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Quantum Chromodynamics

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Noncommutative gauge theories from Yang-Baxter deformations of AdS/CFT

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The AdS/CFT correspondence

type IIB string on $AdS_5 \times S^5$ ⟷ 4D $\mathcal{N}=4$ SU(N) SYM $(N \to \infty)$

Great progress: the discovery of integrability

[For a big review, Beisert et al., 1012.3982]

Integrability is so powerful!

The integrability enables us to compute exactly physical quantities even at finite coupling, without relying on supersymmetries.

EX anomalous dimensions, amplitudes etc.

Indeed, there are many directions of study concerned with this integrability.

Our motive here

Construct various examples of dualities
to which integrability techniques can be applied

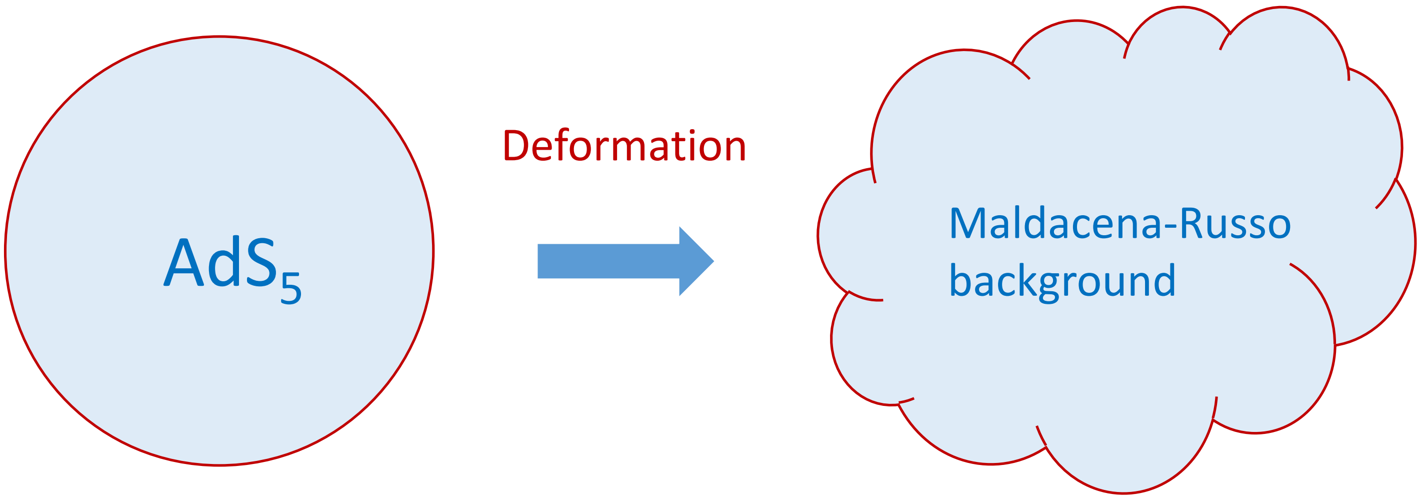
The main subject

Integrable deformations of the $AdS_5 \times S^5$ superstring

Integrable deformations → Deformed $AdS_5 \times S^5$ geometries

(as a 2D non-linear sigma model)

EX



4D N=4 SYM

4D N=4 SYM on NC plane

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu} \quad (\text{const.})$$

By considering integrable deformations, one can construct

- A lot of the bulk geometries on which string theories are integrable
- Deformed gauge theories associated with the deformations

One may find out nice applications to non-perturbative study of QCD.

(Some non-perturbative techniques are based on integrability.)

Along this line, we should employ a systematic way to perform integrable deformations.

Yang-Baxter deformation

The plan of my talk

1. What is Yang-Baxter deformation? (5 mins.)

2. Yang-Baxter deformations of superstring on $AdS_5 \times S^5$ (10 mins.)

3. Non-commutativity and Divergence Formula (5 mins.)

4. Summary and Discussion

1. What is Yang-Baxter deformation?

Yang-Baxter deformations

[Klimcik, 2002, 2008]

Integrable deformation!

An example

G-principal chiral model

Yang-Baxter sigma model

$$S = \int d^2x\, \eta^{\mu\nu} \mathrm{tr}(J_\mu J_\nu) \quad \longrightarrow \quad S^{(\eta)} = \int d^2x\, \eta^{\mu\nu} \mathrm{tr}\left(J_\mu \frac{1}{1-\eta R} J_\nu\right)$$

$$J_\mu = g^{-1}\partial_\mu g, \quad g \in G$$

η : a const. parameter

What is *R*?

$$R \; : \; \mathfrak{g} \longrightarrow \mathfrak{g}$$

a linear op.

a classical r-matrix satisfying

the modified classical Yang-Baxter eq. (mCYBE)

An integrable deformation can be specified by a classical r-matrix.

Strong advantage

Given a classical r-matrix, a Lax pair follows automatically.

No need to construct Lax pair in an intuitive manner case by case

Relation between R-operator and classical r-matrix

A linear R-operator ⟷ A skew-symmetric classical r-matrix

$R \,:\, \mathfrak{g} \longrightarrow \mathfrak{g}$ $\qquad$ $r \in \mathfrak{g} \otimes \mathfrak{g}$

$$R(X) \equiv \langle r_{12}, 1 \otimes X \rangle = \sum_i a_i \langle b_i, X \rangle \qquad \text{for} \quad X \in \mathfrak{g}$$

$$r_{12} = \sum_i a_i \otimes b_i \qquad \text{with} \qquad a_i, b_i \in \mathfrak{g}$$

Two sources of classical r-matrices

1) modified classical Yang-Baxter eq. (mCYBE) ⟵ the original work by Klimcik

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = \underline{-c^2[X, Y]} \qquad (\, c \in \mathbb{C} \,)$$

2) classical Yang-Baxter eq. (CYBE) $\quad (c = 0)$ ⟵ a possible generalization

2. Yang-Baxter deformations of superstring on $AdS_5 \times S^5$

Yang-Baxter deformations of superstring on $AdS_5 \times S^5$

$$S = -\frac{1}{2}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \; P_-^{\alpha\beta} \mathrm{Str}\left[A_\alpha d \circ \frac{1}{1-\eta\,[R]_g \circ d}(A_\beta) \right]$$

classical r-matrices are inserted here.

There are two sources for classical r-matrices:

1) modified classical Yang-Baxter eq. (mCYBE) [Delduc-Magro-Vicedo, 1309.5850]
2) homogeneous classical Yang-Baxter eq. (CYBE) [Kawaguchi-Matsumoto-KY, 1401.4855]

- Kappa invariance : a consistency as string theory at classical level
- Lax pair is constructed : classical integrability

The undeformed limit: $\eta \to 0$ → the Metsaev-Tseytlin action

[Metsaev-Tseytlin, hep-th/9805028]

An outline of supercoset construction

[Arutyunov-Borsato-Frolov, 1507.04239]
[Kyono-KY, 1605.02519]

By taking a representation of the group element and expanding w.r.t. the fermions, the deformed action can be rewritten into the canonical form:

$$S = -\frac{\sqrt{\lambda_c}}{4}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \left[\gamma^{ab} G_{MN} \partial_a X^M \partial_b X^N - \epsilon^{ab} B_{MN} \partial_a X^M \partial_b X^N\right]$$
$$-\frac{\sqrt{\lambda_c}}{2}\, i\bar{\Theta}_I (\gamma^{ab}\delta^{IJ} - \epsilon^{ab}\sigma_3^{IJ})\, e_a^m \Gamma_m D_b^{JK} \Theta_K + \mathcal{O}(\theta^4)$$

In general, the covariant derivative D is given by

[Cvetic-Lu-Pope-Stelle, hep-th/9907202]

$$D_a^{IJ} \equiv \delta^{IJ}\left(\partial_a - \frac{1}{4}\omega_a^{mn}\Gamma_{mn}\right) + \frac{1}{8}\sigma_3^{IJ}\, e_a^m H_{mnp}\Gamma^{np}$$
$$-\frac{1}{8} e^{\Phi}\left[\epsilon^{IJ}\Gamma^p F_p + \frac{1}{3!}\sigma_1^{IJ}\Gamma^{pqr} F_{pqr} + \frac{1}{2\cdot 5!}\epsilon^{IJ}\Gamma^{pqrst} F_{pqrst}\right] e_a^m \Gamma_m$$

From this expression, one can read off all of the fields of type IIB SUGRA.

Summary of the resulting backgrounds

1) The mCYBE case [Delduc-Magro-Vicedo, 1309.5850]

η-deformation or standard *q*-deformation [Arutyunov-Borsato-Frolov, 1312.3542]

The background is not a sol. of the usual SUGRA,

but satisfies the generalized SUGRA. [Arutyunov-Borsato-Frolov, 1507.04239]

[Arutyunov-Frolov-Hoare-Roiban-Tseytlin, 1511.05795]

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2) The CYBE case [Kawaguchi-Matsumoto-KY, 1401.4855]

A certain class of classical r-matrices satisfying

The unimodularity condition [Borsato-Wulff, 1608.03570]

$r^{ij}[b_i, b_j] = 0$ for a classical r-matrix $r = r^{ij} b_i \wedge b_j$

→ Sols. of the standard SUGRA

EX Lunin-Maldacena, Maldacena-Russo backgrounds [Matsumoto-KY, 1404.1838 ,1404.3657] [Kyono-KY, 1605.02519]

Otherwise, the backgrounds are sols. of the generalized SUGRA.

The Essential Point 1):

The unimodularity of classical r-matrix

(Algebraic property)

is closely related to

the on-shell condition of supergravity

(Geometric property)

NOTE: If not unimodular, then the supergravity must be generalized by adding an additional vector field *I*.

i) unimodular example: gravity duals for non-commutative gauge theories

c.f. Seiberg-Witten, 1999

Abelian Jordanian r-matrix: $r = \frac{1}{2}\, p_2 \wedge p_3$

[Matsumoto-KY, 1404.3657]

where $p_\mu \equiv \frac{1}{2}\gamma_\mu - m_{\mu 5}\,,\quad m_{\mu 5} = \frac{1}{4}\left[\gamma_\mu, \gamma_5\right]\,,\quad \gamma_\mu$: a basis of $\mathfrak{su}(2,2)$

Metric: $$ds^2 = \frac{1}{z^2}(-dx_0^2 + dx_1^2) + \frac{z^2}{z^4+\eta^2}(dx_2^2 + dx_3^2) + \frac{dz^2}{z^2} + d\Omega_5^2$$

B-field: $$B_2 = \frac{\eta}{z^4+\eta^2}\, dx^2 \wedge dx^3\,,$$ dilaton: $$\Phi = \frac{1}{2}\log\left(\frac{z^4}{z^4+\eta^2}\right)$$

R-R : $$F_3 = \frac{4\eta}{z^5}\, dx^0 \wedge dx^1 \wedge dz\,,\qquad F_5 = 4\left[\mathrm{e}^{2\Phi}\omega_{AdS_5} + \omega_{\mathrm{S}^5}\right]\,.$$

[Hashimoto-Itzhaki, Maldacena-Russo, 1999]

Note This solution can also be reproduced as a special limit of η-deformed AdS_5 .

[Arutyunov-Borsaro-Frolov, 1507.04239] [Kameyama-Kyono-Sakamoto-KY, 1509.00173]

A relation between classical r-matrices and non-commutativities

Roughly speaking, one can see the following correspondence:

	Classical r-matrix	Non-commutativity	
1.	$r = p^{\mu} \wedge p^{\nu}$	$[x^{\mu}, x^{\nu}] = i\theta^{\mu\nu}$	(contant)
2.	$r = p^{\mu} \wedge n^{\nu\rho}$	$[x^{\mu}, x^{\nu}] = i\theta^{\mu\nu}{}_{\rho}\, x^{\rho}$	(Lie algebraic)
3.	$r = n^{\mu\nu} \wedge n^{\rho\sigma}$	$[x^{\mu}, x^{\nu}] = i\theta^{\mu\nu}{}_{\rho\sigma}\, x^{\rho} x^{\sigma}$	(quadratic)

NOTE: This correspondence nicely agrees with the old result on classification of non-commutative spaces with classical r-matrices.

ii) non-unimodular example: a solution of the generalized SUGRA

$$
\begin{aligned}
r &= E_{24} \wedge (c_1 E_{22} - c_2 E_{44}) \\
&= (p_0 - p_3) \wedge \left[a_1 \left(\frac{1}{2}\gamma_5 - n_{03} \right) - a_2 \left(n_{12} - \frac{i}{2} \mathbf{1}_4 \right) \right]
\end{aligned}
$$

$$
a_1 \equiv \frac{c_1 + c_2}{2} = \mathrm{Re}(c_1) \,, \qquad a_2 \equiv \frac{c_1 - c_2}{2\,i} = \mathrm{Im}(c_1)
$$

The resulting background: [Kyono-KY, 1605.02519]

$$
\begin{aligned}
ds^2 &= \frac{-2dx^+dx^- + d\rho^2 + \rho^2 d\phi^2 + dz^2}{z^2} - 4\eta^2 \left[(a_1^2 + a_2^2)\frac{\rho^2}{z^6} + \frac{a_1^2}{z^4} \right] (dx^+)^2 + ds^2_{\mathrm{S}^5} \,, \\
B_2 &= 8\eta \left[\frac{a_1 x^1 + a_2 x^2}{z^4} dx^+ \wedge dx^1 + \frac{a_1 x^2 - a_2 x^1}{z^4} dx^+ \wedge dx^2 + a_1 \frac{1}{z^3} dx^+ \wedge dz \right] \,, \\
F_3 &= 8\eta \left[\frac{a_2 x^1 - a_1 x^2}{z^5} dx^+ \wedge dx^1 \wedge dz + \frac{a_1 x^1 + a_2 x^2}{z^5} dx^+ \wedge dx^2 \wedge dz + \frac{a_1}{z^4} dx^+ \wedge dx^1 \wedge dx^2 \right] \,, \\
F_5 &= \text{undeformed} \,, \qquad \Phi = \text{const}
\end{aligned}
$$

$$
I = -\frac{2\eta a_1}{z^2}\, dx^+ \,, \qquad Z = 0
$$

c.f, the a1=0 case corresponds to the usual SUGRA solution
[Hubeney-Rangamani-Ross, hep-th/0504034]

This is a solution of the generalized SUGRA!

3. Non-commutativity and Divergence Formula

[Araujo-Bakhmatov-O Colgain-Sakamoto-Sheikh Jabbari-KY, 1702.02861, 1705.02063]

Hereafter, we will focus upon YB deformations of AdS_5,
which are related to non-commutative gauge theories.

How to see the noncommutativity ?

So far, we have considered the closed string picture with (g_{MN}, B_{MN}, g_s).

To see the NC, we should work in the open string picture with $(G_{MN}, \Theta^{MN}, G_s)$.

The relations

$$G_{MN} = (g - Bg^{-1}B)_{MN}$$

$$\Theta^{MN} = -((g+B)^{-1}B(g-B)^{-1})^{MN}$$

$$G_s = g_s \left(\frac{\det(g+B)}{\det g} \right)^{1/2}$$

The open string picture of YB deformations of AdS_5 with homogeneous CYBE:

G_{MN} : the undeformed $AdS_5 \times S^5$ G_s : const.

Only the NC parameter Θ^{MN} depends on the deformation.

→ Classical r-matrices determine non-commutativities

[van Tongeren, 1506.01023, 1610.05677]

[Araujo-Bakhmatov-O Colgain-Sakamoto-Sheikh Jabbari-KY, 1702.02861, 1705.02063]

The relation between SUGRA and noncommutativity

[Araujo-Bakhmatov-O Colgain-Sakamoto-Sheikh Jabbari-KY, 1702.02861, 1705.02063]

The on-shell condition of SUGRA (= the unimodularity condition)

$$\Rightarrow \quad \nabla_M \Theta^{MN} = 0$$

This condition is necessary for the cyclic property of star product

For the generalized SUGRA,

$$\Rightarrow \quad \nabla_M \Theta^{MN} = I^N$$

The extra vector field *I* has been related to the noncommutativity!

This is the first result that relates *I* to a physical quantity like non-commutativity.

The Essential Point 2):

The non-commutative parameter

(Non-commutative geometry)

is closely related to

the on-shell condition of supergravity

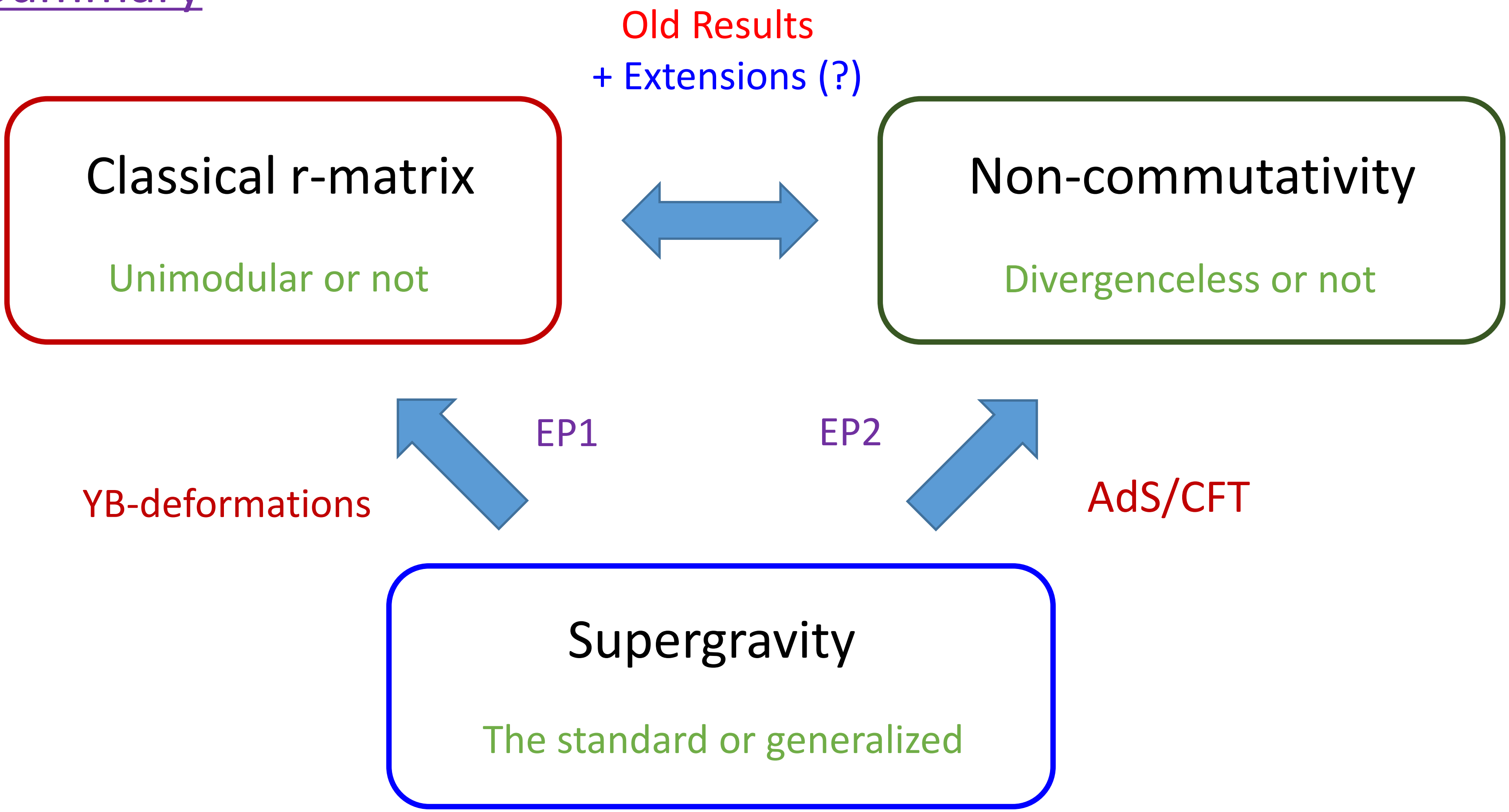
(Geometric property)

c.f., In this relation, the AdS/CFT correspondence is assumed to be valid, though the complete proof of AdS/CFT has not been provided yet.

(Holographic non-commutativity!)

4. Summary and Discussion

Summary



This picture has been established based on Yang-Baxter deformations of type IIB superstring on $AdS_5 \times S^5$.

This would be true apart from YB deformations.

Thank you!

What is the implication of the divergence formula?

This formula was derived by considering YB deformations of $AdS_5 \times S^5$, but this may be much more general.

NOTE: The transformation to the open string metric appears in a different context when considering duality transformations. [Duff, NPB335 (1990) 610]

Then the non-commutativity is called the beta field.

Then, by using the beta field, a certain flux, called Q-flux, can be defined as

$$Q_p{}^{mn} \equiv \partial_p \beta^{mn}$$

[Grana-Minasian-Petrini-Waldram, 0807.4527]

For a constant shift for a direction $x \to x+1$, one can introduce

The monodromy

$$\beta^{mn}(x+1) - \beta^{mn}(x) = \int_x^{x+1} dx'^p \, \partial_p \beta^{mn} = \int_x^{x+1} dx'^p \, Q_p{}^{mn}(x')$$

If this monodromy is non-trivial along the x-direction, this flux is non-geometric.

Our proposal [Sakamoto-Sakatani-KY, 1705.07116]

$I^m \equiv D_n \beta^{mn}$ = trace of Q-flux + Chrsitoffel symbols

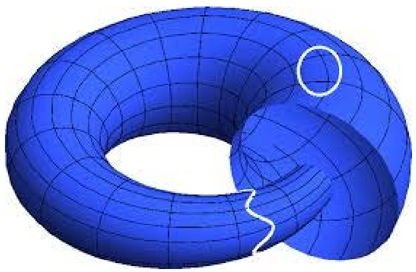
(divergence formula)

When the extra vector field $I \neq 0$, solutions may be non-geometric.

Hence we have checked some YB-deformed backgrounds with $I \neq 0$
and obtained the non-trivial monodromy. [Fernandez Melgarejo-Sakamoto-Sakatani-KY, 1710.06849]

→ (At least some) YB-deformed backgrounds with $I \neq 0$ are T-folds.

NOTE: A T-fold is a generalized notion of a manifold.
It is locally a Riemannian manifold, but the patches
are glued with diffeomorphism and T-duality.



[Blumenhagen, et al., 1510.04059]

In general, solutions of the generalized SUGRA are non-geometric!

Back up

Definitions of the quantities

Maurer-Cartan 1-form

$$A_\alpha \equiv g^{-1}\partial_\alpha g\,, \quad g \in SU(2,2|4)$$,

Projection on the group manifold

$$d \equiv P_1 + 2P_2 - P_3$$

Projection on the world-sheet

$$P_\pm^{\alpha\beta} \equiv \frac{1}{2}(\gamma^{\alpha\beta} \pm \epsilon^{\alpha\beta})$$

$$\begin{cases} \gamma^{\alpha\beta} = \mathrm{diag}(-1,1) \\ \epsilon^{\alpha\beta}\ :\ \text{anti-symm. tensor} \end{cases}$$

A chain of operations

$$R_g(X) \equiv g^{-1}R(gXg^{-1})g\,, \qquad \forall X \in \mathfrak{su}(2,2|4)$$

A group element: $g = g_{\rm b}\, g_{\rm f} \quad \in SU(2,2|4)$

$$g_{\rm b} = {g_{\rm b}}^{{\rm AdS}_5}\, {g_{\rm b}}^{{\rm S}^5}\,,$$

[For a big review, Arutyunov-Frolov, 0901.4937]

$$g_{\rm f} = \exp(\mathbf{Q}^I \theta_I)\,, \quad \mathbf{Q}^I\theta_I \equiv (\mathbf{Q}^{\check\alpha\hat\alpha})^I\,(\theta_{\check\alpha\hat\alpha})_I \quad (I=1,2;\ \check\alpha,\hat\alpha = 1,\ldots,4)$$

When we take a parametrization like

$$\begin{aligned} {g_{\rm b}}^{{\rm AdS}_5} &= \exp\Big[x^0\,P_0 + x^1\,P_1 + x^2\,P_2 + x^3\,P_3\Big]\,\exp\Big[(\log\, z)\,D\Big]\,, \\ {g_{\rm b}}^{{\rm S}^5} &= \exp\Big[\frac{i}{2}(\phi_1\,h_1 + \phi_2\,h_2 + \phi_3\,h_3)\Big]\,\exp\Big[\xi\,\mathbf{J}_{68}\Big]\,\exp\Big[-i\,r\,\mathbf{P}_6\Big]\,, \end{aligned}$$

the metric of AdS_5 x S^5 is given by

(the undeformed case)

$$\begin{aligned} ds^2 &= ds^2_{{\rm AdS}_5} + ds^2_{{\rm S}^5}\,, \\ ds^2_{{\rm AdS}_5} &= \frac{-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2}{z^2} + \frac{dz^2}{z^2}\,, \\ ds^2_{{\rm S}^5} &= dr^2 + \sin^2 r\, d\xi^2 + \cos^2\xi\,\sin^2 r\, d\phi_1^2 + \sin^2 r\,\sin^2\xi\, d\phi_2^2 + \cos^2 r\, d\phi_3^2 \end{aligned}$$

The generalized eqns of type IIB SUGRA

[Arutyunov-Frolov-Hoare-Roiban-Tseytlin, 1511.05795]

[Tseytlin-Wulf, 1605.04884]

$$R_{MN} - \frac{1}{4}H_{MKL}H_N{}^{KL} - T_{MN} + D_M X_N + D_N X_M = 0\,,$$

$$\frac{1}{2}D^K H_{KMN} + \frac{1}{2}F^K F_{KMN} + \frac{1}{12}F_{MNKLP}F^{KLP} = X^K H_{KMN} + D_M X_N - D_N X_M$$

$$R - \frac{1}{12}H^2 + 4D_M X^M - 4X_M X^M = 0\,,$$

$$D^M \mathcal{F}_M - Z^M \mathcal{F}_M - \frac{1}{6}H^{MNK}\mathcal{F}_{MNK} = 0\,, \qquad I^M \mathcal{F}_M = 0\,, \qquad \mathcal{F}_{n_1 n_2 \ldots} = e^{\Phi} F_{n_1 n_2 \ldots}$$

$$D^K \mathcal{F}_{KMN} - Z^K \mathcal{F}_{KMN} - \frac{1}{6}H^{KPQ}\mathcal{F}_{KPQMN} - (I \wedge \mathcal{F}_1)_{MN} = 0\,,$$

$$D^K \mathcal{F}_{KMNPQ} - Z^K \mathcal{F}_{KMNPQ} + \frac{1}{36}\epsilon_{MNPQRSTUVW}H^{RST}\mathcal{F}^{UVW} - (I \wedge \mathcal{F}_3)_{MNPQ} = 0$$

$$T_{MN} \equiv \frac{1}{2}\mathcal{F}_M\mathcal{F}_N + \frac{1}{4}\mathcal{F}_{MKL}\mathcal{F}_N{}^{KL} + \frac{1}{4\times 4!}\mathcal{F}_{MPQRS}\mathcal{F}_N{}^{PQRS} - \frac{1}{4}G_{MN}\left(\mathcal{F}_K\mathcal{F}^K + \frac{1}{6}\mathcal{F}_{PQR}\mathcal{F}^{PQR}\right)$$

Modified Bianchi identities

$$(d\mathcal{F}_1 - Z \wedge \mathcal{F}_1)_{MN} - I^K \mathcal{F}_{MNK} = 0\,,$$

$$(d\mathcal{F}_3 - Z \wedge \mathcal{F}_3 + H_3 \wedge \mathcal{F}_1)_{MNPQ} - I^K \mathcal{F}_{MNPQK} = 0\,,$$

$$(d\mathcal{F}_5 - Z \wedge \mathcal{F}_5 + H_3 \wedge \mathcal{F}_3)_{MNPQRS} + \frac{1}{6}\epsilon_{MNPQRSTUVW}I^T\mathcal{F}^{UVW} = 0$$

New ingredients: $X, \quad I, \quad Z$ 3 vector fields

But $X_M \equiv I_M + Z_M$, so two of them are independent.

Then $I \ \& \ Z$ satisfy the following relations:

$$D_M I_N + D_N I_M = 0\,, \quad D_M Z_N - D_N Z_M + I^K H_{KMN} = 0\,, \quad I^M Z_M = 0$$

Assuming that I is chosen such that the Lie derivative

$$(\mathcal{L}_I B)_{MN} = I^K \partial_K B_{MN} + B_{KN} \partial_M I^K - B_{KM} \partial_N I^K$$

vanishes, the 2nd equation above can be solved by

$$Z_M = \partial_M \Phi - B_{MN} I^N \quad .$$

Thus only I is independent after all.

Note When $I = 0$, the usual type IIB SUGRA is reproduced.

The list of generalizations of Yang-Baxter sigma models (2 classes)

(i) modified classical Yang-Baxter eq. (trigonometric)

1) Principal chiral model [Klimcik, hep-th/0210095, 0802.3518]
2) Symmetric coset sigma model [Delduc-Magro-Vicedo, 1308.3581]
3) Superstring on $AdS_5 x S^5$ [Delduc-Magro-Vicedo, 1309.5850]

(ii) classical Yang-Baxter eq. (rational)

1) Principal chiral model [Matsumoto-KY, 1501.03665]
2) Symmetric coset sigma model [Matsumoto-KY, 1501.03665]
3) Superstring on $AdS_5 x S^5$ [Kawaguchi-Matsumoto-KY, 1401.4855]

NOTE bi-Yang-Baxter sigma models are also constructed. [Klimcik, 0802.3518, 1402.2105]

The supercoset for superstring theory on $AdS_5 \times S^5$

The coset structure of $AdS_5 \times S^5$ is closely related to the integrability.

$$\mathrm{AdS}_5 \times \mathrm{S}^5 = \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$$

: symmetric coset

Z_2-grading → classical integrability

Including fermions

$$\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$$

: super coset

[Metsaev-Tseytlin, 1998]

Z_4-grading → classical integrability

elucidated by
[Bena-Polchinski-Roiban, 2003]

This supercoset is the starting point in the following discussion.