Towards a Holographic Dictionary for the SYK Model

Sumit R. Das

(with A. Ghosh, A. Jevicki, K. Suzuki)

SYK Model

- This is a model of N real fermions which are all connected to each other by a random coupling. The Hamiltonian is

$$H = (i)^{\frac{q}{2}} \sum_{1\le i_1 < i_2 < \cdots < i_q \le N} j_{i_1 i_2 \cdots i_q} \chi_{i_1} \chi_{i_2} \cdots \chi_{i_q} ; \qquad \{\chi_i, \chi_j\} = \delta_{ij}$$

- The couplings are random with a Gaussian distribution with width J

$$< j^2_{i_1 i_2 \cdots i_q} >= \frac{J^2 (q-1)!}{N^{q-1}}$$

- This model is of interest since this displays quantum chaos and thermalization.
- There are good reasons to believe that there is a dual theory in 2 dimensions which has black holes – so this may serve as a valuable toy model

- In the large N limit, the disordered averaging can be performed .It is then useful to introduce a bilocal collective field

$$\Psi(t_1, t_2) \equiv \frac{1}{N} \sum_{i=1}^{N} \chi_i(t_1)\chi_i(t_2)$$

- And re-write the path integral in terms of these new variables (*Jevicki, Suzuki and Yoon)*
- We will consider the Euclidean theory.

- The path integral is now

$$\int \mathcal{D}\Psi(t_1, t_2)\; e^{-S_c[\Psi]}$$

- Where the collective action includes the jacobian for transformation from the original variables to the new bilocal fields

$$S_{\rm col}[\Psi] = \frac{N}{2}\int dt\left[\partial_t \Psi(t,t')\right]_{t'=t} + \frac{N}{2}\,{\rm Tr}\log\Psi - \frac{J^2 N}{2q}\int dt_1 dt_2\, \Psi^q(t_1,t_2)$$

- The equations of motion are the large N Dyson-Schwinger equations

$$\partial_{t_1}\Psi(t_1,t_2) + \delta(t_1 - t_2) - J^2\int dt_3[\Psi(t_3,t_1)]^{q-1}\Psi(t_3,t_2) = 0$$

- At strong coupling – which is the IR of the theory – the first term can be neglected, and there is an emergent reparametrization symmetry, which includes SL(2,R)
- In this limit, the saddle point solution is

$$\Psi_0(t_1,t_2) = \frac{b}{|t_{12}|^{\frac{2}{q}}}{\rm sgn}(t_{12}) \qquad b^q = \frac{\tan(\frac{\pi}{q})}{J^2\pi}\left(\frac{1}{2} - \frac{1}{q}\right)$$

The Strong Coupling Spectrum

- Expand the bilocal action around the large N saddle point

$$\Psi(t_1, t_2) = \Psi_0(t_1, t_2) + \sqrt{\frac{2}{N}}\, \eta(t_1, t_2)$$

- The quadratic action is $S^{(2)} = \int [dt_1 \cdots dt_4] \eta(t_1, t_2)\, K(t_1, t_2; t_3, t_4)\, \eta(t_3, t_4)$
- The kernel needs to be diagonalized – this is done by using SL(2,R) symmetry *(Kitaev*, *Polchinski and Rosenhaus*). The eigenfunctions are

$$u_{\nu,\omega}(t, z) = \mathrm{sgn}(z)\, e^{i\omega t}\, Z_\nu(|\omega z|) \qquad z = (x - y) \qquad t = (x + y)$$

$$Z_\nu(x) = J_\nu(x) + \xi_\nu\, J_{-\nu}(x)\,, \qquad \xi_\nu = \frac{\tan(\pi\nu/2) + 1}{\tan(\pi\nu/2) - 1}$$

- For non-compact time ω is any real number, and

$$\nu = 3/2 + 2n \qquad \nu = ir$$

- These are eigenfunctions of the SL(2,R) Casimir which is a AdS_2 or dS_2 laplacian

$$\left[z^2(-\partial_t^2+\partial_z^2)+\frac{1}{4}\right]e^{-i\omega t}z^{1/2}Z_\nu(\omega z)=\nu^2\ e^{-i\omega t}z^{1/2}Z_\nu(\omega z)$$

- The orthonormality and completeness relations are

$$\int_0^\infty \frac{dx}{x}\,Z_\nu^*(x)\,Z_{\nu'}(x)\;=\;N_\nu\,\delta(\nu-\nu')\qquad N_\nu\;=\;\begin{cases}(2\nu)^{-1} & \text{for }\nu=3/2+2n\\ 2\nu^{-1}\sin\pi\nu & \text{for }\nu=ir\,,\end{cases}$$

$$\int\frac{d\nu}{N_\nu}\,Z_\nu^*(|x|)\,Z_\nu(|x'|)\;=\;x\,\delta(x-x')\,.$$

- The integral here is a shorthand for a sum over discrete modes and an integral over imaginary values.
- We now expand the fluctuation in terms of these modes

$$\eta(t_1,t_2)\;\equiv\;\Phi(t,z)\;=\sum_{\nu,\omega}\tilde{\Phi}_{\nu,\omega}u_{\nu,\omega}(t,z)$$

- This leads to the quadratic action

$$S^{(2)} \sim \int d\nu \int d\omega \tilde{\Phi}^{\star}_{\nu,\omega} [\tilde{\kappa}(\nu) - 1] \tilde{\Phi}_{\nu,\omega}$$

where

$$\tilde{\kappa(\nu)} = -\frac{1}{(q-1)} \frac{\Gamma(\frac{1}{2}+\frac{1}{q})\Gamma(\frac{1}{q})}{\Gamma(\frac{3}{2}-\frac{1}{q})\Gamma(1-\frac{1}{q})} \frac{\Gamma(\frac{5}{4}-\frac{1}{q}+\frac{\nu}{2})\Gamma(\frac{5}{4}-\frac{1}{q}-\frac{\nu}{2})}{\Gamma(\frac{1}{4}+\frac{1}{q}+\frac{\nu}{2})\Gamma(\frac{1}{4}+\frac{1}{q}+\frac{\nu}{2})}$$

- The spectrum is therefore given by the infinitely many solutions of the equation

$$\tilde{\kappa(\nu)} = 1 \qquad \nu = p_m$$

- For any q $p_m = 3/2$ is always a solution. This is a zero mode at strong coupling coming from broken reparametrization symmetry.

The Bilocal Propagator

- The 4 point function of the fermions (2 point function of bilocals) at large J is

$$\mathcal{G}(t,z;t',z') \sim (zz')^{1/2} \int_{-\infty}^{\infty} d\omega \int \frac{d\nu}{N_\nu}\; e^{-i\omega(t-t'} \frac{Z_\nu(\omega z) Z_\nu(\omega z')}{\tilde{\kappa}(\nu)-1}$$

- Performing the integral over ν the propagator can be expressed as a sum over poles

$$\mathcal{G}(t,z;t',z') \sim -\frac{1}{J}|zz'|^{\frac{1}{2}} \sum_m \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \frac{Z_{-p_m}(|\omega|z^>) J_{p_m}(|\omega|z^<)}{N_{p_m}} R_{p_m}$$

- Here $z^>(z^<)$ denotes the greater (smaller) of z and z'.
- R_{p_m} is the residue of the pole at $\nu = p_m$ - analytic expressions derivable.

- Explicit expression for the residue

$$R_{h-1/2} = N_h \left[H_{-1+\frac{h}{2}+\frac{1}{q}} + H_{\frac{1}{2}-\frac{h}{2}-\frac{1}{q}} - H_{\frac{h}{2}-\frac{1}{q}} - H_{-\frac{1}{2}-\frac{h}{2}+\frac{1}{q}} \right]$$

$$N_h = \frac{\left(\sin \pi h + \sin \frac{2\pi}{q}\right) \Gamma\left(\frac{2}{q}\right) \Gamma\left(2 - h - \frac{2}{q}\right) \Gamma\left(1 + h - \frac{2}{q}\right)}{\pi q \Gamma\left(3 - \frac{2}{q}\right)}$$

Large q limit

- In the large q limit $p_m = 3/2$ remains an exact solution. The other poles are (*Gross and Rosenhaus; Maldacena and Stanford*)

$$p_m = 2m + \frac{1}{2} + \frac{2}{q}\frac{2m^2+m+1}{2m^2+m-1} + \cdots \qquad m = 1, 2, \cdots$$

- The residue at $p_m = 3/2$ remains finite

$$R(3/2) = \frac{2}{3} - \frac{1}{q}\left(\frac{5}{2} + \frac{\pi^2}{3}\right) + O(1/q^2)$$

- The residues at the other poles vanish for large q - these do not contribute to the propagator.

$$R(p_m) \to \frac{1}{q}\frac{4(2m^2+m)}{(2m^2+m-1)^2} + O(1/q^2)$$

- However the large q limit is rather subtle. In some sense this becomes 2d Liouville.

- There is a special mode at $p_0 = 3/2$ which – at strong coupling – is a zero mode of the diffeomorphism invariance in the IR.
- This would lead to an infinite contribution at infinite J. For finite J, the diffeo is explicitly broken. The mode has a correction (*Maldacena and Stanford*)

$$p_0 = 3/2 - \alpha \frac{\omega}{J}$$

- This leads to the following enhanced contribution of this mode to the propagator in the zero temperature limit

$$J \int \frac{d\omega}{\omega^2} e^{i\omega(t'_+ - t_+)} \left[\frac{\sin(\omega t_-)}{\omega t_-} - \cos(\omega t_-) \right] \left[\frac{\sin(\omega t'_-)}{\omega t'_-} - \cos(\omega t'_-) \right]$$

$$t_\pm \equiv \frac{t_1 \pm t_2}{2}, \qquad t'_\pm \equiv \frac{t_3 \pm t_4}{2}$$

- The effective theory which reproduces this is a Schwarzian action whose dynamical variable is the parameter of the diffeomorphism.
- Note this is proportional to J - which is why it diverges at strong coupling

- The object $\Phi(t,z)$ appears to be as a field in 1+1 dimensions.
- However the field action in real space is non-polynomial in derivatives.

$$S^{(2)} = \int dtdz\{(z^{1/2}\eta(t,z)\left[\tilde{\kappa}(\sqrt{\mathcal{D}_B}) - 1\right](z^{1/2}\eta(t,z)\}$$

$$\mathcal{D}_B \equiv z^2(-\partial_t^2 + \partial_z^2) + \frac{1}{4}$$

- In fact the form of the propagator looks like a sum of contributions from an infinite number of fields in AdS or maybe dS
- The conformal dimensions of the corresponding operators are given by $h_m = \frac{1}{2} + p_m$
- Indeed, it is believed that the dual theory is possibly an infinite number of matter fields coupled to Jackiw-Teitelboim dilaton gravity (*Engelsoy, Mertens, Verlinde*; *Maldacena, Stanford and Yang*) or Polyakov 2d action (*Mandal, Nayak and Wadia*).
- The gravity theory has AdS_2 as well as black hole solutions.

CAN WE UNDERSTAND THE EMERGENCE OF THE HOLOGRAPHIC DIRECTION DIRECTLY FROM THE LARGE N DEGREES OF FREEDOM ?

The Bilocal Space

- In fact, as a special case of the suggestion of *S.R.D. and A. Jevicki (2003)* in the context of duality between Vasiliev theory and O(N) vector model – the center of mass and the relative coordinates $z = (t_1 - t_2)$ $t = (t_1 + t_2)$ can be thought of as Poincare coordinates in AdS_2
- The SL(2,R) transformations of the two points of the bilocal

$$\begin{aligned} \delta t_1 &= \epsilon_1, \qquad \delta t_1 = \epsilon_2 t_1 \qquad \delta t_1 = \epsilon_3 t_1^2 \\ \delta t_2 &= \epsilon_1, \qquad \delta t_2 = \epsilon_2 y \qquad \delta t_2 = \epsilon_3 t_2^2 \end{aligned}$$

- These become isometries of a Lorentzian AdS_2 space-time

$$\begin{aligned} \delta t &= \epsilon_1 \qquad \delta z = 0 \\ \delta t &= \epsilon_2 t \qquad \delta z = \epsilon_2 z \\ \delta t &= \frac{1}{2}\epsilon_3(t^2 + z^2) \qquad \delta z = \epsilon_3 t z \end{aligned}$$

$$ds^2 = \frac{1}{z^2}[-dt^2 + dz^2]$$

- The bi-local space indeed provides a realization of AdS_2
- However

1. The wavefunctions which appear in the propagator are not the usual AdS_2 wavefunctions. The latter are Bessel functions with real positive order.
2. The "matter" fields must have rather unconventional Kinetic energy terms – since the poles of the propagator have non-trivial residues.
3. More significantly – we are actually working with Euclidean SYK model – in fact the bilocal propagator *does not have a factor of* i which should be there in Lorentzian signature.

   One would expect that the dual theory should have Euclidean signature as well.

- We will see the understanding of the points (1) and (3) are closely related.

A Three dimensional view

(S.R. Das, A. Jevicki and K. Suzuki : JHEP 1709 (2017) 17
(S.R. Das, A. Ghosh, A. Jevicki and K. Suzuki : JHEP 1802 (2018) 162)

- It turns out that the infinite tower of fields can be interpreted as the KK tower of a Horava-Witten compactification of a 3 dimensional theory in a fixed background.
- For q=4 the background is $ds^2 = \frac{1}{z^2}[-dt^2 + dz^2] + (1 + \frac{a}{z})^2 dy^2 \qquad a \sim 1/J$
- The third direction is an interval S^1/Z_2 with Dirichlet boundary conditions
- There is a single scalar field which is subject to a delta function potential

$$S = \int dzdt \left[(\partial_t\phi)^2 - (\partial_z\phi)^2 + \frac{1}{4z^2} - \frac{1}{z^2}\{(\partial_y\phi)^2 + V(y)\phi^2\}\right] \qquad V(y) = V\delta(y)$$

Schrodinger problem in an infinite well with a delta function in the middle

- This reproduces the spectrum exactly.
- A rather non-standard three dimensional propagator between points which lie at y=0 has an exact agreement with the SYK bilocal propagator – including the enhanced contribution of the h=2 mode.
- Here the non-trivial residues which appear in the SYK answer come from non-trivial wavefunctions in the 3rd direction.
- This generalizes rather nontrivially to arbitrary q. The metric is now non-trivial

$$ds^2 = |x|^{\frac{4}{q}-1}\left[\frac{-dt^2+dz^2}{z^2} + \frac{dx^2}{4|x|(1-|x|)}\right]$$

- And there is a nontrivial potential

$$V(x) = \frac{1}{|x|^{\frac{4}{q}-1}}\left[4\left(\frac{1}{q}-\frac{1}{4}\right)^2 + m_0^2 + \frac{2V}{J(x)}\left(1-\frac{2}{q}\right)\delta(x)\right] \qquad J(x) = \frac{|x|^{\frac{2}{q}-1}}{2\sqrt{1-|x|}}$$

- It is significant that not just the spectrum, but the propagator is exactly reproduced by this three dimensional model.
- However there is no reason to expect that the theory has local interactions in these three dimensions – we should view this 3d picture as an unpacking of the infinite number of modes.

Towards an Euclidean Dual

S.R.D, A. Ghosh, A. Jevicki and K. Suzuki, 1712.02725

- Both Lorentzian and Euclidean AdS_2 have the symmetry group SO(1,2) or SO(2,1).
- The bi-local space has a metric $ds^2 = \frac{1}{\eta^2}[-dt^2 + d\eta^2] \quad t \equiv \frac{t_1 + t_2}{2} \quad \eta. \equiv \frac{t_1 - t_2}{2}$
- The isometry generators are

$$\hat{D}_{1+2} = t_1\, p_1 + t_2\, p_2 \,, \qquad \hat{P}_{1+2} = -p_1 - p_2 \,, \qquad \hat{K}_{1+2} = -\, t_1^2\, p_1 \,-\, t_2^2\, p_2$$

- Euclidean AdS_2 has a metric $ds^2 = \frac{d\tau^2 + dz^2}{z^2}$
- Whose isometries are

$$\hat{D}_{\mathrm{EAdS}} = \tau\, p_\tau + z\, p_z \,, \qquad \hat{P}_{\mathrm{EAdS}} = -p_\tau \,, \qquad \hat{K}_{\mathrm{EAdS}} = (z^2 - \tau^2)\, p_\tau \,-\, 2\tau z\, p_z$$

- One might wonder if there is a canonical transformation which relates these generators.
- The answer turns out to be YES

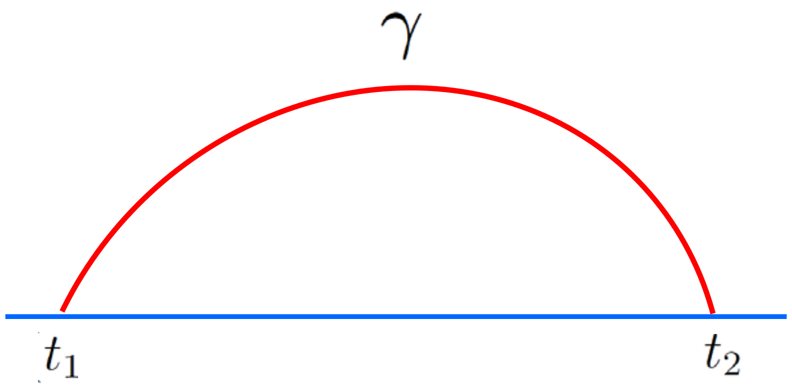
$$\tau = \frac{t_1 p_1 - t_2 p_2}{p_1 - p_2}, \quad p_\tau = p_1 + p_2, \quad z^2 = -\left(\frac{t_1 - t_2}{p_1 - p_2}\right)^2 p_1 p_2, \quad p_z^2 = -4 p_1 p_2$$

- Such canonical transformations lead to integral transformations between corresponding fields living in EAdS_2 and AdS_2 - turns out to be Radon or X-ray transform

$$[\mathcal{R}f](\eta, t) = 2\eta \int_{t-\eta}^{t+\eta} d\tau \int_0^\infty \frac{dz}{z} \delta\left(\eta^2 - (\tau - t)^2 - z^2\right) f(\tau, z)$$

- The procedure is somewhat similar to transforms used to relate linear dilaton backgrounds and black hole backgrounds in two dimensional string theory *(Martinec & Shatashvili; S.R.D., Jevicki and Yoneya; Dhar, Mandal & Wadia)*
- Also to transforms used in relating bilocals to higher spin fields *(R. de Mello Koch, A.Jevicki, K. Jin and J. Rodrigues)*

- The Radon transform of a function on EAdS_2 is the integral of the function evaluated on a geodesic with end-points on the boundary



$$\mathcal{R}f(\tau, z) = \int_\gamma ds\, f(\gamma)$$

$$t \equiv \frac{t_1 + t_2}{2}$$

$$\eta. \equiv \frac{t_1 - t_2}{2}$$

- In fact the bilocal space is somewhat similar to kinematic space defined by *Czech, Lamprou, McCandlish, Mosk and Sully* - however in that case this was on a single time slice.
- The radon transform has been used to relate operators on the boundary to fields in the bulk (*Czech et.al*.; *Bhowmick, Ray and Sen*). Related formulae appear in *(de Boer, Haehl, Heller, Myers, Niemann).*

- The radon transform has the property

$$\nabla^2_{LAdS_2}\mathcal{R} = \mathcal{R}\nabla^2_{EAdS_2}$$

- Remarkably this takes the normalizable eigenfunctions of ∇^2_{EAdS}

$$\phi_{EAdS}(\tau, z) = z^{\frac{1}{2}} e^{-i\omega\tau} K_\nu(\omega z) \qquad \nu = ir$$

to *precisely* the combination of Bessel functions which appear in the SYK problem

$$[\mathcal{R}\phi_{EAdS}](t,\eta) = -\frac{\pi^{3/2}}{\sin(\pi\nu)} \frac{\Gamma(\frac{1}{4}+\frac{\nu}{2})}{\Gamma(\frac{1}{4}+\frac{\nu}{2})} \eta^{\frac{1}{2}} e^{-i\omega t} \left[J_\nu(\omega\eta) + \frac{\tan(\frac{\pi\nu}{2})+1}{\tan(\frac{\pi\nu}{2})-1} J_{-\nu}(\omega\eta) \right] \qquad \nu = ir$$

$$Z_\nu(x) = J_\nu(x) + \xi_\nu J_{-\nu}(x)$$

While for $\nu = \nu_n = 2n + 3/2$

$$\mathcal{R}[\alpha'_{\nu_n} z^{1/2} e^{-ik\tau} I_{\nu_n}(|k|z)] = (2\nu_n\eta)^{1/2} e^{-ikx} J_{\nu_n}(|k|\eta)$$

$$\alpha'_{\nu_n} = \left(\frac{2\nu_n}{\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{3}{4}+\frac{\nu_n}{2})}{\Gamma(\frac{1}{4}+\frac{\nu_n}{2})}$$

- One might have thought that this would mean that the inverse Radon will transform the SYK propagator into the standard Euclidean propagator.
- This is *almost correct*, but not quite.
- Recall the SYK propagator is

$$\mathcal{G}(t,z;t',z') \sim (zz')^{1/2} \int_{-\infty}^{\infty} d\omega \int \frac{d\nu}{N_\nu}\; e^{-i\omega(t-t'} \frac{Z_\nu(\omega z) Z_\nu(\omega z')}{\tilde{\kappa}(\nu) - 1}$$

- We need to perform the inverse transform on the combinations of the Bessel functions and then perform the integration over ν

- Inverse radon transform on the SYK propagator

$$G(\tau, z; \tau', z')$$
$$= \frac{|zz'|^{\frac{1}{2}}}{2\pi J} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega(\tau-\tau')} \left\{ \sum_{m=0}^{\infty} \frac{\Gamma(\frac{3}{4}+\frac{p_m}{2})\Gamma(\frac{3}{4}-\frac{p_m}{2})}{\Gamma(\frac{1}{4}+\frac{p_m}{2})\Gamma(\frac{1}{4}-\frac{p_m}{2})} \frac{p_m}{\widetilde{g}'(p_m)} K_{p_m}(|\omega|z^{>}) I_{p_m}(|\omega|z^{<}) \right.$$
$$\left. + \sum_{n=0}^{\infty} \frac{\Gamma^2(\frac{3}{4}+\frac{\nu_n}{2})}{\Gamma^2(\frac{1}{4}+\frac{\nu_n}{2})} \left(\frac{\nu_n}{\widetilde{g}(\nu_n)-1}\right) I_{\nu_n}(|\omega|z^{<}) \left[2I_{\nu_n}(|\omega|z^{>}) - I_{-\nu_n}(|\omega|z^{>})\right] \right\}. \quad (4.13)$$

- Inverse radon transform on the SYK propagator

$$
\begin{aligned}
&G(\tau, z; \tau', z') \\
&= \frac{|zz'|^{\frac{1}{2}}}{2\pi J} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega(\tau-\tau')} \left\{ \sum_{m=0}^{\infty} \frac{\Gamma(\frac{3}{4}+\frac{p_m}{2})\Gamma(\frac{3}{4}-\frac{p_m}{2})}{\Gamma(\frac{1}{4}+\frac{p_m}{2})\Gamma(\frac{1}{4}-\frac{p_m}{2})} \boxed{\frac{p_m}{\widetilde{g}'(p_m)}} K_{p_m}(|\omega| z^{>}) I_{p_m}(|\omega| z^{<}) \right. \\
&\qquad \left. + \sum_{n=0}^{\infty} \frac{\Gamma^2(\frac{3}{4}+\frac{\nu_n}{2})}{\Gamma^2(\frac{1}{4}+\frac{\nu_n}{2})} \left(\frac{\nu_n}{\widetilde{g}(\nu_n)-1} \right) I_{\nu_n}(|\omega| z^{<}) \left[2 I_{\nu_n}(|\omega| z^{>}) - I_{-\nu_n}(|\omega| z^{>}) \right] \right\}. \qquad (4.13)
\end{aligned}
$$

Residues of the poles – wavefunctions in the 3d picture

- Inverse radon transform on the SYK propagator

$$
\begin{aligned}
&G(\tau, z; \tau', z') \\
&= \frac{|zz'|^{\frac{1}{2}}}{2\pi J} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega(\tau-\tau')} \left\{ \sum_{m=0}^{\infty} \frac{\Gamma(\frac{3}{4}+\frac{p_m}{2})\Gamma(\frac{3}{4}-\frac{p_m}{2})}{\Gamma(\frac{1}{4}+\frac{p_m}{2})\Gamma(\frac{1}{4}-\frac{p_m}{2})} \frac{p_m}{\widetilde{g}'(p_m)} K_{p_m}(|\omega|z^>) I_{p_m}(|\omega|z^<) \right. \\
&\quad \left. + \sum_{n=0}^{\infty} \frac{\Gamma^2(\frac{3}{4}+\frac{\nu_n}{2})}{\Gamma^2(\frac{1}{4}+\frac{\nu_n}{2})} \left(\frac{\nu_n}{\widetilde{g}(\nu_n)-1} \right) I_{\nu_n}(|\omega|z^<) \left[2I_{\nu_n}(|\omega|z^>) - I_{-\nu_n}(|\omega|z^>) \right] \right\} . \qquad (4.13)
\end{aligned}
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Residues of the poles – wavefunctions in the 3d picture

Usual Euclidean propagator for a field

- Inverse radon transform on the SYK propagator

$$
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&= \frac{|zz'|^{\frac{1}{2}}}{2\pi J} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega(\tau-\tau')} \left\{ \sum_{m=0}^{\infty} \frac{\Gamma(\frac{3}{4}+\frac{p_m}{2})\Gamma(\frac{3}{4}-\frac{p_m}{2})}{\Gamma(\frac{1}{4}+\frac{p_m}{2})\Gamma(\frac{1}{4}-\frac{p_m}{2})} \frac{p_m}{\widetilde{g}'(p_m)} K_{p_m}(|\omega|z^>) I_{p_m}(|\omega|z^<) \right. \\
&\left. + \sum_{n=0}^{\infty} \frac{\Gamma^2(\frac{3}{4}+\frac{\nu_n}{2})}{\Gamma^2(\frac{1}{4}+\frac{\nu_n}{2})} \left(\frac{\nu_n}{\widetilde{g}(\nu_n)-1} \right) I_{\nu_n}(|\omega|z^<) \left[2I_{\nu_n}(|\omega|z^>) - I_{-\nu_n}(|\omega|z^>) \right] \right\}. \qquad (4.13)
\end{aligned}
$$

Residues of the poles – wavefunctions in the 3d picture

Usual Euclidean propagator for a field

Additional "Leg Pole" factors. In 3d interpretation another transformation in the 3rd direction

- Similar factors appear in the c=1 matrix model

- Inverse radon transform on the SYK propagator

$$G(\tau, z; \tau', z')$$
$$= \frac{|zz'|^{\frac{1}{2}}}{2\pi J} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega(\tau-\tau')} \left\{ \sum_{m=0}^{\infty} \frac{\Gamma(\frac{3}{4}+\frac{p_m}{2})\Gamma(\frac{3}{4}-\frac{p_m}{2})}{\Gamma(\frac{1}{4}+\frac{p_m}{2})\Gamma(\frac{1}{4}-\frac{p_m}{2})} \frac{p_m}{\widetilde{g}'(p_m)} K_{p_m}(|\omega| z^{>}) I_{p_m}(|\omega| z^{<}) \right.$$
$$\left. + \sum_{n=0}^{\infty} \frac{\Gamma^2(\frac{3}{4}+\frac{\nu_n}{2})}{\Gamma^2(\frac{1}{4}+\frac{\nu_n}{2})} \left(\frac{\nu_n}{\widetilde{g}(\nu_n)-1}\right) I_{\nu_n}(|\omega| z^{<}) \left[2 I_{\nu_n}(|\omega| z^{>}) - I_{-\nu_n}(|\omega| z^{>})\right] \right\}. \qquad (4.13)$$

?? ????? Discrete states

The Black Hole Background

- So far we have dealt with the zero temperature theory.
- The finite temperature theory is described as usual by compactifying the Euclidean time – the answers at infinite coupling can be obtained by performing a reparamterization

$$t = \tan(\frac{\pi\theta}{\beta})$$

- This leads to the metric on the space of bilocals (kinematic space)

$$-\frac{4dt_1dt_2}{|t_1 - t_2|^2} \rightarrow \boxed{\frac{-dt^2 + d\rho^2}{\sin^2\rho}} \qquad \text{FTSYK} \quad \rho = \frac{\pi}{\beta}(\theta_1 - \theta_2) \quad t = \frac{\pi}{\beta}(\theta_1 + \theta_2)$$

- This is NOT the metric of a Lorentzian AdS_2 black hole (or rather AdS_2 Rindler). The latter is

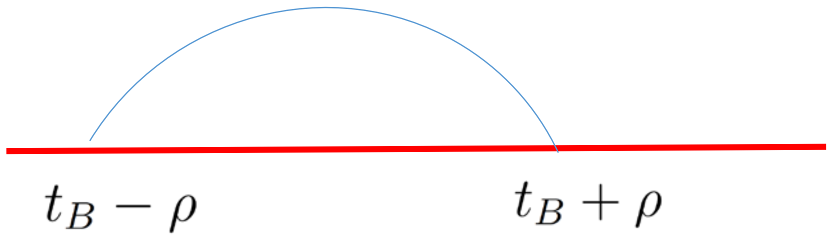
$$\frac{-dt^2 + d\rho^2}{\sinh^2\rho}$$

- The Euclidean continuation of this is

$$ds_B^2 = \frac{d\tau_B^2 + d\xi^2}{\sinh^2 \xi} \qquad -\pi \le \tau_B \le \pi$$

- This is actually diffeomorphic to the whole upper half plane (just as in flat space).
- In these coordinates a geodesic which joins two points on the boundary is

$$\cosh \xi = \sec \rho \ \cos(t_B - \tau_B)$$



- The radon transform of a function is

$$[\mathcal{R}f](t_B, \rho) = \int_{t_B-\rho}^{t_B+\rho} d\tau_B \int \frac{d\xi}{\sinh \xi} \delta(\cosh \xi - \sec \rho \ \cos(t_B - \tau_B)) f(\tau_B, \xi)$$

- This radon transform correctly intertwines the laplacians on Euclidean BH and on the lorentzian metric which appears in the space of bilocals at finite temperature

$$\nabla^2_{FTSYK}\mathcal{R} = \mathcal{R}\nabla^2_{EBH_2}$$

- The eigenvalue problem which needs to be solved to diagonalize the finite temperature kernel (the finite temperature analogs of the $Z_\nu(\omega z)$) are not known for arbitrary finite q.
- However they are known for $q = \infty$ *(Maldacena and Stanford)*. The operator which needs to be diagonalized is precisely ∇^2_{FTSYK}

$$\left[\sin^2\rho(-\partial_t^2 + \partial_\rho^2) + \frac{1}{4}\right]\psi = \nu^2\psi$$

- The solutions which satisfy the periodicity conditions and regularity are

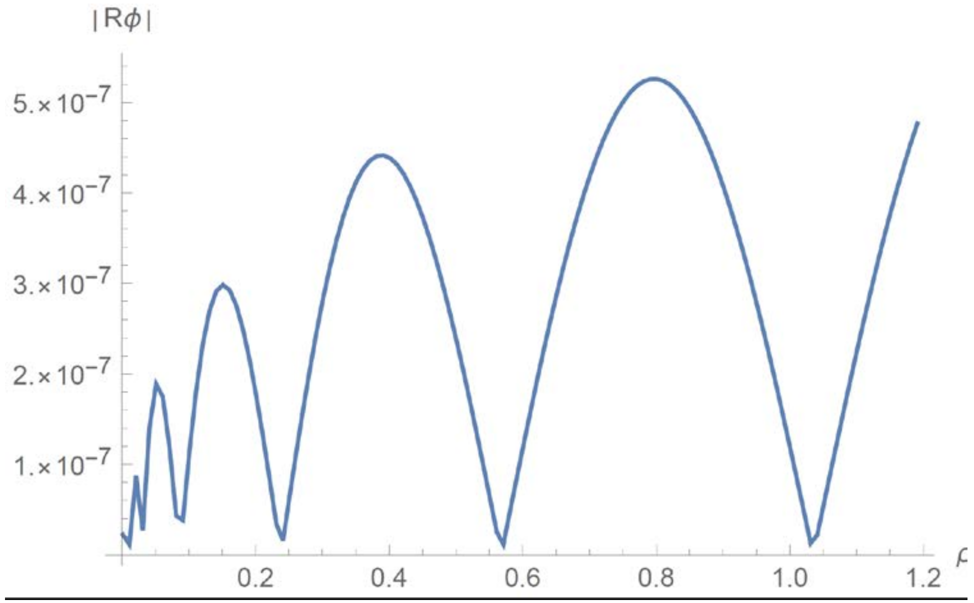
$$\psi_{n,\nu}(t,\rho) = d_{n,\nu}e^{-int}(\cos\rho)(\sin\rho)^{\nu+\frac{1}{2}}\ {}_2F_1[\frac{1}{2}(\frac{3}{2}+\nu-n), \frac{1}{2}(\frac{3}{2}+\nu+n); \frac{3}{2}; \cos^2\rho]$$

$$\nu = ir \qquad \nu = \frac{3}{2} + 2n$$

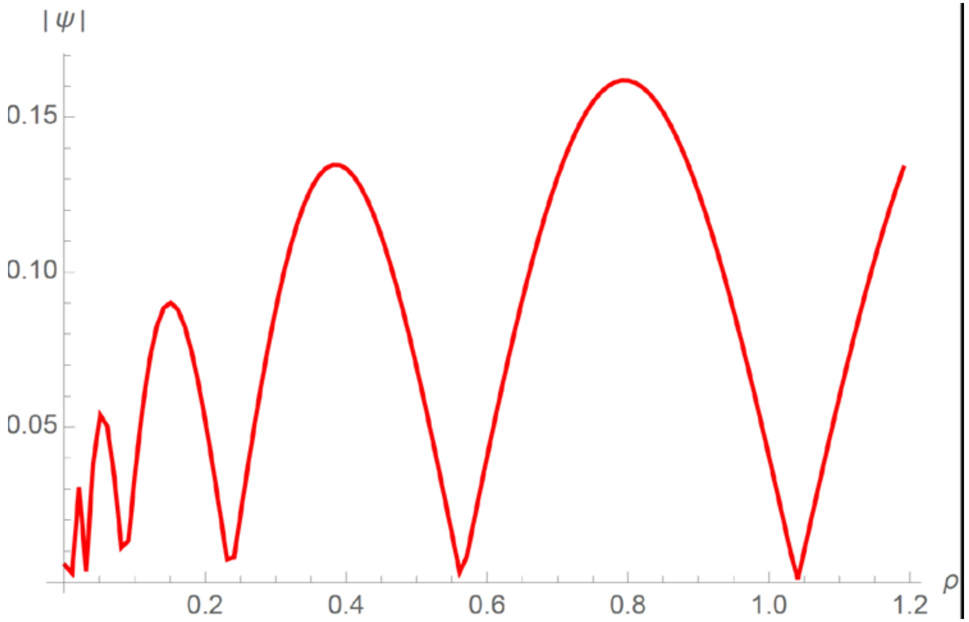
- On the other hand the eigenfunctions of $\nabla^2_{EBH_2}$ which are regular everywhere are, for $\nu = ir$ given by

$$\phi_{n,\nu} = c_{n,\nu} e^{-in\tau_B} (\sinh\xi)^{1/2} Q^{\nu}_{|n|-\frac{1}{2}}(\cosh\xi) \qquad \nu = ir$$

- We have calculated the radon transform of this numerically for several values of r and n and compared them with the eigenfunctions of ∇^2_{FTSYK} with the same values



Radon Transform of Euclidean BH eigenfunction for r=3 and n=5



eigenfunction of ∇^2_{FTSYK} for r=3 and n=5

- We do not have analytic expressions for the normalization factors – so we compared the ratios of these two quantities for different values of ρ

$n = 2, r = 3\ (\nu = ir, h = 1/2 + \nu)$:

ρ_1	ρ_2	$\frac{[\mathcal{R}\phi_{n,\nu}](\rho_1)}{[\mathcal{R}\phi_{n,\nu}](\rho_2)}$	$\frac{\psi_{n,h}(\rho_1)}{\psi_{n,h}(\rho_2)}$
0.3	0.7	−0.725792	−0.738056
0.2	0.5	−2.21603	−2.44764
0.4	0.1	−1.11162	−1.14059
0.7	0.2	−4.70768	−4.422

$n = 5, r = 2.5\ (\nu = ir, h = 1/2 + \nu)$:

ρ_1	ρ_2	$\frac{[\mathcal{R}\phi_{n,\nu}](\rho_1)}{[\mathcal{R}\phi_{n,\nu}](\rho_2)}$	$\frac{\psi_{n,h}(\rho_1)}{\psi_{n,h}(\rho_2)}$
0.3	0.7	−0.833928	−0.842058
0.1	0.6	0.890446	0.919449
0.4	0.1	−1.32968	−1.34681
0.7	0.2	−4.90662	−5.07062

- The agreement gets substantially better with higher accuracy.
- We can say with confidence that the radon transform of the regular eigenfunctions of $\nabla^2_{EBH_2}$ are indeed the correct eigenfunctions of ∇^2_{FTSYK}

Epilogue

- We clearly do not understand the whole story. But we expect that our observations will play a useful role in determining the bulk dual of the SYK model.
- The necessity of a radon transform to relate the space of bilocals of SYK to the dual Euclidean space seems to be a key ingredient.
- The unpacking of the modes by a 3d Horava-Witten should be also useful
- The uncanny resemblance of the form of the propagator with that of “macroscopic loops” in the c=1 Matrix Model prompts a conjecture that the dual theory contains “discrete states” – pretty much like discrete states in two dimensional string theory.
- However we do not yet know what is this dual theory !

THANK YOU