Elements of Bi-Local Holography

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Vector Model / Higher Spin Gravity

- Large N

- d=3 : * $$\mathcal{L} = (\partial\vec{\phi})\cdot(\partial\vec{\phi}) + \frac{\lambda}{2N}(\vec{\phi}\cdot\vec{\phi})^2$$

  UV : $\lambda = 0$ IR : $\lambda = \infty$

  *

  Vasiliev HS Theory

  [Klebanov & Polyakov '02] [Sezgin & Sundell '02] [Giombi & Yin '09]

- d=2 : AdS_3 HS / Minimal Model [Gaberdiel & Gopakumar '10]

- d=1 SYK model: AdS_2 HS Gravity

Higher Spin Theory : s=0,1,2,3..

- AdS{d+1} HS fields : Fronsdal 80'

$$(\Box - m^2) h_{\mu_1 \cdots \mu_s} + s \nabla_{(\mu_1} \nabla^{\nu} h_{\mu_2 \cdots \mu_s)\nu} - \frac{s(s-1)}{2(d+2s-3)} g_{(\mu_1 \mu_2} \nabla^{\nu_1} \nabla^{\nu_2} h_{\mu_3 \cdots \mu_s)\nu_1 \nu_2} = 0$$

- Fradkin, Vasiliev (1996-2000) :Nonlinear Eqs/HS-gauge symmetry
- No known Action, Feynman rules , Loops?

Gauge Reduction for HS Theory

- Physical Realization :obtained by solving the tracelessness & de Donder gauge conditions:

$$\eta^{\hat{\mu}_1\hat{\mu}_2} H_{\hat{\mu}_1\hat{\mu}_2\cdots\hat{\mu}_s} = 0$$

$$\left(\partial_z - \frac{1}{z}\right) H_{z\hat{\mu}_2\cdots\hat{\mu}_x} + \partial_\mu H_{\mu\hat{\mu}_2\cdots\hat{\mu}_s} = 0$$

For AdS3 results in

$$H(x^i, z, \theta) = \sum_{s=1}^{\infty} e^{\pm is\theta} H_{(\pm s)}(x^i, z)$$

Physical space of hs: AdS + S

Bi-local (Re)Construction

$$\Psi(x_1, x_2) = \frac{1}{N} \sum_{i=1}^{N} \phi_i(x_1)\phi_i(x_2)$$

CFT_d => AdS_{d+1} + Spin (S^{d-1})

- Sumit Das,AJ (2006),R .de Mello Koch, Kenta Suzuki, J.Rodrigues, J.Yoon(2011-)

Bi-local Field: $\Psi(x_1^\mu, x_2^\mu) \Leftrightarrow H(x^\mu, z; S)$

$$d + d = (d+1) + (d-1)$$

- Map to Higher Spins in AdS_(d+1) .

Large N Action

- Collective Action:

$$S_{\rm col} = \frac{N}{2}\int d^dx\left[\left(\nabla_x^2\Psi(x,x')\right)_{x'=x} + \frac{\lambda}{2}\Psi^2(x,x)\right] - \frac{N}{2}\,{\rm Tr}\log\Psi$$

- AJ Sakita 80' matrix theories Recent :Eff action Gurau

- Saddle-point:

$$\Psi(x_1,x_2) = \Psi_0(x_1,x_2) + \frac{1}{\sqrt{N}}\overline{\Psi}(x_1,x_2)$$

Quadratic Fluctuations:

$$S_{(2)} = \frac{1}{4}\int\prod_{a=1}^{4} d^dx_a\,\overline{\Psi}(x_1,x_2)\,\widehat{\mathcal{L}}_{\rm bi}\,\overline{\Psi}(x_3,x_4)$$

Quadratic Kernel

- Conformal symmetry: Casimirs of SO(2,3)

$$
\begin{aligned}
\widehat{\mathcal{L}}_{\mathrm{bi}} &\approx C_4 + \frac{1}{4} C_2^2 \\
&\approx \frac{1}{4} |x_1 - x_2|^4 \frac{\partial}{\partial x_1} \cdot \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \cdot \frac{\partial}{\partial x_2}
\end{aligned}
$$

with

$$
\begin{aligned}
C_2 &\equiv \frac{1}{2} L_{AB} L^{AB} \\
C_4 &\equiv \frac{1}{4} L_A{}^B L_B{}^C L_C{}^D L_D{}^A - \frac{1}{2} C_2^2
\end{aligned}
$$

Eigenvalue Problem

- For the Bi-local Laplacian:

$$\widehat{\mathcal{L}}_{\rm bi}\,\psi_{h,s}(x_1,x_2) \;=\; \lambda(h,s)\,\psi_{h,s}(x_1,x_2)$$

- The eigenfunctions are conformal spinning 3-pt funcs:

$$\psi_{h,s}(\vec{x}_1,\vec{x}_2;\vec{x}_3) \;=\; \frac{(V^{\mu} z_{\mu})^{s}}{|x_1-x_3|^{h+s}|x_2-x_3|^{h+s}|x_1-x_2|^{2\Delta+s-h}}$$

and the eigenvalues are

$$\lambda(h,s) \;=\; \frac{1}{4}\,(h^2-s^2)^2$$

d=2 Example

- Eigenfunctions :in terms of by Bessel functions:

$$\psi_{h,s}(\vec{x}_1,\vec{x}_2;\vec{p}) \;=\; |\eta\bar{\eta}|^{\frac{1}{2}}\, e^{-i\vec{p}\cdot\vec{x}}\, K_{\frac{h+s-1}{2}}(ip\eta)\, K_{\frac{h-s-1}{2}}(i\bar{p}\bar{\eta})$$

Exact eigenvalues

$$\lambda(h,s) \;=\; \frac{1}{4}(h+s)(h+s-2)(h-s)(h-s-2)$$

where

$$\vec{x} \equiv \frac{\vec{x}_1+\vec{x}_2}{2}\,, \qquad \vec{\eta} \equiv \frac{\vec{x}_1-\vec{x}_2}{2}$$

Momentum Space

- Fourier Transform of the eigenfunctions (for CFT_2) can be written as

$$\psi_{h,s}(\vec{k}_1, \vec{k}_2; \vec{p}) = \delta(\vec{k}_1 + \vec{k}_2 - \vec{p}) F(k^+) F(k^-)$$

$$F(k^+)F(k^-) = \frac{e^{i\frac{s}{2}(\arcsin\frac{k^+}{p^+} - \arcsin\frac{k^-}{p^-})} \, e^{i\frac{h}{2}(\arcsin\frac{k^+}{p^+} + \arcsin\frac{k^-}{p^-})}}{\sqrt{(k^+)^2 - (p^+)^2}\sqrt{(k^-)^2 - (p^-)^2}}$$

where $k^\pm \equiv k_1^\pm - k_2^\pm \,, \quad k_i^\pm \equiv \frac{1}{2}(k_i^0 \pm k_i^1)$

AdS_3 * S^1 Wave Functions:Mom Space

- The Laplacian:

$$\mathcal{L}_{\mathrm{AdS}_3\times S^1} = \frac{1}{2}\Big[\Big(z\sqrt{|p|^2 - p_z^2}\,\Big)^2 - p_\theta^2\Big]$$

In momentum-space $z \to \frac{\partial}{\partial p_z}$, , $p_\theta \to \frac{\partial}{\partial \theta}$

$$\mathcal{L}_{\mathrm{AdS}_3\times S^1} = \frac{1}{2}\left[\frac{\partial^2}{\partial\phi^2} - \frac{\partial^2}{\partial\theta^2}\right] \qquad \phi \equiv \arccos\left(\frac{p_z}{2|p|}\right)$$

- With : wave functions $\psi_{h,s} = e^{\pm ih\phi}e^{\pm is\theta}$

- Compare with Bi-local function:

Momentum Map

- Comparing the Eigenfunctions (in momentum space) leads to

$$\vec{p} = \vec{k}_1 + \vec{k}_2$$

$$p_z = 2\sqrt{k_1^+ k_1^-} - 2\sqrt{k_2^+ k_2^-}$$

$$\theta = \frac{1}{2}\left(\arcsin\frac{k^+}{p^+} - \arcsin\frac{k^-}{p^-}\right)$$

In Fourier (Momentum) Space, just change of (momentum) variables:

$$\widetilde{\Psi}(k_1^i, k_2^i) = \widetilde{H}(p^i, p_z, \theta)$$

Momentum space in ads/cft (also by Skenderis, et al,2014'

Configuration Space Transform

- By chain rule

$$z = \frac{\sqrt{q_1^+ q_2^+}(u_2^- - u_1^-) + \sqrt{q_1^- q_2^-}(u_2^+ - u_1^+)}{2\left(\sqrt{q_1^+ q_2^-} + \sqrt{q_1^- q_2^+}\right)}$$

$$p^\theta = \sqrt{q_1^- q_2^-}(u_2^+ - u_1^+) - \sqrt{q_1^+ q_2^+}(u_2^- - u_1^-)$$

- The fields are transformed

$$\overline{\Psi}(x_1^i, x_2^i) = \int \mathcal{M}(x_1^i, x_2^i | x^i, z, S)\, H(x^i, z, S)$$

$$H(x^i, z, S) = \int \mathcal{M}^{-1}(x^i, z, S | x_1^i, x_2^i)\, \overline{\Psi}(x_1^i, x_2^i)$$

- Off-shell/

BI-LOCAL (WITTEN) DIAGRAMS

- After expanding the bi-local field as

$$\Psi(x_1, x_2) = \Psi_0(x_1, x_2) + \frac{1}{\sqrt{N}} \eta(x_1, x_2)$$

The Action leads to vertices

$$\begin{aligned} S[\eta] = \frac{N}{4} \int \prod_{a=1}^{4} d^d x_a \, \eta(x_1, x_2) \mathcal{K}(x_1, x_2; x_3, x_4) \eta(x_3, x_4) \\ - \frac{N}{6} \mathrm{Tr}\Big(\Psi_0^{-1} \star \eta \star \Psi_0^{-1} \star \eta \star \Psi_0^{-1} \star \eta \Big) \\ + \frac{N}{8} \mathrm{Tr}\Big(\Psi_0^{-1} \star \eta \star \Psi_0^{-1} \star \eta \star \Psi_0^{-1} \star \eta \star \Psi_0^{-1} \star \eta \Big) + \cdots \end{aligned}$$

Bi-local Propagator

- Using the Spinning 3-pt funcs for complete basis, bi-local propagator is given by

$$\mathcal{D}(x_1, x_2; x_3, x_4) \propto \sum_{h,s,\vec{p}} \frac{\psi^*_{h,s}(x_1, x_2; p)\psi_{h,s}(x_3, x_4; p)}{|x_1 - x_2||x_3 - x_4|\, \lambda_{h,s}}$$

- All-order Vertices V_n : by geometric series

- Through Momentum space Map : AdS HS

-to appear(Kenta Suzuki,R de Mello Koch, J yoon 2018'

Propagator

- Orthogonality/Completnes of the Bessel basis

$$\int d^2x_1 d^2x_2\, |x_{12}|^{-4} \psi^*_{h,s}(x_1,x_2;p)\psi_{h',s'}(x_1,x_2;p')$$
$$= 16\pi^6\, \delta^2(p-p')\, \frac{\delta(h-h')\delta(s-s')}{\nu\bar{\nu}\sin(\pi\nu)\sin(\pi\bar{\nu})}$$

where

$$\nu \equiv \frac{h+s-1}{2}, \qquad \bar{\nu} \equiv \frac{h-s-1}{2}$$

Leads to the bi-local propagator:

$$\mathcal{D}(x_1,x_2;x_3,x_4) = \sum_{h,s,\vec{p}} \frac{\nu\bar{\nu}\sin(\pi\nu)\sin(\pi\bar{\nu})}{36\pi^6\lambda_{h,s}} \frac{\psi^*_{h,s}(x_1,x_2;p)\psi_{h,s}(x_3,x_4;p)}{|x_1-x_2|^2|x_3-x_4|^2}$$

Cubic Vertex

- The cubic vertex in bi-local CFT_2:

$$V^{+}_{(3)}(k_1, k'_1; k_2, k'_2; k_3, k'_3) \;\propto\; \delta(k^{+}_1 + k'^{+}_2)\delta(k^{+}_2 + k'^{+}_3)\delta(k^{+}_3 + k'^{+}_1)$$

Through bi-local map : AdS_3 Higher-spin cubic vertex

$$\begin{aligned} & V^{+}_{(3)}(p_1, \phi_1, \theta_1; p_2, \phi_2, \theta_2; p_3, \phi_3, \theta_3) \\ \propto\ & \delta(p^{+}_1 + p^{+}_2 + p^{+}_3)\,\delta\Big(p^{+}_2\,(1 + \sin(\phi_2 + \theta_2)) + p^{+}_3\,(1 - \sin(\phi_3 + \theta_3))\Big) \\ & \times\ \delta\Big(p^{+}_1\,(1 - \sin(\phi_1 + \theta_1)) + p^{+}_3\,(1 + \sin(\phi_3 + \theta_3))\Big) \end{aligned}$$

Diagrams: 4-point Func

- We consider bi-local 4-pt function:

$$\Big\langle \eta(x_1, x_1')\eta(x_2, x_2')\eta(x_3, x_3')\eta(x_4, x_4') \Big\rangle$$

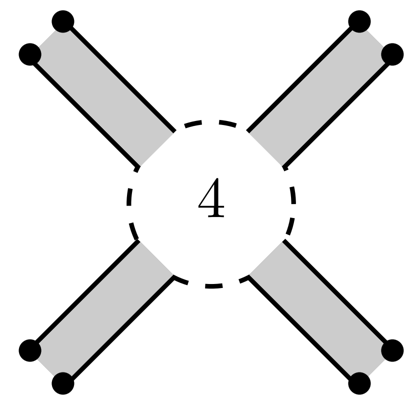
which generates any higher spin 4-pt functions through

$$\widehat{D}_s(x, x') \equiv \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(s-k)!} (\hat{z} \cdot \partial_x)^{s-k} (\hat{z} \cdot \partial_{x'})^k$$

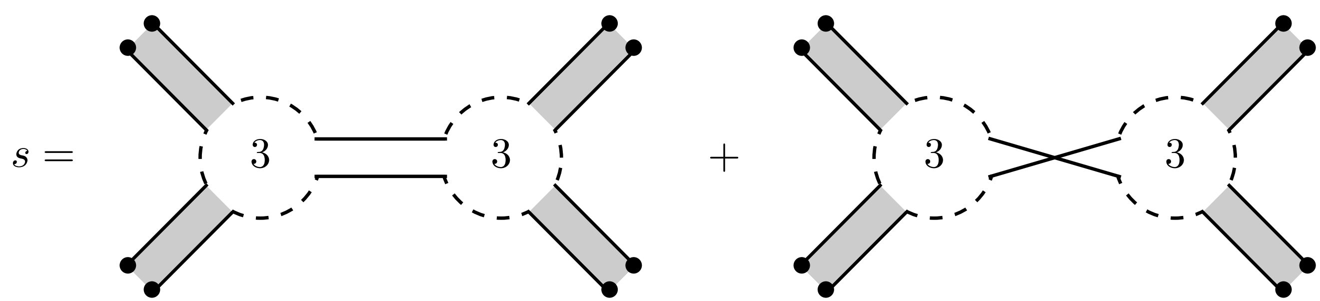
on external legs//

Bi-local Diagrams

- For tree-level 4-pt func,
  we have the contact diagram: 1/N



- And s-, t- u-channel exchange diagrams: 1/N



S-channel Contribution

- S-channel contribution can be written as:

$$\left\langle \eta(x_1, x_1')\eta(x_2, x_2')\eta(x_3, x_3')\eta(x_4, x_4') \right\rangle_s$$

$$= \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \frac{dh}{2\pi i} \sum_{s=0}^{\infty} \frac{\rho(h,s)}{\lambda_{h,s}} \int d^d y \; A(x_1, x_1'; x_2, x_2'; y, h, s) A(y, h^*, s; x_3,$$

where A represents a 3-pt bi-local functions, written as

$$A(x_1, x_1'; x_2, x_2'; y, h, s) \; = \; g_3 \, \Phi_0(x_1', x_2) \, \psi_{h,s}(x_1, x_2'; y)$$
$$+ \; (3 \text{ permutations of external po}$$

Conformal Block

- The Remaining integral leads to the Conformal Block Expansion:

$$\Big\langle \eta(x_1,x_1')\eta(x_2,x_2')\eta(x_3,x_3')\eta(x_4,x_4')\Big\rangle_s$$

$$\sim \frac{\tilde{g}_3^2\,\Phi_0(x_1',x_2')\,\Phi_0(x_3',x_4')}{|x_{12}|^{2\Delta-2}|x_{34}|^{2\Delta-2}} \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \frac{dh}{2\pi i} \sum_{s=0}^{\infty} \frac{\rho(h,s)}{\lambda_{h,s}}$$

$$\times \Big[F_{h,s}(u,v) \; + \; F_{h^*,s}(u,v)\Big] \; + \; (15 \text{ permutations})$$

where $F_{h,s}(u,v)$ is conformal partial wave with

$$u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

S-channel Contribution

- The s-channel contribution to the bi-local 4-pt function:

$$\Big\langle \eta(y_1,y_1')\eta(y_2,y_2')\eta(y_3,y_3')\eta(y_4,y_4')\Big\rangle_s$$
$$= \widetilde{g}_3^2\,\Phi_0(y_1',y_2)\,\Phi_0(y_3',y_4)\int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty}\frac{dh}{2\pi i}\sum_{s=0}^{\infty}\frac{\rho(h,s)}{\lambda_{h,s}}$$
$$\times\int d^2y\,\psi_{h,s}(y_1,y_2';y)\psi_{h^*,s}(y_3,y_4';y)\;+\;(15\text{ permutations})$$

Evaluation

- Introducing (anti-) holomorphic coordinates by

$y \equiv y^1 + iy^2, \;\; \bar{y} \equiv y^1 - iy^2$, the intermediate state integral becomes

$$\int d^2y\; \psi_{h,s}(y_1, y_2; y)\psi_{h^*,s}(y_3, y_4; y)$$

$$= \frac{1}{(y_{12})^{\Delta-2-(\frac{h+s}{2})}(y_{34})^{\Delta-2-(\frac{h+s}{2})}(\bar{y}_{12})^{\Delta-2-(\frac{h-s}{2})}(\bar{y}_{34})^{\Delta-2-(\frac{h-s}{2})}}$$

$$\times \int_{-\infty}^{\infty} dy\; \frac{1}{(y-y_1)^{\frac{h+s}{2}}(y-y_2)^{\frac{h+s}{2}}(y-y_3)^{\frac{h^*+s}{2}}(y-y_4)^{\frac{h^*+s}{2}}}$$

$$\times \int_{-\infty}^{\infty} d\bar{y}\; \frac{1}{(\bar{y}-\bar{y}_1)^{\frac{h-s}{2}}(\bar{y}-\bar{y}_2)^{\frac{h-s}{2}}(\bar{y}-\bar{y}_3)^{\frac{h^*-s}{2}}(\bar{y}-\bar{y}_4)^{\frac{h^*-s}{2}}}$$

Gives:Conformal Block Expansion

- The integral is performed in [Dolan & Osborn '11] as

$$\int d^2y\ \psi_{h,s}(y_1,y_2;y)\psi_{h^*,s}(y_3,y_4;y)$$
$$= \frac{\pi}{|y_{12}|^{2\Delta}|y_{34}|^{2\Delta}}\frac{\Gamma(1-\frac{h+s}{2})\Gamma(\frac{h+s}{2})}{\Gamma(\frac{h-s}{2})\Gamma(1-\frac{h-s}{2})}$$
$$\times\left[\frac{1}{K_{\frac{h^*+s}{2},\frac{h^*-s}{2}}}\frac{\Gamma^2(\frac{h-s}{2})}{\Gamma^2(\frac{h^*+s}{2})}F_{\frac{h+s}{2},\frac{h-s}{2}}(\eta,\bar\eta)+\frac{1}{K_{\frac{h+s}{2},\frac{h-s}{2}}}\frac{\Gamma^2(\frac{h^*-s}{2})}{\Gamma^2(\frac{h+s}{2})}F_{\frac{h^*+s}{2},\frac{h^*-s}{2}}(\eta,\bar\eta)\right]$$

with

$$F_{\beta,\bar\beta}(x,\bar x)\ \equiv\ x^{\beta}\ {}_2F_1(\beta,\beta;2\beta;x)\,\bar x^{\bar\beta}\ {}_2F_1(\bar\beta,\bar\beta;2\bar\beta;\bar x)$$

and conf cross ratios

$$\eta\ \equiv\ \frac{y_{12}y_{34}}{y_{13}y_{24}}\,,\qquad \bar\eta\ \equiv\ \frac{\bar y_{12}\bar y_{34}}{\bar y_{13}\bar y_{24}}$$

Result

- s,t,u-channel diagrams give

$$\left\langle \eta(y_1,y_1')\eta(y_2,y_2')\eta(y_3,y_3')\eta(y_4,y_4')\right\rangle_s$$

$$= \frac{\pi\,\widetilde{g}_3^2\,\Phi_0(y_1',y_2')\,\Phi_0(y_3',y_4')}{|y_{12}|^{2\Delta}|y_{34}|^{2\Delta}} \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \frac{dh}{2\pi i} \sum_{s=0}^{\infty} \frac{\rho(h,s)}{\lambda_{h,s}} \frac{\Gamma(1-\frac{h+s}{2})\Gamma(\frac{h+s}{2})}{\Gamma(\frac{h-s}{2})\Gamma(1-\frac{h-s}{2})}$$

$$\times\left[\frac{1}{K_{\frac{h^*+s}{2},\frac{h^*-s}{2}}}\frac{\Gamma^2(\frac{h-s}{2})}{\Gamma^2(\frac{h^*+s}{2})}F_{\frac{h+s}{2},\frac{h-s}{2}}(\eta,\bar{\eta}) + \frac{1}{K_{\frac{h+s}{2},\frac{h-s}{2}}}\frac{\Gamma^2(\frac{h^*-s}{2})}{\Gamma^2(\frac{h+s}{2})}F_{\frac{h^*+s}{2},\frac{h^*-s}{2}}(\eta,\bar{\eta})\right]$$

$$+\ (15\ \text{permutations})$$

- Sum over Conformal Partial Waves (Operator product expansion
- General Spinning Conformal Blocks

Question of Locality

- No-go Result: [Sleight & Taronna '17] [Ponomarev '17]
  quartic coupling of higher spin theory contains
  $1/\Box_{\rm AdS}$ type non-localities.

- In contrast, bi-local theory leads to

$$S = \eta \cdot K \cdot \eta - V_3 \cdot \eta\eta\eta + V_4 \cdot \eta\eta\eta\eta + \cdots$$

Momentum space) vertices $V_3 = \prod_{i=1}^{3} \delta_i(\cdots)\,, \qquad V_4 = \prod_{i=1}^{4} \delta_i(\cdots)$

and

$$K = C_4 + \frac{1}{4} C_2^2 \approx \left(\Box_{\rm AdS}\right)^2$$

Summary

- Described a Constructive Approach to Vectorial Duality
- Bilocal (Re)Construction of Bulk AdS
- Diagrams : All Order n-point Vertices
- Of-shell (Action) for Higher Spin Gravity
- Reproduce : Conf Partial Wave Exp:Conf Blocks