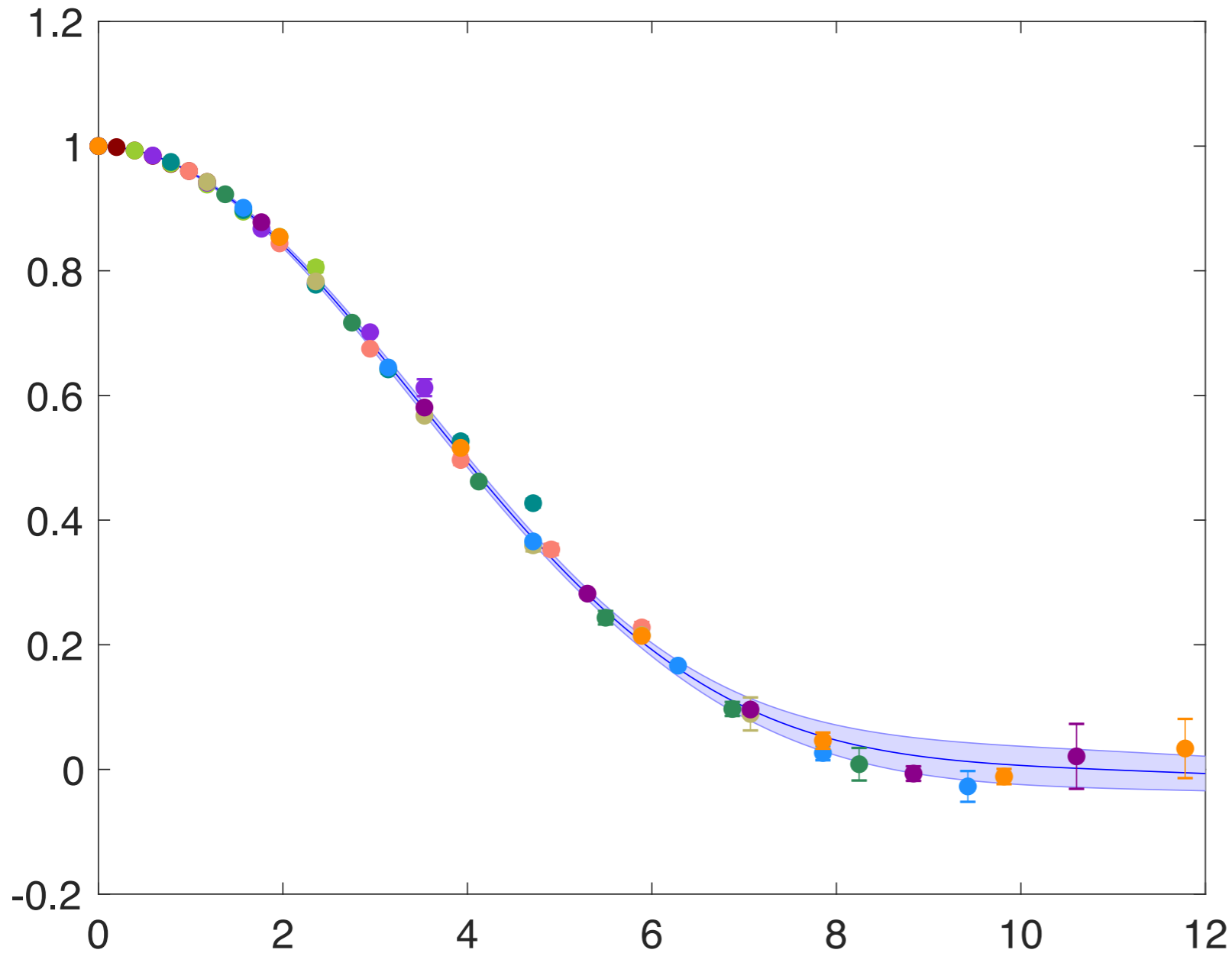
Data corresponding to $z/a = 1, 2, 3, 4$



Evolved to 1GeV

$$\mathfrak{M}(\nu, z^2) = \int_0^1 d\alpha \mathfrak{E}(\alpha, z^2 \mu^2, \alpha_s(\mu)) \mathcal{Q}(\alpha\nu, \mu)$$

$$\mu = 1 \text{ GeV}$$



Fit to:

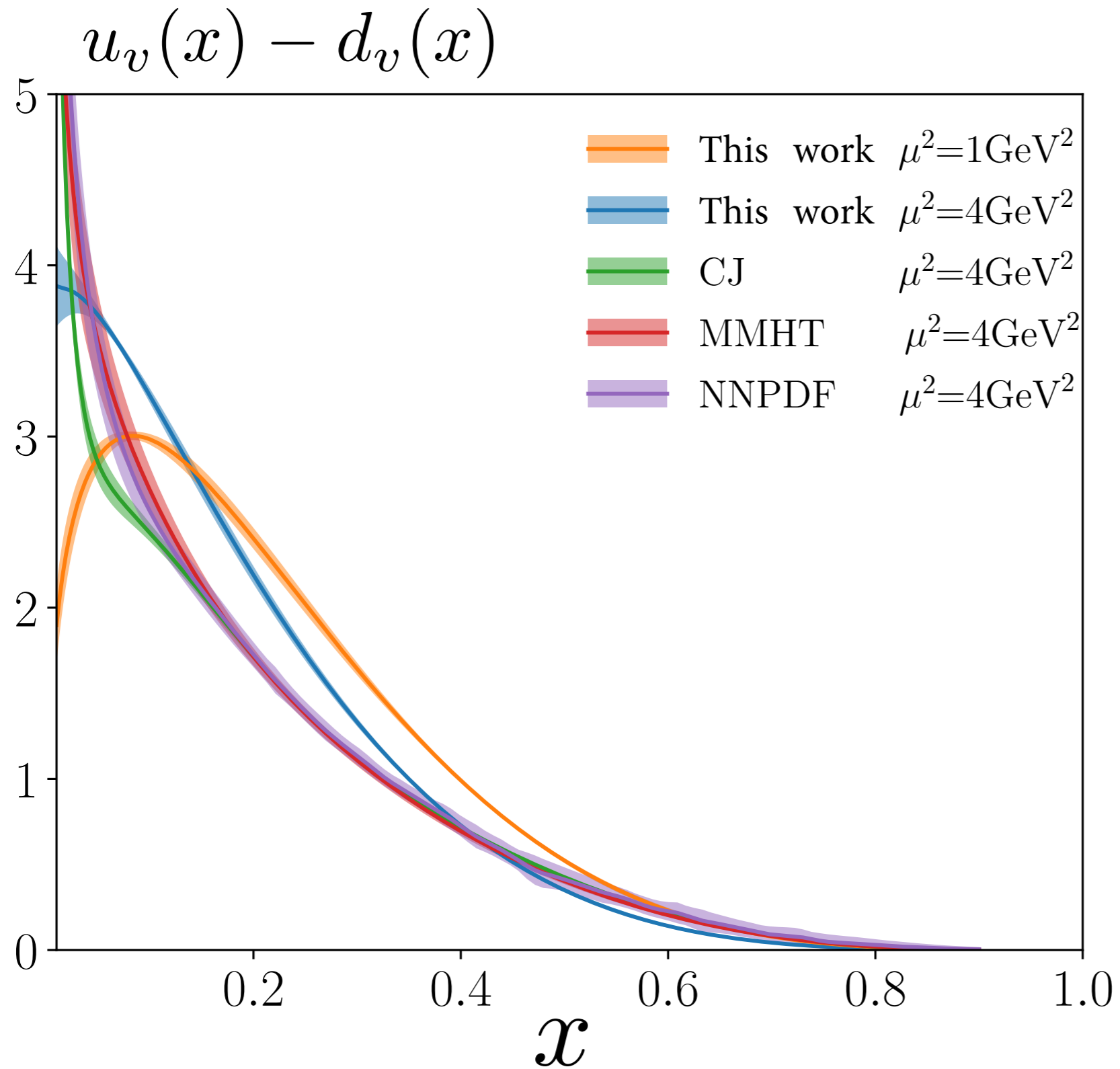
$$N(a, b) x^a (1-x)^b$$

$$a = 0.36(6)$$

$$b = 3.95(22)$$

Ignoring polynomial corrections

$$\mathcal{Q}(\nu, \mu^2) = \int_0^1 dx \cos(\nu x) q_\nu(x, \mu^2)$$



Thanks to N. Sato for making this figure

Summary

- Methods for obtaining parton distribution from Lattice QCD have now emerged
- An approach based on pseudo-PDFs has been proposed
 - Renormalization is handled in a simple way
 - Light cone limit is obtained by computing real space matrix elements at short Euclidean distances
 - All hadron momenta are useful in obtaining PDFs
- WM/JLab: first numerical tests are available in quenched approximation indicating the feasibility of the method
 - Results consistent with DGLAP evolution
- Dynamical fermion simulations are on the way
- Lattice spacing effect under study (quenched)
- Probing the small x region (or large Ioffe time) remains a challenge
 - Large Ioffe time may be probed with high momentum which requires a small lattice spacing (JLab anisotropic gauge ensembles?)
- Correctly applying evolution, matching and controlling polynomial corrections is essential for obtaining reliable results