The Schwarzian and black hole physics

Thomas Mertens

Ghent University

Based on arXiv:1606.03438 with J. Engelsöy and H. Verlinde
arXiv:1705.08408 with G.J. Turiaci and H. Verlinde
arXiv:1801.09605
arXiv:1804.09834 with H. Lam, G.J. Turiaci and H. Verlinde

Outline

Motivation: SYK at low energies

SYK-model: N Majorana fermions with all-to-all random interactions of q fermions

Motivation: SYK at low energies

SYK-model: N Majorana fermions with all-to-all random interactions of q fermions

IR symmetry of solution to SYK Dyson-Schwinger equations for two-point function $\langle T\psi_i(\tau_1)\psi_i(\tau_2)\rangle$:

$$G(\tau_1, \tau_2) \rightarrow f'(\tau_1)^\Delta f'(\tau_2)^\Delta G(f(\tau_1), f(\tau_2)), \quad \Delta = 1/q$$

Motivation: SYK at low energies

SYK-model: N Majorana fermions with all-to-all random interactions of q fermions

IR symmetry of solution to SYK Dyson-Schwinger equations for two-point function $\langle T\psi_i(\tau_1)\psi_i(\tau_2)\rangle$:

$$G(\tau_1, \tau_2) \to f'(\tau_1)^\Delta f'(\tau_2)^\Delta G(f(\tau_1), f(\tau_2)), \quad \Delta = 1/q$$

IR conformal symmetry is spontaneously broken (by specific solution G_c) and explicitly broken (by UV)

Motivation: SYK at low energies

SYK-model: N Majorana fermions with all-to-all random interactions of q fermions

IR symmetry of solution to SYK Dyson-Schwinger equations for two-point function $\langle T\psi_i(\tau_1)\psi_i(\tau_2)\rangle$:

$$G(\tau_1,\tau_2) \to f'(\tau_1)^\Delta f'(\tau_2)^\Delta G(f(\tau_1), f(\tau_2)), \quad \Delta = 1/q$$

IR conformal symmetry is spontaneously broken (by specific solution G_c) and explicitly broken (by UV)

Effective Action (master fields Σ, G) $\sim N \Leftrightarrow$ suppressed at $N \to \infty$, unless also $\beta J \to \infty \Rightarrow$ Schwarzian action

$\Rightarrow$ Path-integrate over low-energy reparametrizations f of the classical solution with Schwarzian action

Motivation: SYK at low energies

SYK-model: N Majorana fermions with all-to-all random interactions of q fermions

IR symmetry of solution to SYK Dyson-Schwinger equations for two-point function $\langle T\psi_i(\tau_1)\psi_i(\tau_2)\rangle$:

$$G(\tau_1,\tau_2) \to f'(\tau_1)^\Delta f'(\tau_2)^\Delta G(f(\tau_1), f(\tau_2)), \quad \Delta = 1/q$$

IR conformal symmetry is spontaneously broken (by specific solution G_c) and explicitly broken (by UV)

Effective Action (master fields Σ, G) $\sim N \Leftrightarrow$ suppressed at $N \to \infty$, unless also $\beta J \to \infty \Rightarrow$ Schwarzian action

$\Rightarrow$ Path-integrate over low-energy reparametrizations f of the classical solution with Schwarzian action

G_c has $SL(2,\mathbb{R})$-symmetry $\Rightarrow$ not to be path integrated $\equiv$ gauge redundancy

Motivation: 2d dilaton gravity

Jackiw-Teitelboim (JT) gravity is model of 2d dilaton gravity
JT appears in s-wave dim. red. from near-extremal black holes

Motivation: 2d dilaton gravity

Jackiw-Teitelboim (JT) gravity is model of 2d dilaton gravity
JT appears in s-wave dim. red. from near-extremal black holes

JT is holographically dual to Schwarzian theory

Motivation: 2d dilaton gravity

Jackiw-Teitelboim (JT) gravity is model of 2d dilaton gravity
JT appears in s-wave dim. red. from near-extremal black holes

JT is holographically dual to Schwarzian theory

Derivation:

Motivation: 2d dilaton gravity

Jackiw-Teitelboim (JT) gravity is model of 2d dilaton gravity
JT appears in s-wave dim. red. from near-extremal black holes

JT is holographically dual to Schwarzian theory

Derivation:

$$S = \frac{1}{16\pi G} \int d^2x \sqrt{-g}\Phi(R+2) + \frac{1}{8\pi G} \int d\tau \sqrt{-\gamma}\Phi_{bdy} K$$

Path integrate over $\Phi \Rightarrow R = -2$: Geometry fixed as AdS_2

Motivation: 2d dilaton gravity

Jackiw-Teitelboim (JT) gravity is model of 2d dilaton gravity
JT appears in s-wave dim. red. from near-extremal black holes

JT is holographically dual to Schwarzian theory

Derivation:

$$S = \frac{1}{16\pi G}\int d^2x\sqrt{-g}\Phi(R+2) + \frac{1}{8\pi G}\int d\tau\sqrt{-\gamma}\Phi_{bdy}K$$

Path integrate over $\Phi \Rightarrow R = -2$: Geometry fixed as AdS_2

- Consider boundary curve $(f(\tau), z = \epsilon\dot{f}(\tau))$ as UV cutoff, carving out a shape from AdS, $f =$ time reparametrization

Motivation: 2d dilaton gravity

Jackiw-Teitelboim (JT) gravity is model of 2d dilaton gravity
JT appears in s-wave dim. red. from near-extremal black holes

JT is holographically dual to Schwarzian theory

Derivation:

$$S = \frac{1}{16\pi G}\int d^2x\sqrt{-g}\Phi(R+2) + \frac{1}{8\pi G}\int d\tau\sqrt{-\gamma}\Phi_{bdy}K$$

Path integrate over $\Phi \Rightarrow R = -2$: Geometry fixed as AdS_2

- Consider boundary curve $(f(\tau), z = \epsilon \dot{f}(\tau))$ as UV cutoff, carving out a shape from AdS, f = time reparametrization
- Fix asymptotics of dilaton $\Phi_{bdy} \sim 1/\epsilon$

Using $K = 1 + \epsilon^2 \{f, \tau\}$

$$\boxed{\Rightarrow S = -C\int dt\,\{f,\tau\}}, \quad C \to \infty \equiv G \to 0$$

Motivation: 2d dilaton gravity

Jackiw-Teitelboim (JT) gravity is model of 2d dilaton gravity
JT appears in s-wave dim. red. from near-extremal black holes

JT is holographically dual to Schwarzian theory

Derivation:

$$S = \frac{1}{16\pi G}\int d^2x\sqrt{-g}\Phi(R+2) + \frac{1}{8\pi G}\int d\tau\sqrt{-\gamma}\Phi_{bdy}K$$

Path integrate over $\Phi \Rightarrow R=-2$: Geometry fixed as AdS_2

- Consider boundary curve $(f(\tau), z=\epsilon\dot{f}(\tau))$ as UV cutoff, carving out a shape from AdS, $f =$ time reparametrization
- Fix asymptotics of dilaton $\Phi_{bdy}\sim 1/\epsilon$

Using $K = 1+\epsilon^2\{f,\tau\}$

$$\boxed{\Rightarrow S = -C\int dt\,\{f,\tau\}}, \quad C\to\infty \equiv G\to 0$$

Compare to CS / WZW, 3d gravity / Liouville topological dualities

The Schwarzian theory: goal

Main goal:
Compute all correlation functions:

$$\left\langle \mathcal{O}_{\ell_1}\mathcal{O}_{\ell_2}\dots\right\rangle_\beta = \frac{1}{Z}\int_{\mathcal{M}} [\mathcal{D}f]\mathcal{O}_{\ell_1}\mathcal{O}_{\ell_2}\dots\ e^{C\int_0^\beta d\tau \left(\left\{f,\tau\right\}+\frac{2\pi^2}{\beta^2}f'^2\right)}$$

with action

$$S[f] = -C\int_0^\beta d\tau\left\{F,\tau\right\}, \qquad F \equiv \tan\left(\frac{\pi f(\tau)}{\beta}\right)$$
$$= -C\int_0^\beta d\tau\left(\left\{f,\tau\right\}+\frac{2\pi^2}{\beta^2}f'^2\right)$$

where $\left\{f,\tau\right\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ the Schwarzian derivative

The Schwarzian theory: goal

Main goal:
Compute all correlation functions:

$$\langle \mathcal{O}_{\ell_1} \mathcal{O}_{\ell_2} \dots \rangle_\beta = \frac{1}{Z} \int_{\mathcal{M}} [\mathcal{D}f] \mathcal{O}_{\ell_1} \mathcal{O}_{\ell_2} \dots \; e^{C \int_0^\beta d\tau \left(\left\{ f, \tau \right\} + \frac{2\pi^2}{\beta^2} f'^2 \right)}$$

with action

$$S[f] = -C \int_0^\beta d\tau \left\{ F, \tau \right\}, \qquad F \equiv \tan\left(\frac{\pi f(\tau)}{\beta}\right)$$
$$= -C \int_0^\beta d\tau \left(\left\{ f, \tau \right\} + \frac{2\pi^2}{\beta^2} f'^2 \right)$$

where $\left\{ f, \tau \right\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ the Schwarzian derivative
with $\mathcal{M} = \mathrm{Diff}(S^1)/SL(2,\mathbb{R}), \qquad f(\tau+\beta) = f(\tau) + \beta$

The Schwarzian theory: goal

Main goal:
Compute all correlation functions:

$$\langle \mathcal{O}_{\ell_1}\mathcal{O}_{\ell_2}\dots\rangle_\beta = \frac{1}{Z}\int_{\mathcal{M}}[\mathcal{D}f]\mathcal{O}_{\ell_1}\mathcal{O}_{\ell_2}\dots\, e^{C\int_0^\beta d\tau\left(\left\{f,\tau\right\}+\frac{2\pi^2}{\beta^2}f'^2\right)}$$

with action

$$S[f] = -C\int_0^\beta d\tau\left\{F,\tau\right\}, \qquad F \equiv \tan\left(\frac{\pi f(\tau)}{\beta}\right)$$
$$= -C\int_0^\beta d\tau\left(\left\{f,\tau\right\}+\frac{2\pi^2}{\beta^2}f'^2\right)$$

where $\left\{f,\tau\right\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ the Schwarzian derivative
with $\mathcal{M} = \mathrm{Diff}(S^1)/SL(2,\mathbb{R}), \qquad f(\tau+\beta) = f(\tau)+\beta$

In particular: finding gravitational physics in exact correlators

Bilocal operators:

$$\mathcal{O}_\ell(\tau_1,\tau_2) \equiv \left(\frac{F'(\tau_1)F'(\tau_2)}{(F(\tau_1)-F(\tau_2))^2}\right)^\ell \equiv \left(\frac{f'(\tau_1)f'(\tau_2)}{\frac{\beta}{\pi}\sin^2\frac{\pi}{\beta}[f(\tau_1)-f(\tau_2)]}\right)^\ell$$

Think of this expression as two-point function $\mathcal{O}_\ell(\tau_1,\tau_2) = \langle \mathcal{O}(\tau_1)\mathcal{O}(\tau_2)\rangle_{\mathsf{CFT}}$ of some 1D 'matter CFT' at finite temperature coupled to the Schwarzian theory

Stanford and Witten demonstrated that Schwarzian partition function is one-loop exact Stanford-Witten '17:

Partition function: Duistermaat-Heckman

Stanford and Witten demonstrated that Schwarzian partition function is one-loop exact Stanford-Witten '17:

$$Z(\beta) = \left(\frac{\pi}{\beta}\right)^{3/2} \exp\left(\frac{\pi^2}{\beta}\right) = \int_0^{+\infty} dk^2 \sinh(2\pi k) e^{-\beta k^2} \quad (C = 1/2)$$

Density of states $\rho(E) = \sinh(2\pi\sqrt{E})$

Partition function: Duistermaat-Heckman

Stanford and Witten demonstrated that Schwarzian partition function is one-loop exact Stanford-Witten '17:

$$Z(\beta) = \left(\frac{\pi}{\beta}\right)^{3/2} \exp\left(\frac{\pi^2}{\beta}\right) = \int_0^{+\infty} dk^2 \sinh(2\pi k) e^{-\beta k^2} \quad (C = 1/2)$$

Density of states $\rho(E) = \sinh(2\pi\sqrt{E})$

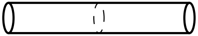
$\Rightarrow$ Cardy scaling at high energies $\rho(E) \sim e^{2\pi\sqrt{E}}$

$\Rightarrow$ 2d CFT origin ?

What about correlators ? Need other techniques

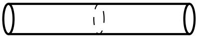
Partition function: Vacuum character

Observation:

$$\mathrm{Tr}_0\left(q^{L_0}\right) \equiv \chi_0(q) = \frac{q^{\frac{1-c}{24}}(1-q)}{\eta(\tau)} =$$  $, \quad q = e^{-2\pi t}$

Observation:

$$\mathrm{Tr}_0\left(q^{L_0}\right) \equiv \chi_0(q) = \frac{q^{\frac{1-c}{24}}(1-q)}{\eta(\tau)} =$$

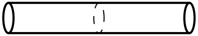


$$, \quad q = e^{-2\pi t}$$

Take double-scaling limit $c \to \infty$, $\qquad t = 12\pi\beta/c \to 0$

Partition function: Vacuum character

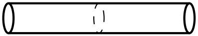
Observation:

$\mathrm{Tr}_0\left(q^{L_0}\right) \equiv \chi_0(q) = \frac{q^{\frac{1-c}{24}}(1-q)}{\eta(\tau)} =$ , $q = e^{-2\pi t}$

Take double-scaling limit $c \to \infty, \qquad t = 12\pi\beta/c \to 0$

$$Z(\beta) \sim \left(\frac{\pi}{\beta}\right)^{3/2} \exp\left(\frac{\pi^2}{\beta}\right)$$

Observation:

$$\mathrm{Tr}_0\left(q^{L_0}\right) \equiv \chi_0(q) = \frac{q^{\frac{1-c}{24}}(1-q)}{\eta(\tau)} =$$  $, \; q = e^{-2\pi t}$

Take double-scaling limit $c \to \infty$, $\qquad t = 12\pi\beta/c \to 0$

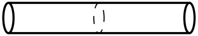
$$Z(\beta) \sim \left(\frac{\pi}{\beta}\right)^{3/2} \exp\left(\frac{\pi^2}{\beta}\right)$$

Schwarzian limit from 2d CFT: Cylinder radius $\to 0$ while $c \to \infty$ keeping product fixed

From Liouville perspective:

Partition function: Vacuum character

Observation:

$\mathrm{Tr}_0\left(q^{L_0}\right) \equiv \chi_0(q) = \frac{q^{\frac{1-c}{24}}(1-q)}{\eta(\tau)} =$ , $q = e^{-2\pi t}$

Take double-scaling limit $c \to \infty, \qquad t = 12\pi\beta/c \to 0$

$Z(\beta) \sim \left(\frac{\pi}{\beta}\right)^{3/2} \exp\left(\frac{\pi^2}{\beta}\right)$

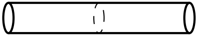
Schwarzian limit from 2d CFT: Cylinder radius $\to 0$ while $c \to \infty$ keeping product fixed

From Liouville perspective:

$\chi_0(q) =$ Cyl. amplitude of Liouville between ZZ $= \langle ZZ|\tilde{q}^{L_0}|ZZ\rangle$

Partition function: Vacuum character

Observation:

$$\mathrm{Tr}_0\left(q^{L_0}\right) \equiv \chi_0(q) = \frac{q^{\frac{1-c}{24}}(1-q)}{\eta(\tau)} =$$  $, \quad q = e^{-2\pi t}$

Take double-scaling limit $c \to \infty, \qquad t = 12\pi\beta/c \to 0$

$$Z(\beta) \sim \left(\frac{\pi}{\beta}\right)^{3/2} \exp\left(\frac{\pi^2}{\beta}\right)$$

Schwarzian limit from 2d CFT: Cylinder radius $\to 0$ while $c \to \infty$ keeping product fixed

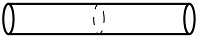
From Liouville perspective:

$\chi_0(q) =$ Cyl. amplitude of Liouville between ZZ $= \langle ZZ|\tilde{q}^{L_0}|ZZ\rangle$

$$|ZZ\rangle = \int_0^\infty dP\, \Psi_{\mathrm{ZZ}}(P)\, \|P\rangle\rangle \quad \Rightarrow \quad \chi_0 = \int_0^\infty dP\, |\Psi_{\mathrm{ZZ}}(P)|^2\ e^{-\beta\frac{P^2}{b^2}}$$

Partition function: Vacuum character

Observation:

$$\mathrm{Tr}_0\left(q^{L_0}\right) \equiv \chi_0(q) = \frac{q^{\frac{1-c}{24}}(1-q)}{\eta(\tau)} =$$  $, \quad q = e^{-2\pi t}$

Take double-scaling limit $c \to \infty, \qquad t = 12\pi\beta/c \to 0$

$$Z(\beta) \sim \left(\frac{\pi}{\beta}\right)^{3/2} \exp\left(\frac{\pi^2}{\beta}\right)$$

Schwarzian limit from 2d CFT: Cylinder radius $\to 0$ while $c \to \infty$ keeping product fixed

From Liouville perspective:

$\chi_0(q) =$ Cyl. amplitude of Liouville between ZZ $= \langle ZZ|\tilde{q}^{L_0}|ZZ\rangle$

$$|ZZ\rangle = \int_0^\infty dP\, \Psi_{\mathrm{ZZ}}(P)\, \| P\rangle\rangle \quad \Rightarrow \quad \chi_0 = \int_0^\infty dP\, |\Psi_{\mathrm{ZZ}}(P)|^2 \; e^{-\beta \frac{P^2}{b^2}}$$

with density

$$|\Psi_{\mathrm{ZZ}}(P)|^2 = \sinh\left(2\pi b P\right) \sinh\left(\frac{2\pi P}{b}\right) \qquad c = 1 + 6(b + \tfrac{1}{b})^2$$

In Schwarzian limit: $P = bk$, $b \to 0$

Path-integral link Liouville and Schwarzian

One can prove via Liouville phase space path integral between ZZ-branes:
$\int_{\phi(0)=\phi(T)} [\mathcal{D}\phi] [\mathcal{D}\pi_\phi] \, e^{\int_0^T d\tau \int d\sigma \left(i\pi_\phi \dot{\phi} - \mathcal{H}(\phi, \pi_\phi) \right)}$

- Field redefinition $(\phi, \pi_\phi) \to (A, B)$ Gervais-Neveu '82:

$$e^{\phi} = -8 \frac{A_\sigma B_\sigma}{(A-B)^2}, \quad \pi_\phi = \frac{A_{\sigma\sigma}}{A_\sigma} - \frac{B_{\sigma\sigma}}{B_\sigma} - 2 \frac{A_\sigma + B_\sigma}{(A-B)}$$

Path-integral link Liouville and Schwarzian

One can prove via Liouville phase space path integral between ZZ-branes:
$\int_{\phi(0)=\phi(T)} [\mathcal{D}\phi] [\mathcal{D}\pi_\phi] \, e^{\int_0^T d\tau \int d\sigma \left(i\pi_\phi \dot{\phi} - \mathcal{H}(\phi,\pi_\phi) \right)}$

- Field redefinition $(\phi, \pi_\phi) \to (A, B)$ Gervais-Neveu '82:

$$e^{\phi} = -8 \frac{A_\sigma B_\sigma}{(A-B)^2}, \quad \pi_\phi = \frac{A_{\sigma\sigma}}{A_\sigma} - \frac{B_{\sigma\sigma}}{B_\sigma} - 2\frac{A_\sigma + B_\sigma}{(A-B)}$$

$$\Rightarrow \mathcal{H} = -\tfrac{c}{24\pi} \left\{ A(\sigma,\tau), \sigma \right\} - \tfrac{c}{24\pi} \left\{ B(\sigma,\tau), \sigma \right\}$$

- Schwarzian limit: $\pi_\phi \dot{\phi} \to 0$
- ZZ-brane boundary conditions: A, B written in terms of 1 doubled field F

Path-integral link Liouville and Schwarzian

One can prove via Liouville phase space path integral between ZZ-branes:

$\int_{\phi(0)=\phi(T)} [\mathcal{D}\phi] \, [\mathcal{D}\pi_\phi] \, e^{\int_0^T d\tau \int d\sigma \left(i\pi_\phi \dot{\phi} - \mathcal{H}(\phi,\pi_\phi) \right)}$

- Field redefinition $(\phi, \pi_\phi) \to (A, B)$ Gervais-Neveu '82:

$$e^\phi = -8 \frac{A_\sigma B_\sigma}{(A-B)^2}, \quad \pi_\phi = \frac{A_{\sigma\sigma}}{A_\sigma} - \frac{B_{\sigma\sigma}}{B_\sigma} - 2\frac{A_\sigma + B_\sigma}{(A-B)}$$

$$\Rightarrow \mathcal{H} = -\frac{c}{24\pi} \left\{ A(\sigma,\tau), \sigma \right\} - \frac{c}{24\pi} \left\{ B(\sigma,\tau), \sigma \right\}$$

- Schwarzian limit: $\pi_\phi \dot{\phi} \to 0$
- ZZ-brane boundary conditions: A, B written in terms of 1 doubled field F

Liouville-Schwarzian Dictionary:

- $T(w) \;\to\; -\frac{c}{24\pi} \left\{ F(\sigma), \sigma \right\}$
- $e^\phi \;\to\; \frac{F_1' F_2'}{(F_1 - F_2)^2}$

2-point correlator

Idea: $V_\ell(\tau_1,\tau_2) \equiv e^{2\ell\phi(\tau_1,\tau_2)} \longleftrightarrow \mathcal{O}_\ell(\tau_1,\tau_2)$

2-point correlator

Idea: $V_\ell(\tau_1,\tau_2) \equiv e^{2\ell\phi(\tau_1,\tau_2)} \longleftrightarrow \mathcal{O}_\ell(\tau_1,\tau_2)$

$$\langle ZZ|\, V_\ell \,|ZZ\rangle = \int dPdQ\, \Psi^*_{ZZ}(P)\Psi_{ZZ}(Q)\langle\langle P||V_\ell||Q\rangle\rangle$$

2-point correlator

Idea: $V_\ell(\tau_1,\tau_2) \equiv e^{2\ell\phi(\tau_1,\tau_2)} \longleftrightarrow \mathcal{O}_\ell(\tau_1,\tau_2)$

$$\langle ZZ|\, V_\ell\, |ZZ\rangle = \int dPdQ\, \Psi^*_{ZZ}(P)\Psi_{ZZ}(Q)\langle\langle P||V_\ell||Q\rangle\rangle$$

Two ways of computing matrix element:

- Schwarzian limit: $||Q\rangle\rangle \to |Q\rangle$
  $\Rightarrow$ 3-point function on sphere $\Rightarrow$ large c limit of DOZZ formula

2-point correlator

Idea: $V_\ell(\tau_1,\tau_2) \equiv e^{2\ell\phi(\tau_1,\tau_2)} \longleftrightarrow \mathcal{O}_\ell(\tau_1,\tau_2)$

$$\langle ZZ|\, V_\ell \,|ZZ\rangle = \int dPdQ\, \Psi^*_{ZZ}(P)\Psi_{ZZ}(Q)\langle\langle P||V_\ell||Q\rangle\rangle$$

Two ways of computing matrix element:

- Schwarzian limit: $||Q\rangle\rangle \to |Q\rangle$
  $\Rightarrow$ 3-point function on sphere $\Rightarrow$ large c limit of DOZZ formula
- Computation in minisuperspace regime of 2d Liouville CFT:
  $\langle P|\, e^{-(\beta-\tau)H} e^{2\ell\phi} e^{-\tau H}\,|Q\rangle$
  Explicitly evaluate minisuperspace integrals

2-point correlator

Idea: $V_\ell(\tau_1,\tau_2) \equiv e^{2\ell\phi(\tau_1,\tau_2)} \longleftrightarrow \mathcal{O}_\ell(\tau_1,\tau_2)$

$\langle ZZ| V_\ell |ZZ\rangle = \int dPdQ\, \Psi^*_{ZZ}(P)\Psi_{ZZ}(Q)\langle\langle P|| V_\ell ||Q\rangle\rangle$

Two ways of computing matrix element:

- Schwarzian limit: $||Q\rangle\rangle \rightarrow |Q\rangle$
  $\Rightarrow$ 3-point function on sphere $\Rightarrow$ large c limit of DOZZ formula
- Computation in minisuperspace regime of 2d Liouville CFT:
  $\langle P| e^{-(\beta-\tau)H} e^{2\ell\phi} e^{-\tau H} |Q\rangle$
  Explicitly evaluate minisuperspace integrals

Both agree:

$$\boxed{G^\beta_\ell(\tau_1,\tau_2) = \frac{1}{Z(\beta)} \int d\mu(k_1)d\mu(k_2)\, e^{-\tau k_1^2-(\beta-\tau)k_2^2}\, \frac{\Gamma\left(\ell \pm i(k_1 \pm k_2)\right)}{2\sqrt{\pi}\Gamma(2\ell)}}$$

$d\mu(k) \equiv dk^2 \sinh(2\pi k)$

Application: Semiclassics for light operators

Semi-classical regime $C \to \infty$, $\ell \ll C$: $k_1 \sim k_2 \gg 1$
Redefine $k_1^2 = M + \omega, k_2^2 = M, \qquad M \gg \omega$

$$G_\ell^\pm \sim \int_0^\infty dM e^{2\pi\sqrt{M} - \frac{\beta}{2C}M} \int \frac{d\omega}{2\pi} e^{\pm i\frac{\tau}{2C}\omega + \pi\frac{\omega}{2\sqrt{M}}} \frac{\Gamma(\ell \pm i\frac{\omega}{2\sqrt{M}})}{\Gamma(2\ell)} (2\sqrt{M})^{2\ell-1}$$

Application: Semiclassics for light operators

Semi-classical regime $C \to \infty$, $\ell \ll C$: $k_1 \sim k_2 \gg 1$
Redefine $k_1^2 = M + \omega, k_2^2 = M, \qquad M \gg \omega$

$$G_\ell^\pm \sim \int_0^\infty dM e^{2\pi\sqrt{M} - \frac{\beta}{2C}M} \int \frac{d\omega}{2\pi} e^{\pm i \frac{\tau}{2C}\omega + \pi\frac{\omega}{2\sqrt{M}}} \frac{\Gamma(\ell \pm i\frac{\omega}{2\sqrt{M}})}{\Gamma(2\ell)} (2\sqrt{M})^{2\ell-1}$$

Interpretation:

- M-integral has saddle: $M_0 = 4\pi^2 C^2/\beta^2$, the JT black hole $E(T)$-relation
  Remaining ω-integral is done explicitly to yield:
  $G_\ell^{\pm,cl}(\tau_1, \tau_2) = \left(\frac{\pi}{\beta \sinh \frac{\pi}{\beta}\tau_{12}}\right)^{2\ell}$

Application: Semiclassics for light operators

Semi-classical regime $C \to \infty$, $\ell \ll C$: $k_1 \sim k_2 \gg 1$
Redefine $k_1^2 = M + \omega, k_2^2 = M, \qquad M \gg \omega$

$$G_\ell^\pm \sim \int_0^\infty dM e^{2\pi\sqrt{M} - \frac{\beta}{2C}M} \int \frac{d\omega}{2\pi} e^{\pm i \frac{\tau}{2C}\omega + \pi \frac{\omega}{2\sqrt{M}}} \frac{\Gamma(\ell \pm i \frac{\omega}{2\sqrt{M}})}{\Gamma(2\ell)} (2\sqrt{M})^{2\ell-1}$$

Interpretation:

- M-integral has saddle: $M_0 = 4\pi^2 C^2/\beta^2$, the JT black hole $E(T)$-relation
  Remaining ω-integral is done explicitly to yield:
  $G_\ell^{\pm,cl}(\tau_1, \tau_2) = \left(\frac{\pi}{\beta \sinh \frac{\pi}{\beta}\tau_{12}}\right)^{2\ell}$
- Γ-functions give poles: $\ell - i\frac{\omega}{2\sqrt{M}} = -n$
  Matches with quasi-normal modes of AdS_2 BH metric Keeler-Ng '14: $\omega = -i\frac{2\pi}{\beta}(n+\ell)$

Application: Semiclassics for light operators

Semi-classical regime $C \to \infty$, $\ell \ll C$: $k_1 \sim k_2 \gg 1$
Redefine $k_1^2 = M + \omega, k_2^2 = M, \qquad M \gg \omega$

$$G_\ell^\pm \sim \int_0^\infty dM e^{2\pi\sqrt{M} - \frac{\beta}{2C}M} \int \frac{d\omega}{2\pi} e^{\pm i \frac{\tau}{2C}\omega + \pi\frac{\omega}{2\sqrt{M}}} \frac{\Gamma(\ell \pm i\frac{\omega}{2\sqrt{M}})}{\Gamma(2\ell)} (2\sqrt{M})^{2\ell-1}$$

Interpretation:

- M-integral has saddle: $M_0 = 4\pi^2 C^2/\beta^2$, the JT black hole $E(T)$-relation
  Remaining ω-integral is done explicitly to yield:
  $G_\ell^{\pm,cl}(\tau_1,\tau_2) = \left(\frac{\pi}{\beta \sinh \frac{\pi}{\beta}\tau_{12}}\right)^{2\ell}$
- Γ-functions give poles: $\ell - i\frac{\omega}{2\sqrt{M}} = -n$
  Matches with quasi-normal modes of AdS_2 BH metric Keeler-Ng '14: $\omega = -i\frac{2\pi}{\beta}(n+\ell)$
- Quantum black hole M emits and reabsorbs excitation with mass $\sim \ell$ and energy ω

Time-ordered 4-point function

$\langle ZZ| V_{\ell_1} V_{\ell_2} |ZZ\rangle$

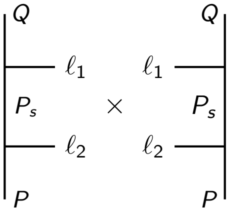
Evaluated using Conformal blocks

Time-ordered 4-point function

$\langle ZZ| \, V_{\ell_1} V_{\ell_2} \, |ZZ\rangle$

Evaluated using Conformal blocks

$$G_{\ell_1\ell_2} = \int dP \, dQ \, dP_s \; \Psi^*_{\mathrm{ZZ}}(P)\Psi_{\mathrm{ZZ}}(Q) \;\times$$

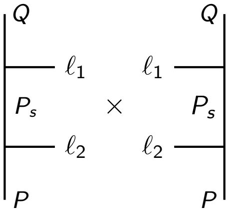


Time-ordered 4-point function

$\langle ZZ| \, V_{\ell_1} V_{\ell_2} \, |ZZ\rangle$

Evaluated using Conformal blocks

$$G_{\ell_1\ell_2} = \int dP \, dQ \, dP_s \; \Psi^*_{\mathrm{ZZ}}(P)\Psi_{\mathrm{ZZ}}(Q) \;\times$$



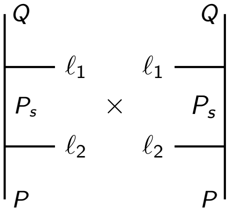
Conformal blocks dominated by primary in the intermediate channel $\Rightarrow$ reduce to DOZZ OPE coefficients

Time-ordered 4-point function

$\langle ZZ|\, V_{\ell_1} V_{\ell_2}\, |ZZ\rangle$

Evaluated using Conformal blocks

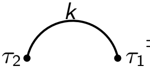
$$G_{\ell_1\ell_2} = \int dP\, dQ\, dP_s\ \Psi^*_{\mathrm{ZZ}}(P)\Psi_{\mathrm{ZZ}}(Q)\ \times$$

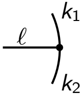


Conformal blocks dominated by primary in the intermediate channel ⇒ reduce to DOZZ OPE coefficients

$$G^{\beta}_{\ell_1\ell_2} = \int dk_1^2 dk_4^2 dk_s^2\ \sinh 2\pi k_1 \sinh 2\pi k_4 \sinh 2\pi k_s$$
$$\times\ e^{-k_1^2(\tau_2-\tau_1)-k_4^2(\tau_4-\tau_3)-k_s^2(\beta-\tau_2+\tau_3-\tau_4+\tau_1)}\ \frac{\Gamma(\ell_2 \pm ik_4 \pm ik_s)\,\Gamma(\ell_1 \pm ik_1 \pm ik_s)}{\Gamma(2\ell_1)\,\Gamma(2\ell_2)}$$

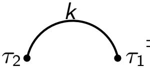
Rules:

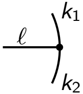
- 

$= e^{-k^2(\tau_2 - \tau_1)}$, 

$= \gamma_\ell(k_1, k_2) = \sqrt{\frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)}}$
- Integrate over intermediate momenta k_i with measure $d\mu(k) = dk^2 \sinh(2\pi k)$

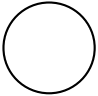
Rules:

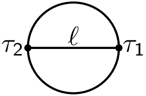
- 

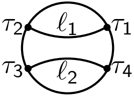
$= e^{-k^2(\tau_2 - \tau_1)}$, 

$= \gamma_\ell(k_1, k_2) = \sqrt{\frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)}}$
- Integrate over intermediate momenta k_i with measure $d\mu(k) = dk^2 \sinh(2\pi k)$

Examples:

$Z(\beta) =$ 

$\langle \mathcal{O}_\ell(\tau_1, \tau_2) \rangle =$ 

$\langle \mathcal{O}_{\ell_1}(\tau_1, \tau_2)\, \mathcal{O}_{\ell_2}(\tau_3, \tau_4) \rangle =$ 

Note: non-perturbative in Schwarzian coupling C

OTO four-point correlator (1)

Swapping two operators in 4-point correlator, means the conformal block is dominated by its primary in a different channel:

$$\mathcal{F}_{P_s}\left[\begin{smallmatrix}2 & 3\\ 1 & 4\end{smallmatrix}\right](z') = \int dP_t \; R_{P_s P_t}\left[\begin{smallmatrix}2 & 3\\ 1 & 4\end{smallmatrix}\right] \; \mathcal{F}_{P_t}\left[\begin{smallmatrix}3 & 2\\ 1 & 4\end{smallmatrix}\right](1/z')$$

OTO four-point correlator (1)

Swapping two operators in 4-point correlator, means the conformal block is dominated by its primary in a different channel:

$$\mathcal{F}_{P_s}\left[\begin{smallmatrix} 2 & 3 \\ 1 & 4 \end{smallmatrix}\right](z') = \int dP_t \; R_{P_s P_t}\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \; \mathcal{F}_{P_t}\left[\begin{smallmatrix} 3 & 2 \\ 1 & 4 \end{smallmatrix}\right](1/z')$$

Blocks unknown in general, but R-matrix is known as quantum 6j-symbol of $U_q(SL(2,\mathbb{R}))$ Teschner-Ponsot '99

OTO four-point correlator (1)

Swapping two operators in 4-point correlator, means the conformal block is dominated by its primary in a different channel:

$$\mathcal{F}_{P_s}\left[\begin{smallmatrix} 2 & 3 \\ 1 & 4 \end{smallmatrix}\right](z') = \int dP_t \; R_{P_s P_t} \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \; \mathcal{F}_{P_t}\left[\begin{smallmatrix} 3 & 2 \\ 1 & 4 \end{smallmatrix}\right](1/z')$$

Blocks unknown in general, but R-matrix is known as quantum 6j-symbol of $U_q(SL(2,\mathbb{R}))$ Teschner-Ponsot '99
$\Rightarrow$ Schwarzian limit of quantum 6j-symbol is 6j-symbol with 4 continuous and 2 discrete $SL(2,\mathbb{R})$ labels

OTO four-point correlator (1)

Swapping two operators in 4-point correlator, means the conformal block is dominated by its primary in a different channel:

$$\mathcal{F}_{P_s}\left[\begin{smallmatrix} 2 & 3 \\ 1 & 4 \end{smallmatrix}\right](z') = \int dP_t \; R_{P_s P_t}\left[\begin{smallmatrix} 2 & 3 \\ 1 & 4 \end{smallmatrix}\right] \; \mathcal{F}_{P_t}\left[\begin{smallmatrix} 3 & 2 \\ 1 & 4 \end{smallmatrix}\right](1/z')$$

Blocks unknown in general, but R-matrix is known as quantum 6j-symbol of $U_q(SL(2,\mathbb{R}))$ Teschner-Ponsot '99

$\Rightarrow$ Schwarzian limit of quantum 6j-symbol is 6j-symbol with 4 continuous and 2 discrete $SL(2,\mathbb{R})$ labels

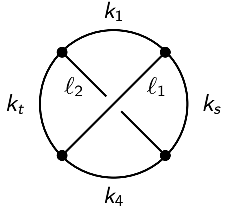
Explicitly:

$$R_{P_s P_t}\left[\begin{smallmatrix} 2 & 3 \\ 1 & 4 \end{smallmatrix}\right] \sim \begin{Bmatrix} \ell_1 & k_2 & k_s \\ \ell_3 & k_4 & k_t \end{Bmatrix} =$$

$$\sqrt{\frac{\Gamma(\ell_1+ik_2\pm ik_s)\Gamma(\ell_3-ik_2\pm ik_t)\Gamma(\ell_1-ik_4\pm ik_t)\Gamma(\ell_3+ik_4\pm ik_s)}{\Gamma(\ell_1-ik_2\pm ik_s)\Gamma(\ell_3+ik_2\pm ik_t)\Gamma(\ell_1+ik_4\pm ik_t)\Gamma(\ell_3-ik_4\pm ik_s)}}$$

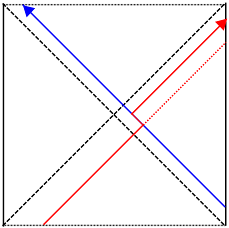
$$\times \int_{-i\infty}^{i\infty} \frac{du}{2\pi i} \, \frac{\Gamma(u)\Gamma(u-2ik_s)\Gamma(u+ik_{2+4-s+t})\Gamma(u-ik_{s+t-2-4})\Gamma(\ell_1+ik_{s-2}-u)\Gamma(\ell_3+ik_{s-4}-u)}{\Gamma(u+\ell_1-ik_{s-2})\Gamma(u+\ell_3-ik_{s-4})}$$

At the Schwarzian level, this procedure is captured by the diagram:



$$e^{-k_1^2(\tau_3-\tau_1)-k_t^2(\tau_3-\tau_2)-k_4^2(\tau_4-\tau_2)-k_s^2(\beta-\tau_4+\tau_1)}$$
$$\times\gamma_{\ell_1}(k_1,k_s)\gamma_{\ell_2}(k_s,k_4)\gamma_{\ell_1}(k_4,k_t)\gamma_{\ell_2}(k_t,k_1)\times\left\{\begin{matrix}\ell_1 & k_1 & k_s\\ \ell_2 & k_4 & k_t\end{matrix}\right\}$$

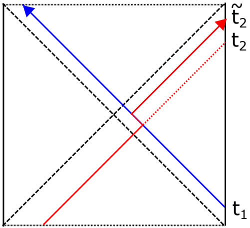
Application: Shockwaves from semiclassics



Time delay:

$$\tilde{t}_2 - t_2 \sim e^{\lambda_M (t_2 - t_1)}, \quad \lambda_M = \frac{2\pi}{\beta_M}$$

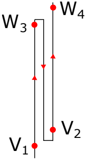
Application: Shockwaves from semiclassics



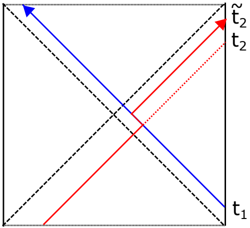
Time delay:

$$\tilde{t}_2 - t_2 \sim e^{\lambda_M (t_2 - t_1)}, \quad \lambda_M = \frac{2\pi}{\beta_M}$$

Shenker-Stanford: OTO-correlator $\langle V_1 W_3 V_2 W_4 \rangle$ in boundary theory captures this behavior Shenker-Stanford '15



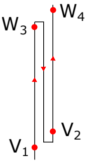
Application: Shockwaves from semiclassics



Time delay:

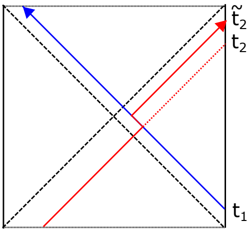
$$\tilde{t}_2 - t_2 \sim e^{\lambda_M (t_2 - t_1)}, \quad \lambda_M = \frac{2\pi}{\beta_M}$$

Shenker-Stanford: OTO-correlator $\langle V_1 W_3 V_2 W_4 \rangle$ in boundary theory captures this behavior Shenker-Stanford '15



$\langle V_1 W_3 V_2 W_4 \rangle$, written using shockwaves in the AdS_2 bulk as

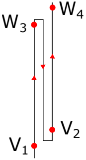
$$\int_0^{+\infty} dq_+ \int_0^{+\infty} dp_- \Psi_1^*(q_+) \Phi_3^*(p_-) \mathcal{S}(p_-, q_+) \Psi_2(q_+) \Phi_4(p_-)$$



Time delay:

$\tilde{t}_2 - t_2 \sim e^{\lambda_M (t_2 - t_1)}, \quad \lambda_M = \frac{2\pi}{\beta_M}$

Shenker-Stanford: OTO-correlator $\langle V_1 W_3 V_2 W_4 \rangle$ in boundary theory captures this behavior Shenker-Stanford '15



$\langle V_1 W_3 V_2 W_4 \rangle$, written using shockwaves in the AdS_2 bulk as

$$\int_0^{+\infty} dq_+ \int_0^{+\infty} dp_- \Psi_1^*(q_+)\Phi_3^*(p_-)\mathcal{S}(p_-, q_+)\Psi_2(q_+)\Phi_4(p_-)$$

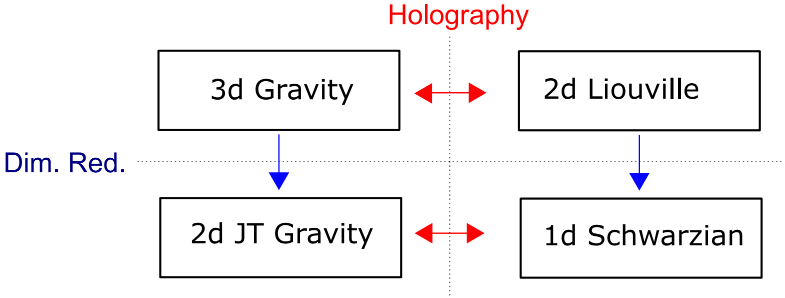
- $\Psi, \Phi =$ Kruskal wavefunctions = bulk-to-boundary propagators
- $\mathcal{S} = \exp\left(\frac{i\beta}{4\pi C}\, p_- q_+\right)$ the Dray-'t Hooft shockwave $\mathcal{S}$-matrix

Application: Shockwaves from the exact OTO correlator

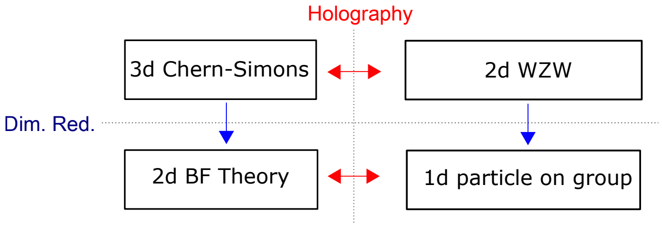
Large C limit of complete OTO 4-point function, with light ℓ, gives full eikonal shockwave expressions

Exact match $\Rightarrow$ Derived full shockwave in semiclassical regime!

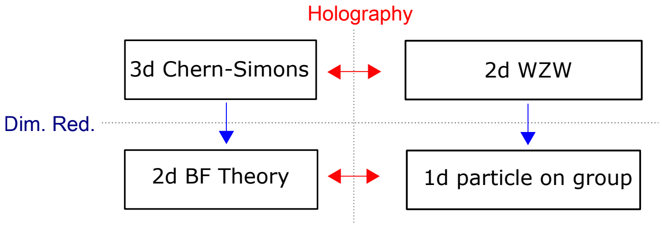
Summary: Structural link theories



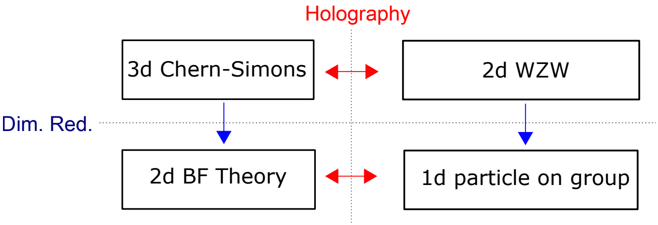
Generalization: Group models



Generalization: Group models

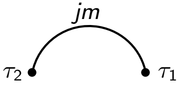


Relevant for SYK-type models with internal symmetries

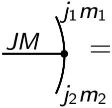


Relevant for SYK-type models with internal symmetries

Correlators determined using 2d WZW CFT techniques



$$= \quad e^{-\,C_j\,(\tau_2-\tau_1)}\ ,$$



$$= \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix}$$

Wilson line perpective

JT is Hamiltonian reduction of $\mathrm{SL}(2,\mathbb{R})$ BF

Wilson line perpective

JT is Hamiltonian reduction of $\mathrm{SL}(2,\mathbb{R})$ BF

with action $S = \int dt \mathrm{Tr}\Phi F$

Boundary-anchored Wilson lines are bilocal operators: in progress

$\left\langle \mathcal{P}e^{\int A} \right\rangle = \left\langle g(\tau_f) g^{-1}(\tau_i) \right\rangle$

Reduce to Schwarzian bilocals for the constrained $\mathrm{SL}(2,\mathbb{R})$ case

Wilson line perpective

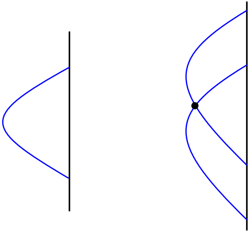
JT is Hamiltonian reduction of $\mathrm{SL}(2,\mathbb{R})$ BF

with action $S = \int dt \mathrm{Tr} \Phi F$

Boundary-anchored Wilson lines are bilocal operators: in progress

$$\left\langle \mathcal{P} e^{\int A} \right\rangle = \langle g(\tau_f) g^{-1}(\tau_i) \rangle$$

Reduce to Schwarzian bilocals for the constrained $\mathrm{SL}(2,\mathbb{R})$ case



Crossing of Wilson lines $\Rightarrow$ $6j$-symbol of group G

Conclusion

Schwarzian QM is relevant as

- Low-energy universal gravity sector of all SYK-type models
  Cfr. Liouville compared to holographic 2d CFT

Conclusion

Schwarzian QM is relevant as

- Low-energy universal gravity sector of all SYK-type models
  Cfr. Liouville compared to holographic 2d CFT
- holographically dual to JT dilaton gravity

Conclusion

Schwarzian QM is relevant as

- Low-energy universal gravity sector of all SYK-type models
  Cfr. Liouville compared to holographic 2d CFT
- holographically dual to JT dilaton gravity

Theory is exactly solvable and displays a wide variety of black hole quantum physics:

- Virtual black hole intermediate states
- Quasinormal modes and shockwaves in the semi-classical regime

Conclusion

Schwarzian QM is relevant as

- Low-energy universal gravity sector of all SYK-type models
  Cfr. Liouville compared to holographic 2d CFT
- holographically dual to JT dilaton gravity

Theory is exactly solvable and displays a wide variety of black hole quantum physics:

- Virtual black hole intermediate states
- Quasinormal modes and shockwaves in the semi-classical regime

Naturally embedded within Liouville theory

Thank you!