A more functional bootstrap

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Non-perturbative QCD, IAP, 2018

Based on 1803.10233 *+ ongoing with* **D. Mazac**

The bootstrap manifesto:

To constrain and determine
quantum field theories
from basic principles

Unitarity

Locality

Crossing

Critical exponents of the 3d Ising universality class?

What is the mass of a string propagating on a Calabi-Yau?

CONFORMAL FIELD THEORY

$$c_p \simeq (T - T_c)^{-\alpha}$$

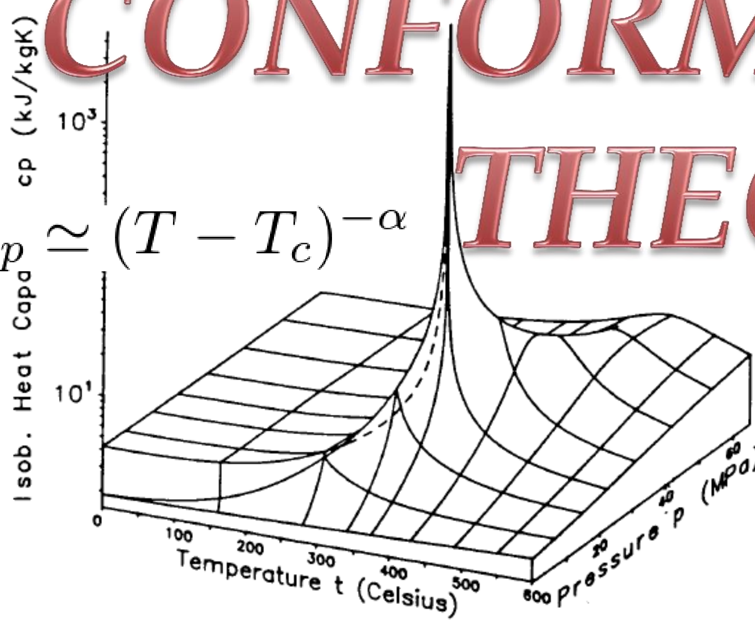


Fig. 5: Isobaric heat capacity c_p over the p, T plane for water

How does the mass of a bound state vary with coupling?

The conformal bootstrap

- We consider four point correlators in a CFT, e.g.:

$$\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle$$

- **Locality**: Operator Product Expansion

$$\sigma \times \sigma = 1 + \mathcal{O}_1 + \partial\mathcal{O}_1 + \ldots + \mathcal{O}_2 + \ldots$$

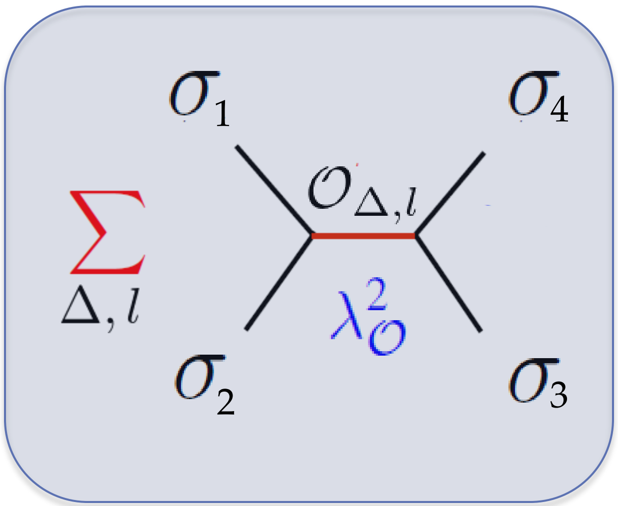
- **Unitarity**: Real fields implies *real* couplings, positive norms

$$\langle \sigma_1(z)\sigma_2(0)\rangle = \frac{\delta_{1,2}}{z^{2\Delta_\sigma}} \qquad \langle \sigma(\infty)\sigma(1)\mathcal{O}(0)\rangle = \lambda_{\mathcal{O}}$$

Conformal symmetry fixes three-point function up to constant

The conformal bootstrap

- **Crossing** Symmetry (combined with OPE):



$$\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle$$

$$\sim \lambda_{\mathcal{O}}\,\mathcal{O}_{\Delta,l} \quad \sim \lambda_{\mathcal{O}}\,\mathcal{O}_{\Delta,l}$$

The conformal bootstrap

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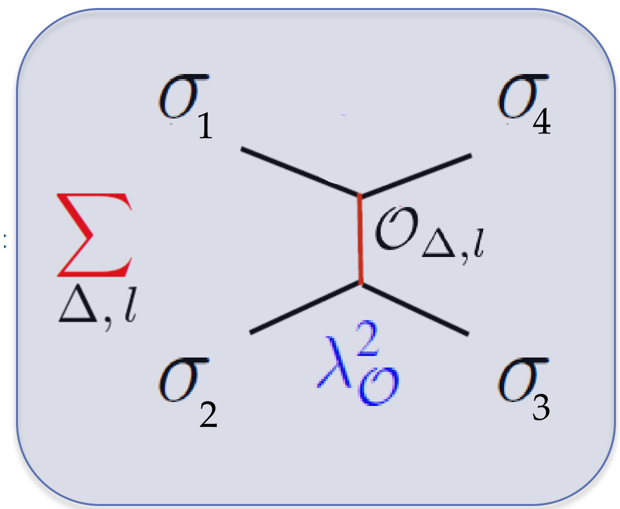

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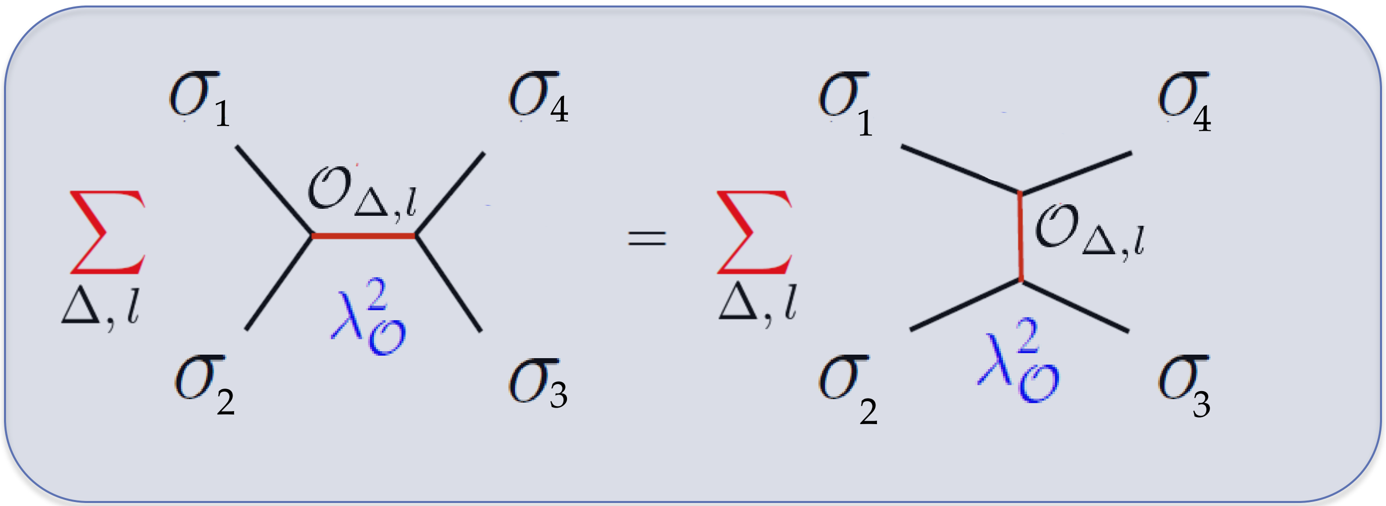
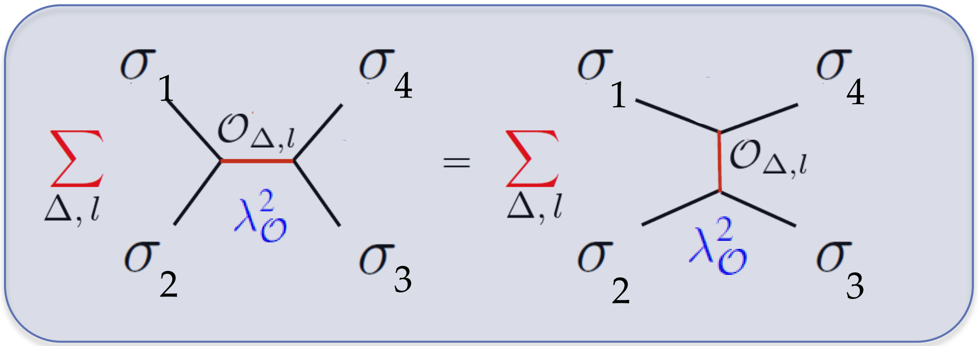


$$\sum_{\Delta,l} \lambda^2_{\mathcal{O}} \, F^{\sigma}_{\Delta,l}(z) = 0$$

Kinematically determined by conformal symmetry

The (1D) conformal bootstrap



Morally, s-t channel duality

$$\sum_{\Delta \in S} \lambda^2_{\Delta} F^{\sigma}_{\Delta}(z) = -F^{\sigma}_0(z)$$

← *Identity operator contribution*

In D=1 (no spin!)

$$G_{\Delta}(z) = z^{\Delta}\, {}_2F_1(\Delta, \Delta, 2\Delta, z)$$

Crossing vector

$$F^{\sigma}_{\Delta}(z) = \frac{G_{\Delta}(z)}{z^{2\Delta_\sigma}} - \frac{G_{\Delta}(1-z)}{(1-z)^{2\Delta_\sigma}}$$

Conformal block

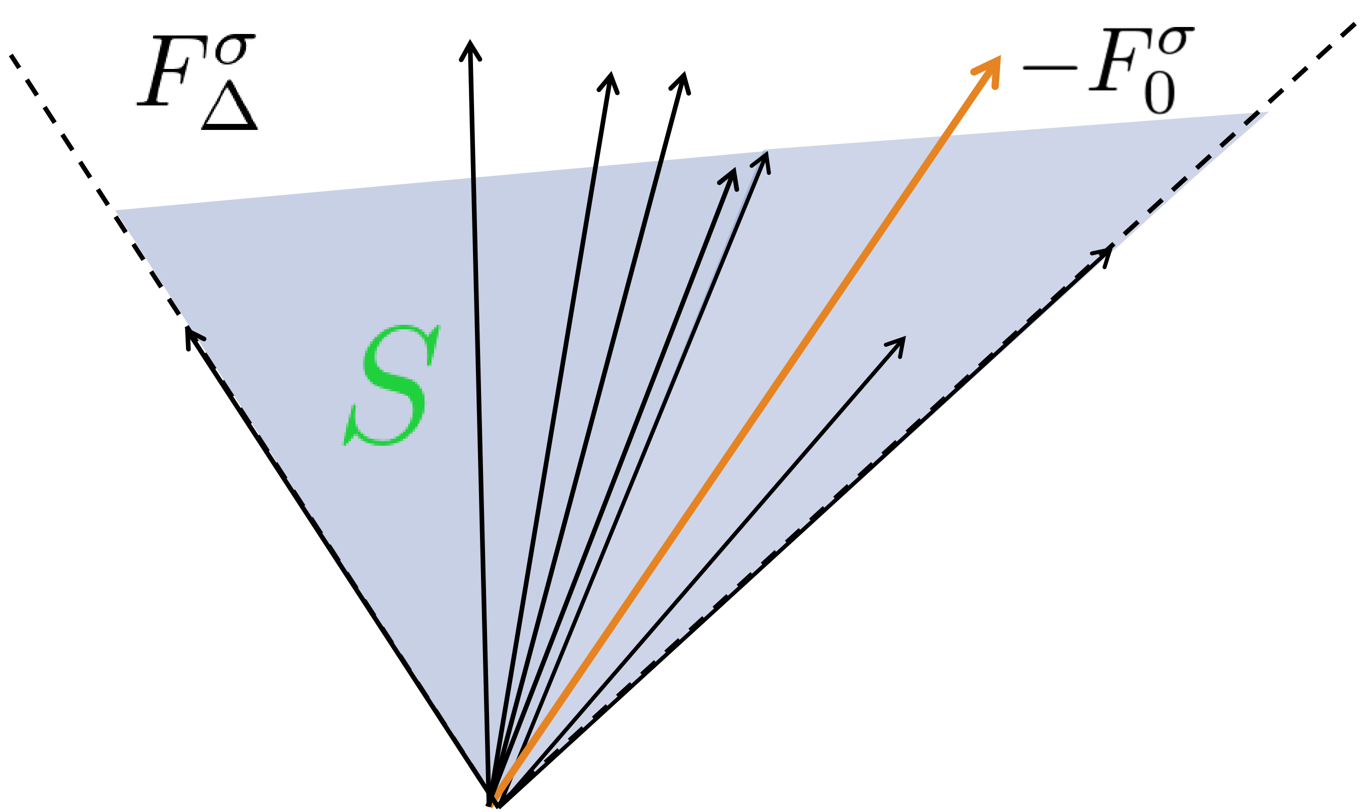
$$\left(\sim \frac{1}{s - m^2}\right)$$

Extracting information from crossing

$$\sum_{\Delta \in S} \lambda_\Delta^2 \, F_\Delta^\sigma(z) \stackrel{?}{=} -F_0^\sigma(z)$$

- Make assumptions on possible intermediate states and try to get a contradiction.
- Such contradictions are possible thanks to positivity of coefficients.
- They are made explicit by the construction of *linear functionals.*

$$\sum_{\Delta \in S} \lambda_{\Delta}^{2} F_{\Delta}^{\sigma}(z) \stackrel{?}{=} -F_{0}^{\sigma}(z)$$



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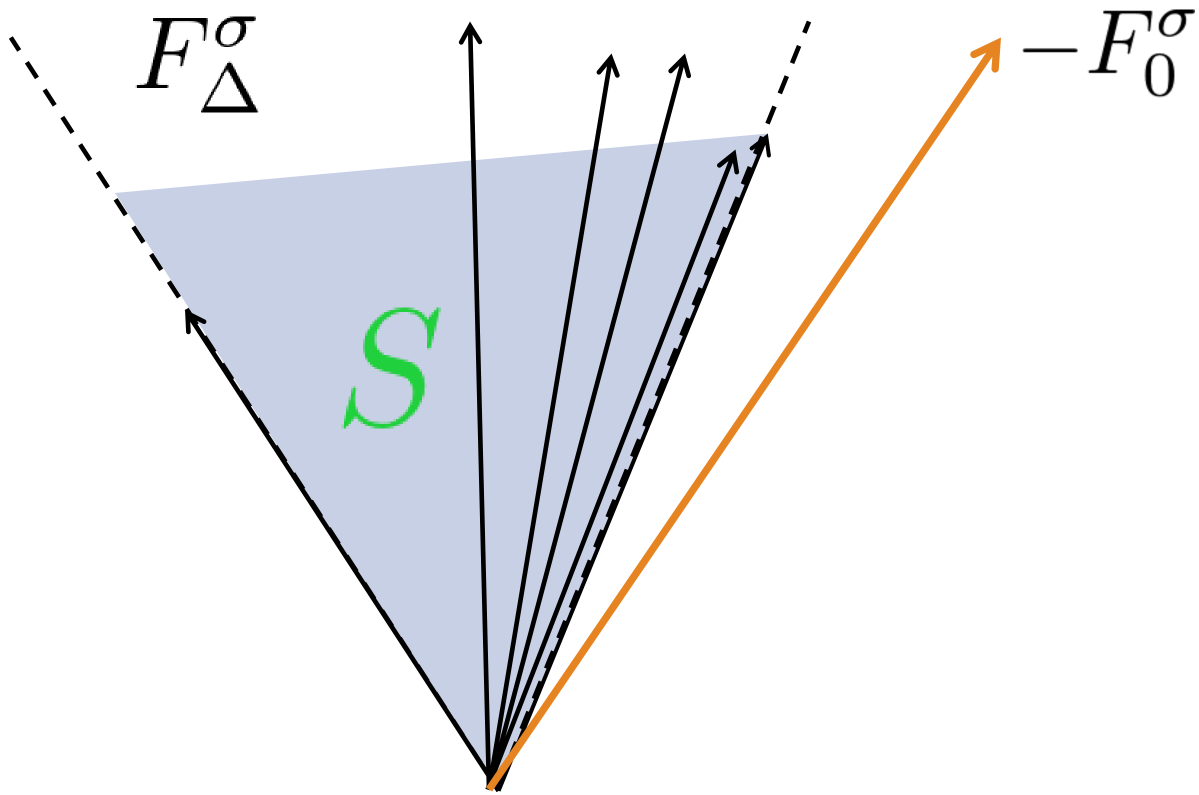

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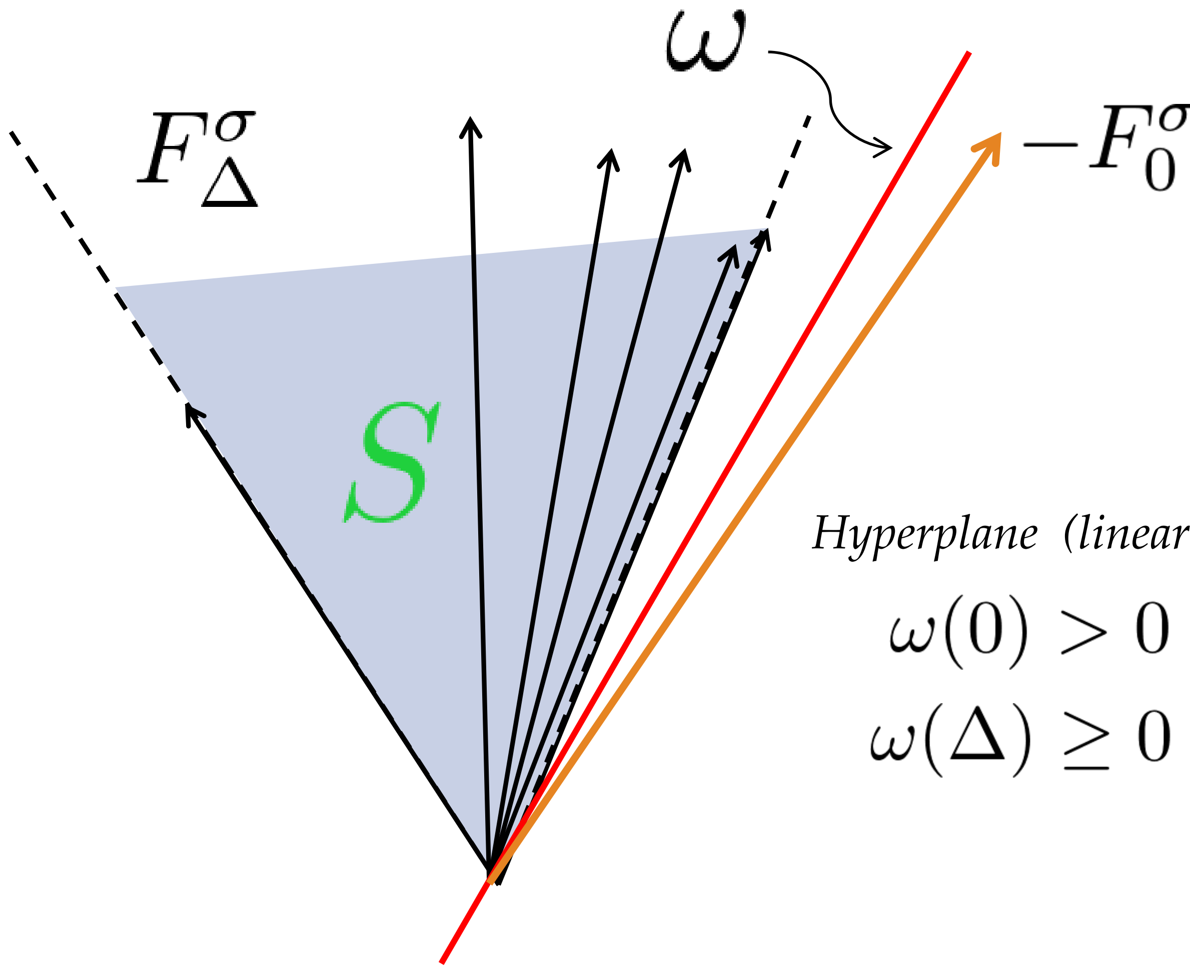


Hyperplane (linear functional):

$$\omega(0) > 0$$

$$\omega(\Delta) \geq 0 \quad \forall_{\Delta \in S}$$

Bounding CFT data

- To bound dimension of leading operator in OPE,

$$\sigma \times \sigma = 1 + \mathcal{O} + \dots$$

construct a functional satisfying:

$$\beta(0) > 0, \quad \beta(\Delta > \Delta_0) \geq 0$$

- This leads to contradiction when applied to hypothetic crossing solution

$$0 < \beta \left[F_0^\sigma + \sum_{\Delta > \Delta_0} \lambda_\Delta^2 F_\Delta^\sigma \right] = 0 \quad \Rightarrow \Delta_{\mathcal{O}} \leq \Delta_0$$

- Optimal bound is obtained by lowering Δ_0 until no such functional exists.

Bounding CFT data

- To place upper bound on OPE coefficient we construct a different kind of functional:

$$\alpha(\Delta_b) > 0, \quad \alpha(\Delta \geq \Delta_0) \geq 0$$

- This leads to:

$$\alpha\left[F_0^\sigma + \lambda_b^2\, F_{\Delta_b}^\sigma + \sum_{\Delta > \Delta_0} \lambda_\Delta^2 F_\Delta^\sigma\right] = 0 \Rightarrow \lambda_b^2 \leq -\frac{\alpha(0)}{\alpha(\Delta_b)}$$

- Optimal bound is obtained by minimizing the ratio over all functionals.

Bounding CFT data

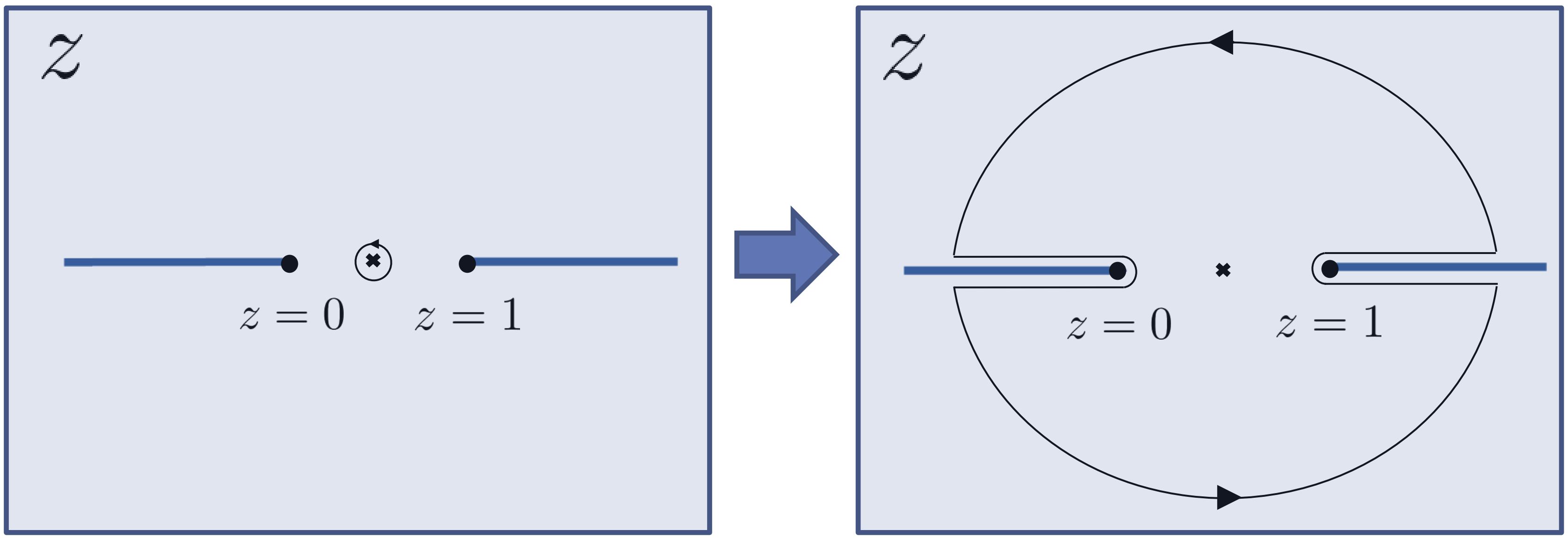
- This approach was pioneered by Rattazzi, Rychkov,Tonni and Vichi in their landmark '08 paper + follow ups.
- Basic approach still in use today: use Taylor series

$$\omega\left[\mathcal{F}\right] = \sum_{i=0}^{N} \omega_i\, \partial_z^{2i+1} \mathcal{F}(z)\Big|_{z=\frac{1}{2}}$$

- Functionals are constructed numerically via numerical optimization algorithms.

- We propose a new class of functionals, where it is possible to obtain exact, optimal results.

Functional ansatz I

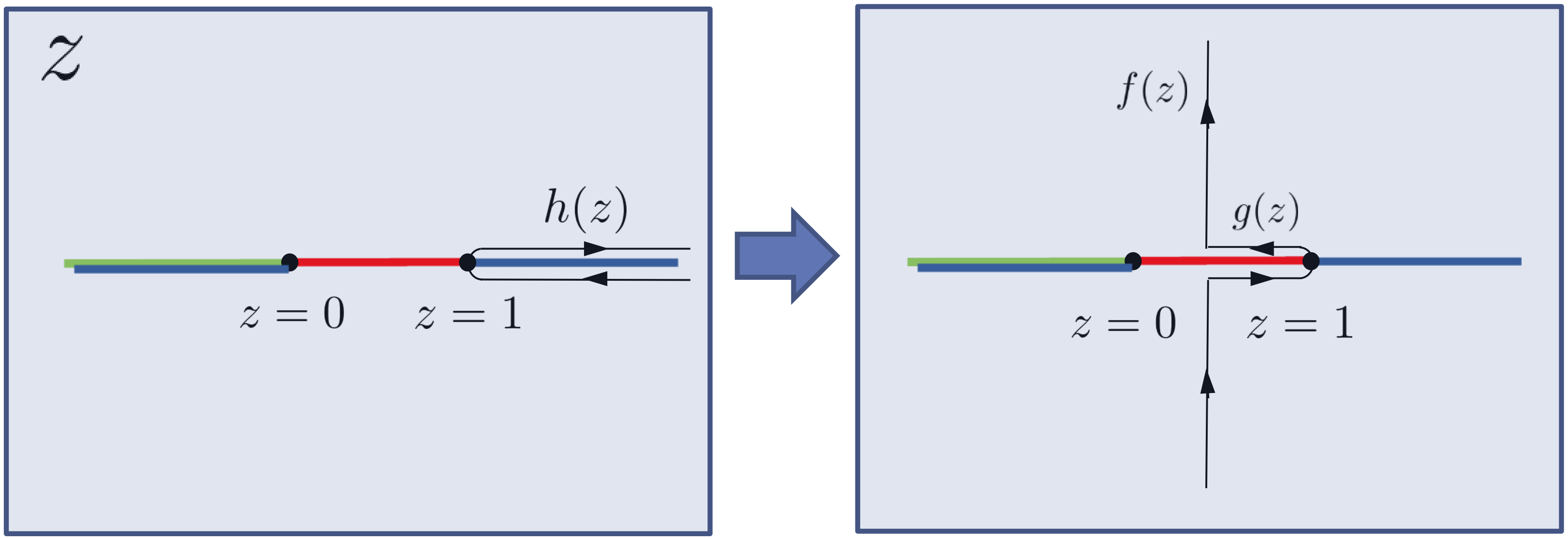


$$\int \rho(w) F_\Delta(w) \to \int \rho(w) \oint \frac{F_\Delta(z)}{z-w} \to \int_1^{+\infty} dz\, h(z) \text{Disc}\left[F_\Delta(z)\right]$$

$$\left(h(z) = \int dw \rho(w) \frac{2w-1}{(z-w)(1-z-w)} \right)$$

The functions F are antisymmetric in their argument

Functional ansatz II



$$\omega(\Delta) := \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dz\, f(z) F_\Delta(z) + \int_{\frac{1}{2}}^{1} dz\, g(z) F_\Delta(z)$$

Functional ansatz III

$$\omega(\Delta) := \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dz\, f(z) F_\Delta(z) + \int_{\frac{1}{2}}^{1} dz\, g(z) F_\Delta(z)$$

- For functionals to be well defined, need appropriate boundary conditions for kernels at 1 and infinity.
- As a remnant of the original definition, we demand the *gluing condition*:

$$\mathrm{Re} f(z) = -g(z) - g(1-z), \quad 0 < z < 1$$

Why is this good?

- After some contour manipulations (valid for large enough Δ):

$$\Rightarrow \omega(\Delta) = \mathfrak{g}(\Delta) - \mathrm{Re}\left[e^{-i\pi(\Delta - 2\Delta_\sigma)}\mathfrak{f}(\Delta)\right]$$

$$\mathfrak{g}(\Delta) := \int_0^1 dz\, g(z) \frac{G_\Delta(z)}{z^{2\Delta_\sigma}} \qquad \mathfrak{f}(\Delta) := \int_1^{+\infty} dz\, f(z + i0^+) \frac{G_\Delta(\frac{z-1}{z})}{(z-1)^{2\Delta_\sigma}}$$

- We should find suitable $f(z), g(z)$ so that:

$$\mathfrak{g}(\Delta) \geq |\mathfrak{f}(\Delta)| \Rightarrow \omega(\Delta) \geq 0$$

- *Optimal* functionals minimize this inequality in as large as range as possible.

Recall:

$$G_\Delta(z) = z^\Delta\, {}_2F_1(\Delta, \Delta, 2\Delta, z)$$

$$F_\Delta(z) = \frac{G_\Delta(z)}{z^{2\Delta_\sigma}} - \frac{G_\Delta(1-z)}{(1-z)^{2\Delta_\sigma}}$$

Why is this good?

$$\mathfrak{f}(\Delta) = |\mathfrak{f}(\Delta)| e^{-i\pi\delta(\Delta)} \qquad \mathfrak{g}(\Delta) \sim |\mathfrak{f}(\Delta)|$$

$$\omega(\Delta) \sim \mathfrak{g}(\Delta) \left[\sin\frac{\pi}{2}[\Delta - 2\Delta_\sigma - \delta(\Delta)]\right]^2$$

- The positivity constraints are saturated at a discrete set of conformal dimensions, i.e. the functional has zeros at this points.
- The *optimal* functional zeros provide a solution to crossing – solution provides an obstruction to further optimization.
- Thus at least some CFT scaling dimensions are encoded in the kernels.

Free case

- A simple way to obtain a good functional is to set:

$$g(z) = -\frac{f\left(\frac{1}{1-z}\right)}{(1-z)^{2-2\Delta_\sigma}} \qquad f(z) < 0, \quad z > 1$$

- This implies exactly $\mathfrak{g}(\Delta) = -\mathfrak{f}(\Delta) \geq 0$

$$\omega(\Delta) = \mathfrak{g}(\Delta)\left[\sin\frac{\pi}{2}[\Delta - 2\Delta_\sigma - 1]\right]^2$$

- The functional zeros match with spectrum of *generalized free fermion*:

$$\Delta_n = 1 + 2\Delta_\sigma + 2n, \quad n \in \mathbb{N}$$

$$\left(\langle\sigma^4\rangle \sim \frac{1}{z^{2\Delta_\sigma}} + \frac{1}{(1-z)^{2\Delta_\sigma}} - 1 = \frac{1}{z^{2\Delta_\sigma}} + \sum_{n=0}^{+\infty} c_n^{\text{gff}} \frac{G_{\Delta_n}(z)}{z^{2\Delta_\sigma}}\right)$$

Just Wick contractions, free massive field in AdS2

Free case

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- This implies exactly $\mathfrak{g}(\Delta) = -\mathfrak{f}(\Delta) \geq 0$

$$\omega(\Delta) = \mathfrak{g}(\Delta)\left[\sin\frac{\pi}{2}[\Delta - 2\Delta_\sigma - 1]\right]^2$$

- The gluing condition becomes an equation for the kernel:

$$\mathrm{Re} f(z) = (1-z)^{2\Delta_\sigma - 2} f\left(\tfrac{1}{1-z}\right) + z^{2\Delta_\sigma - 2} f\left(\tfrac{1}{z}\right)$$

Free case

- This can be solved completely with appropriate b.c.'s.:

$$f_\beta(z) = -\kappa(\Delta_\phi)\frac{2z-1}{w^{3/2}}\left[{}_3\tilde{F}_2\left(-\frac{1}{2},\frac{3}{2},2\Delta_\phi+\frac{3}{2};\Delta_\phi+1,\Delta_\phi+2;-\frac{1}{4w}\right)+\right.$$

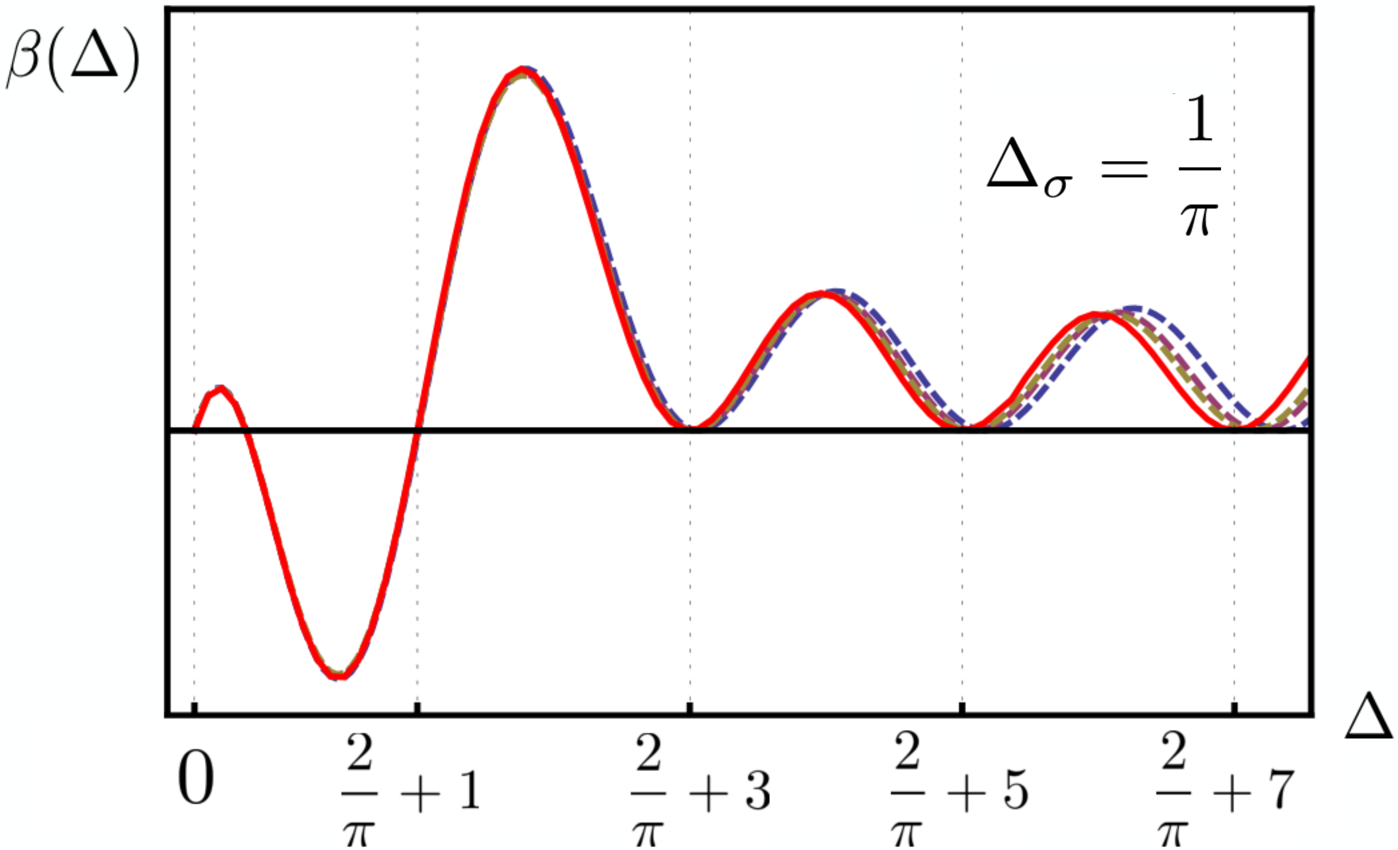
$$w \equiv z(z-1) \qquad \left.+\frac{9}{16w}\,{}_3\tilde{F}_2\left(\frac{1}{2},\frac{5}{2},2\Delta_\phi+\frac{5}{2};\Delta_\phi+2,\Delta_\phi+3;-\frac{1}{4w}\right)\right],$$

- Functional action annihilates identity and is nonnegative for:

$$\Delta \geq \Delta_0 = 1 + 2\Delta_\sigma$$

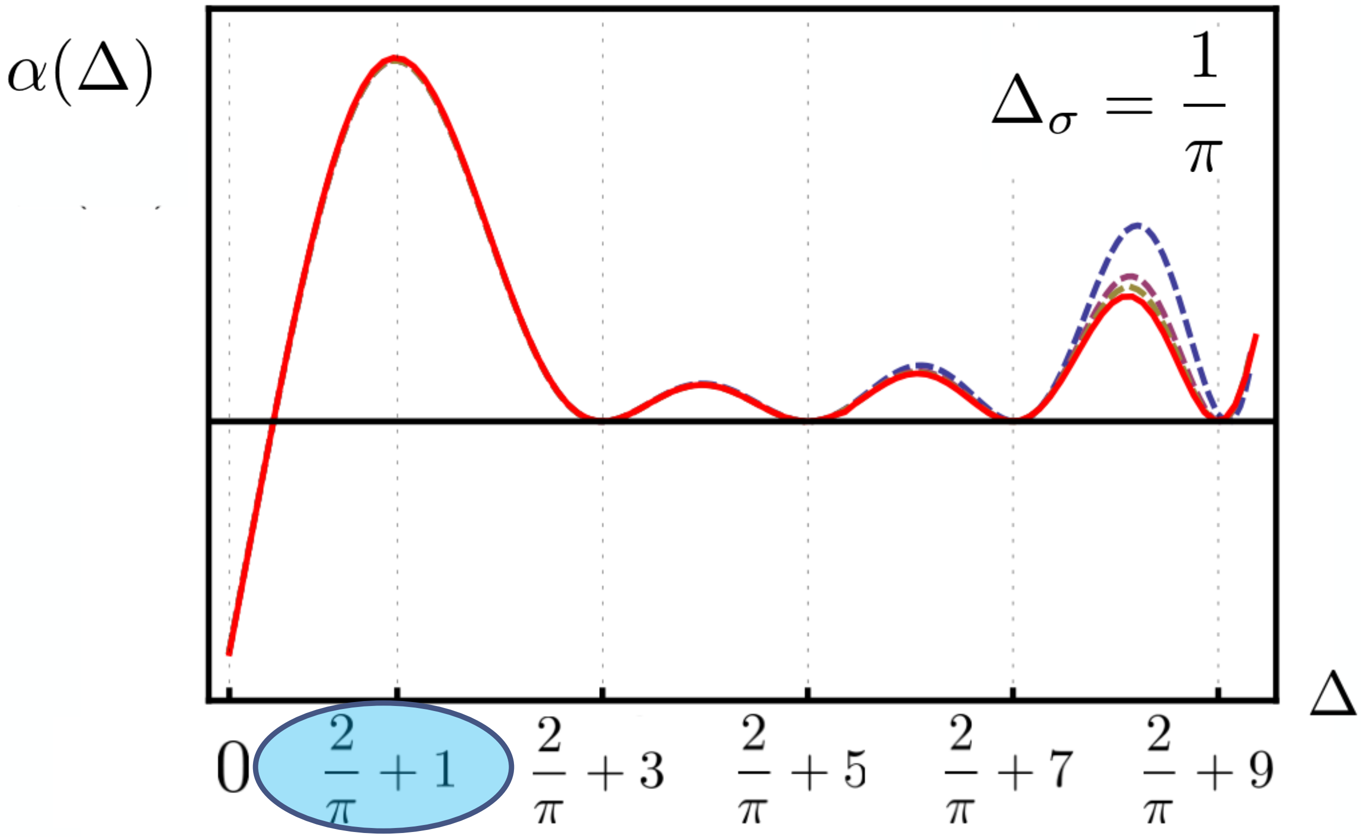
- This bounds the gap to the leading scalar. The bound is optimal since it is saturated by the known generalized free fermion CFT.
- Solution with different b.c.'s provides OPE bounds.

Gapmax: exact vs numerics



$$\Delta_n = 1 + 2\Delta_\sigma + 2n, \quad n \in \mathbb{N}$$

OPEmax: exact vs numerics

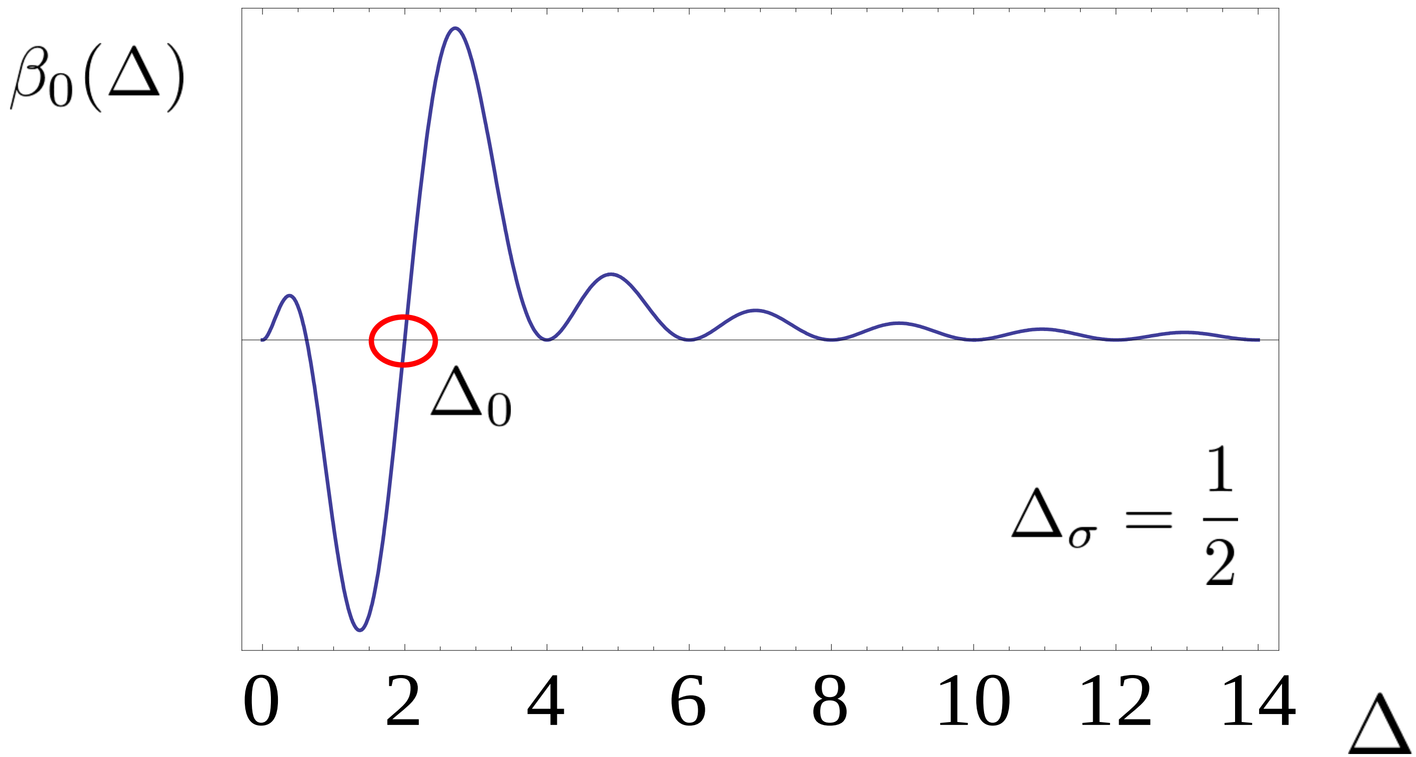


Δ_b

$$\Delta_n = 1 + 2\Delta_\sigma + 2n, \quad n \in \mathbb{N}$$

Who ordered *that*?

- Functional equation has infinitely many more solutions!



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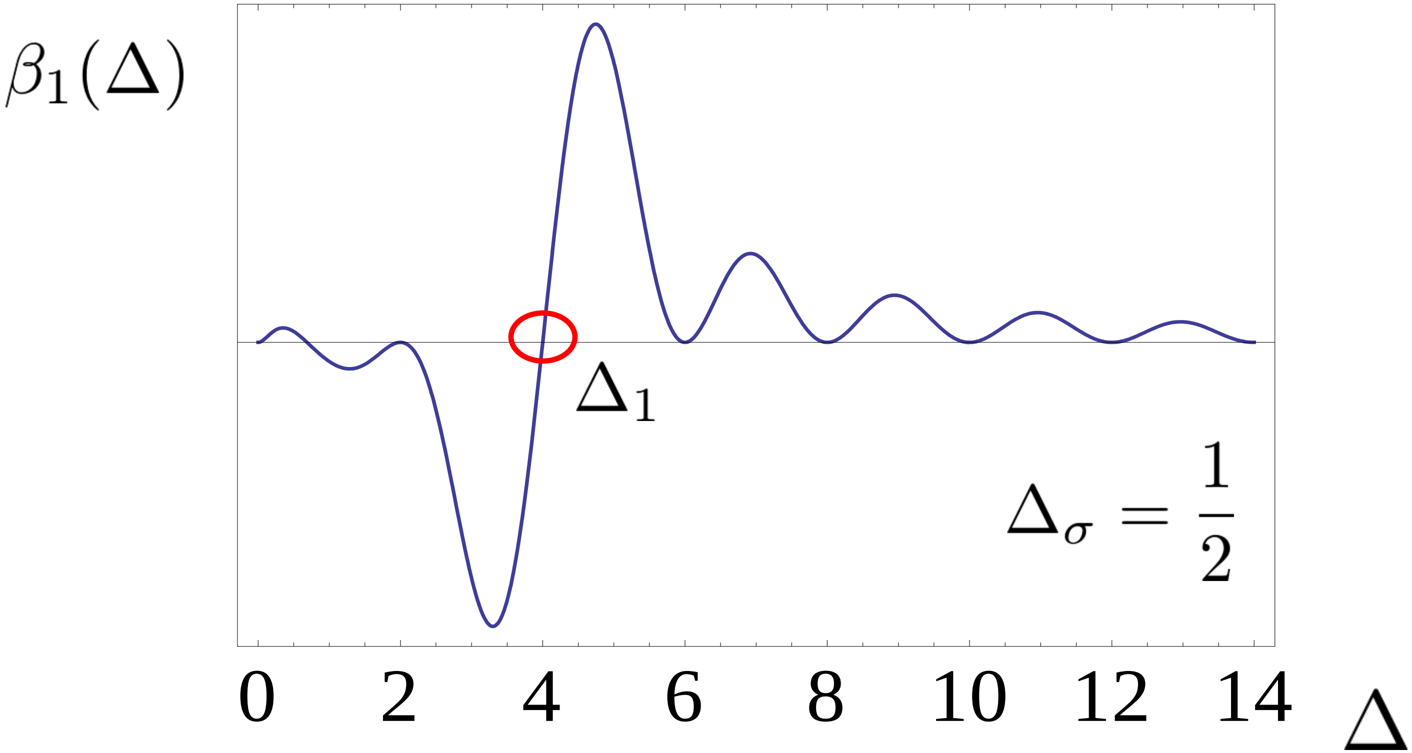

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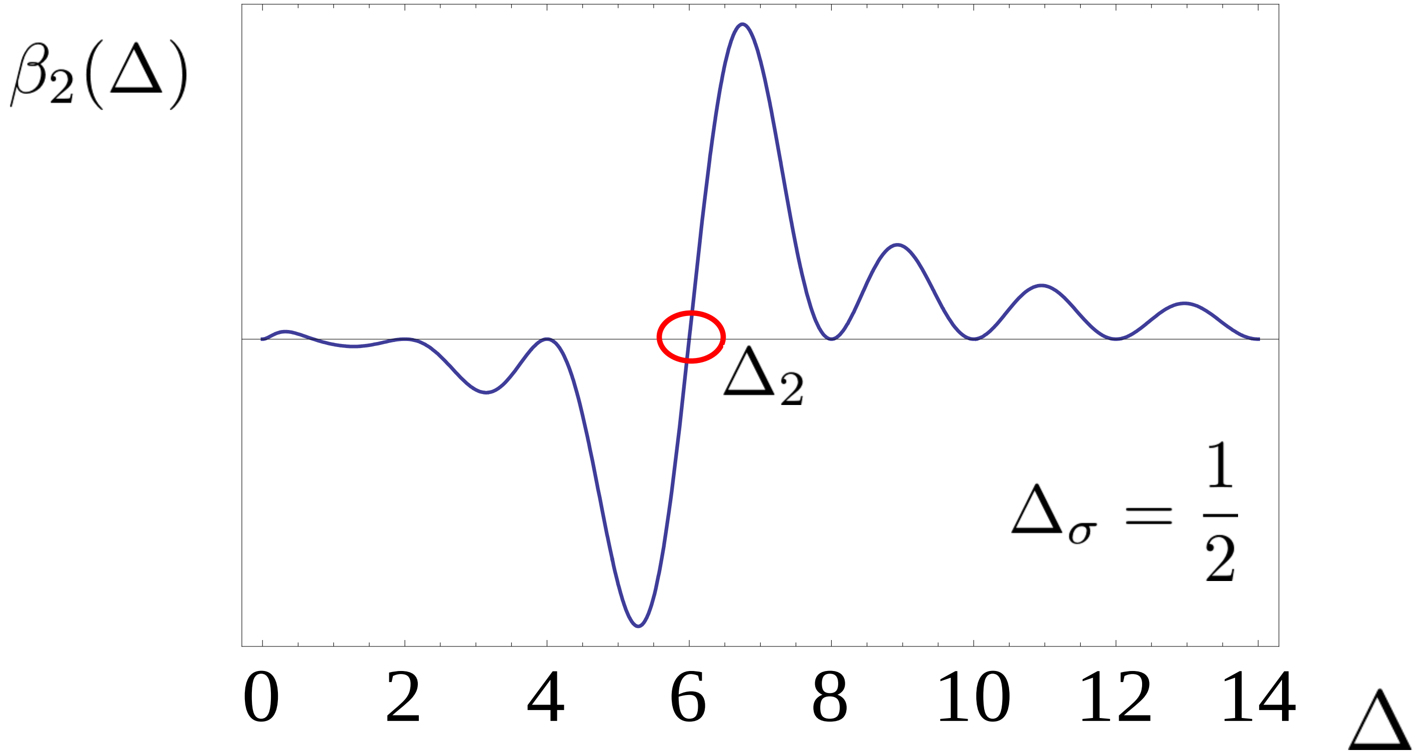

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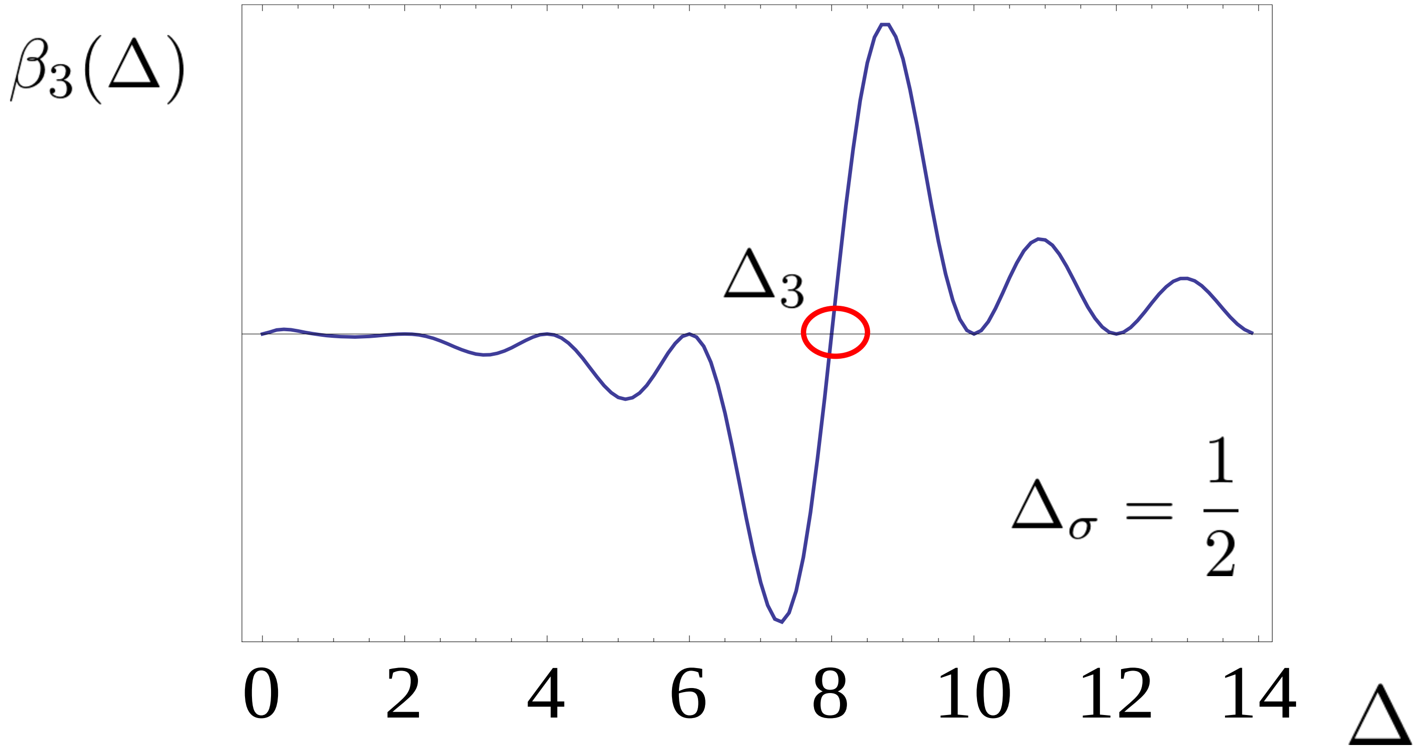

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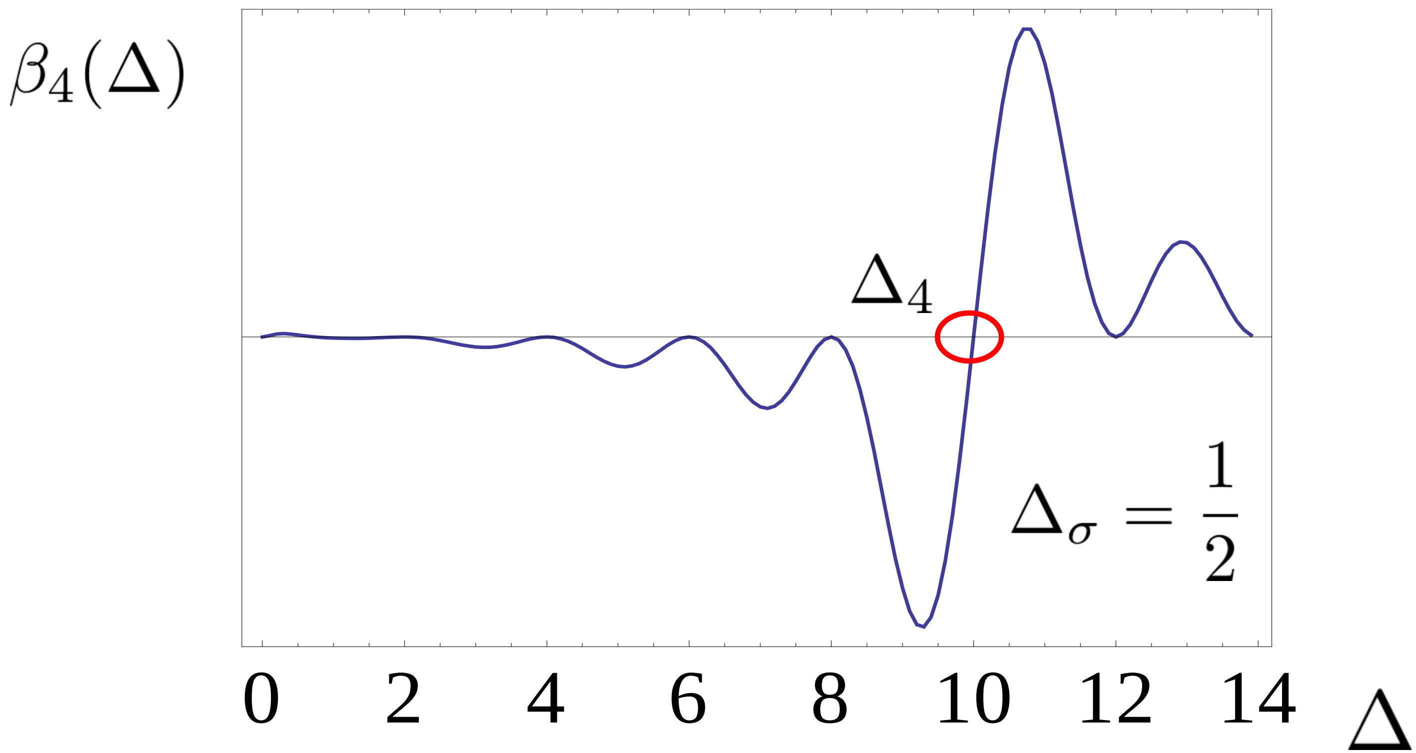

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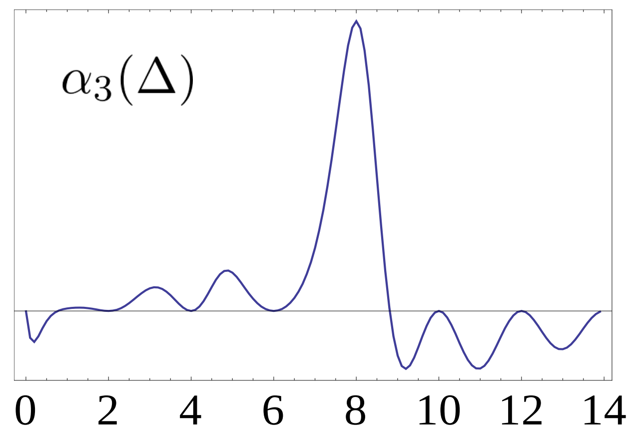
A basis for crossing

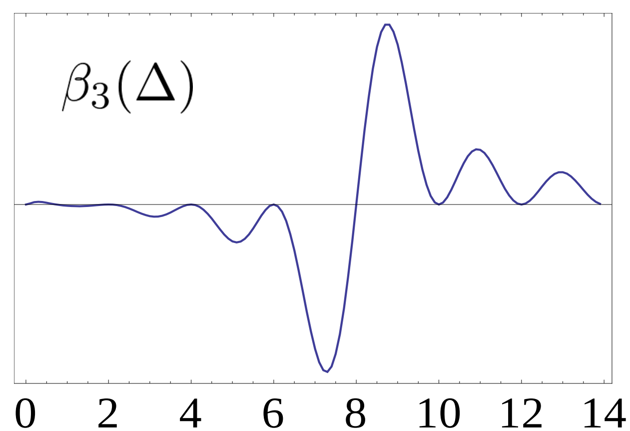
OPEmax type functionals

$$\alpha_n(\Delta_m) = \delta_{n,m}, \quad \alpha'_n(\Delta_m) = 0$$

$$\beta_n(\Delta_m) = 0, \quad \beta'_n(\Delta_m) = \delta_{n,m}$$

Gapmax type functionals





A basis for crossing

OPEmax type functionals

$$\alpha_n(\Delta_m) = \delta_{n,m}, \quad \alpha'_n(\Delta_m) = 0$$

- They tell us how to deform free solution

$$\sum_{n=0}^{+\infty} \left[\delta c_n^{\text{free}} F_{\Delta_n} + \delta\Delta_n \, c_n^{\text{free}} F'_{\Delta_n}\right] = \delta S$$

$$\beta_n(\Delta_m) = 0, \quad \beta'_n(\Delta_m) = \delta_{n,m}$$

Gapmax type functionals

Namely:

$$\delta c_n^{\text{free}} = \alpha_n(\delta S)$$

$$\delta\Delta_n = \beta_n(\delta S)/c_n^{\text{free}}$$

A basis for crossing

- Deforming the free solution by a new operator, we find the identity:

$$F_\Delta(z) = \sum_{n=0}^{+\infty} \left[\alpha_n(\Delta) F_{\Delta_n}(z) + \beta_n(\Delta) F'_{\Delta_n}(z)\right]$$

- For any solution to crossing we must have:

$$0 = F_0 + \sum_{m=0}^{+\infty} c_m F_{\tilde{\Delta}_m} \quad \Rightarrow \quad \begin{aligned} \sum_{m=0}^{+\infty} c_m \alpha_n(\tilde{\Delta}_m) &= c_n^{\text{free}} \\ \sum_{m=0}^{+\infty} c_m \beta_n(\tilde{\Delta}_m) &= 0 \end{aligned}$$

Extremal Flows

- We can now perturb around the free solution[1] and compute anomalous dimensions and OPE coefficients.
- Results match with perturbation theory in AdS2 to the order we checked. No integrals to perform, just (nested) series.

$$\Delta_0 \equiv 2 + g$$

$$\Delta_1 = 4 + \frac{1}{6} g + \left(\frac{317}{144} - \frac{5}{3}\zeta(3) \right) g^2 + \left[10\zeta(5) + \frac{\pi^4}{18} - \frac{299}{18}\zeta(3) + \frac{1225}{2592}\pi^2 - \frac{6995}{3888} \right] g^3$$

$$\vdots$$

Two-loop result from AdS perspective!

[1]*We perturb the free boson whose functionals are very similar to the ones shown.*

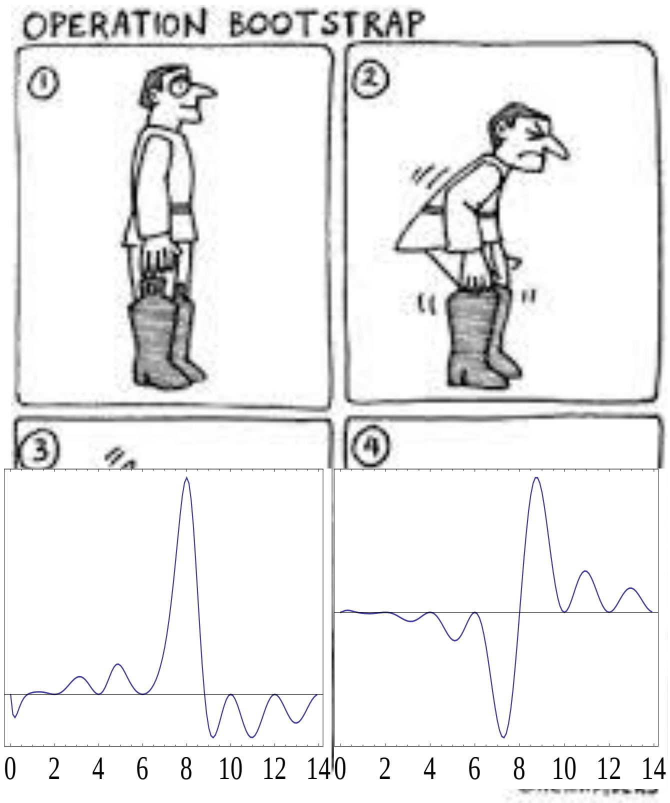
Conclusions

- We have proposed a class of functionals to analyse crossing symmetry sum rules.
- The class seems to be general enough to capture optimal functionals in a variety of cases.
- We have proposed a “basis” of the crossing equation which reformulates the problem in terms of an infinite set of functionals.
- We have recovered AdS2 perturbation theory using our approach.

Outlook

- Physical meaning of functionals?
- Functionals/Basis in higher dimensions?
- Connection to the Polyakov-like bootstrap?
- Non-perturbative solutions?

Thank you!



OPEmax at large Δ

- In the limit of large dimensions, transforms localize, e.g.:

$$\mathfrak{g}(\Delta) \overset{\Delta,\Delta_\sigma\to\infty}{\sim} \mu(\Delta,\Delta_\sigma)\, g(s/4), \quad s \equiv \left(\frac{\Delta}{\Delta_\sigma}\right)^2$$

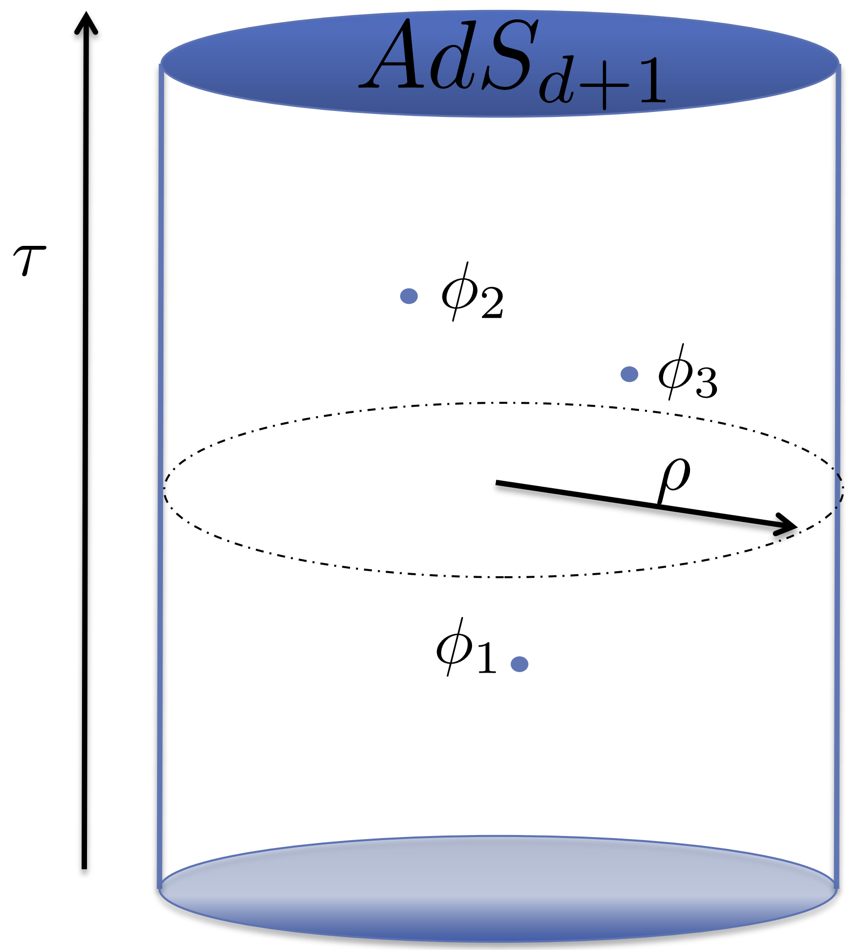
- Hence we can solve the problem by equating kernels as in free case. Zeros of functional determined by phase of *f* kernel directly.

$$\omega_{\Delta_\phi}(\Delta) \sim 2\mu(\Delta_\phi, s)\sin^2\left[\frac{\pi}{2}\left(\Delta - 2\Delta_\phi + \delta\left(\tfrac{s}{4}\right)\right)\right] \left|f\left(\tfrac{s}{4}\right)\right|$$

$$f(z) = \frac{2z-1}{[z(z-1)]^{1/2}(z-z_b)(z-1+z_b)S(z)}$$

$$c_b^2 \leq \sqrt{64\pi\Delta_\phi}\,\frac{m^{3/2}\sqrt{2-m}}{|m^2-2|\sqrt{2+m}}\left[\frac{2^{2(m+2)}}{(2-m)^{2-m}(2+m)^{2+m}}\right]^{-\Delta_\phi}$$

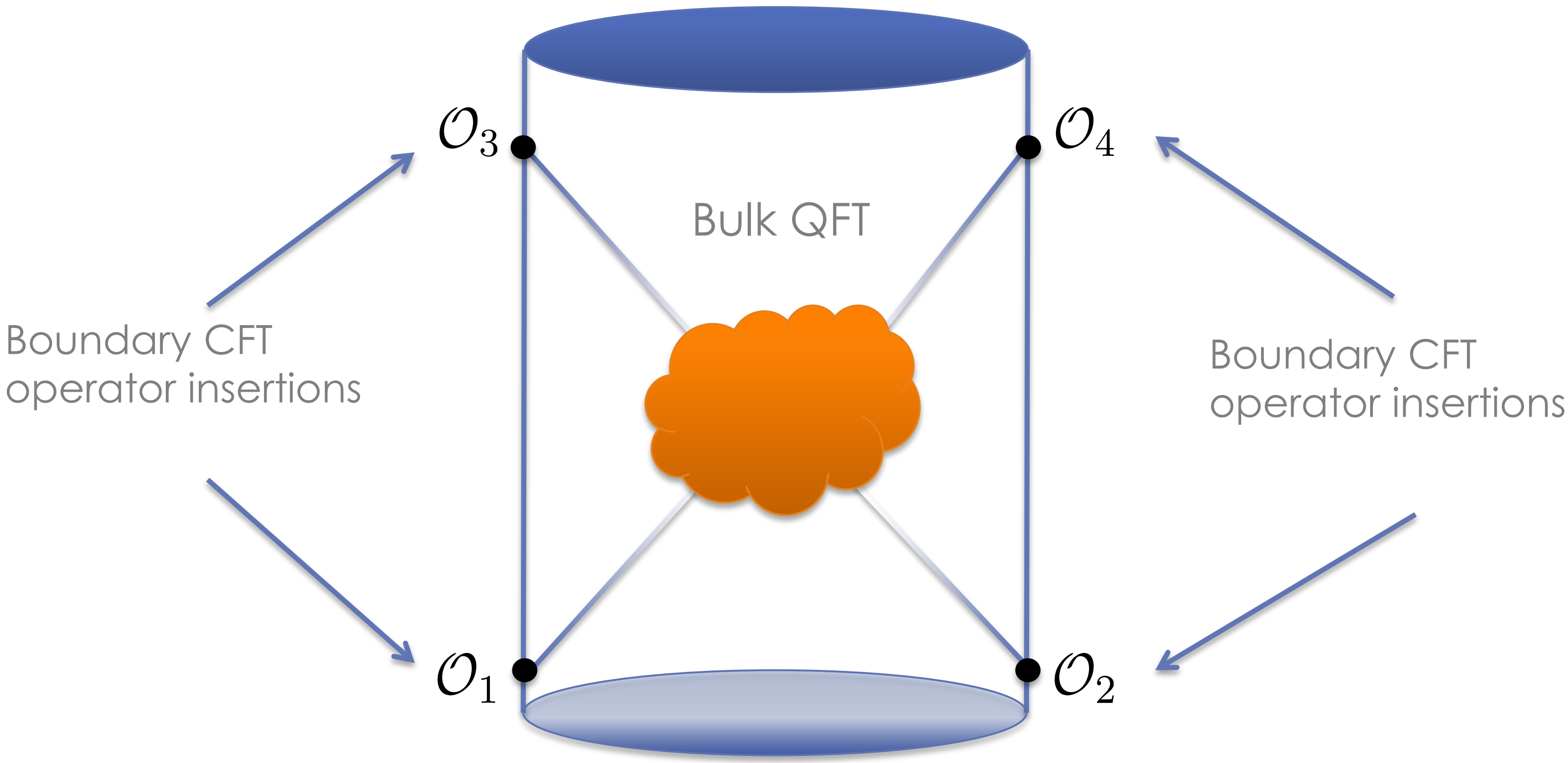
QFT in a box (also a CFT!)



- Our box is anti- de Sitter space.
- Poincare symmetry of QFT in d+1 deformed to SO(d,2)
- These are the symmetries of a conformal field theory in d dimensions.
- Pushing bulk operators to the AdS boundary at spatial infinity defines CFT operators.

Scattering experiments

- We set up a bulk scattering experiment by sourcing with boundary insertions.

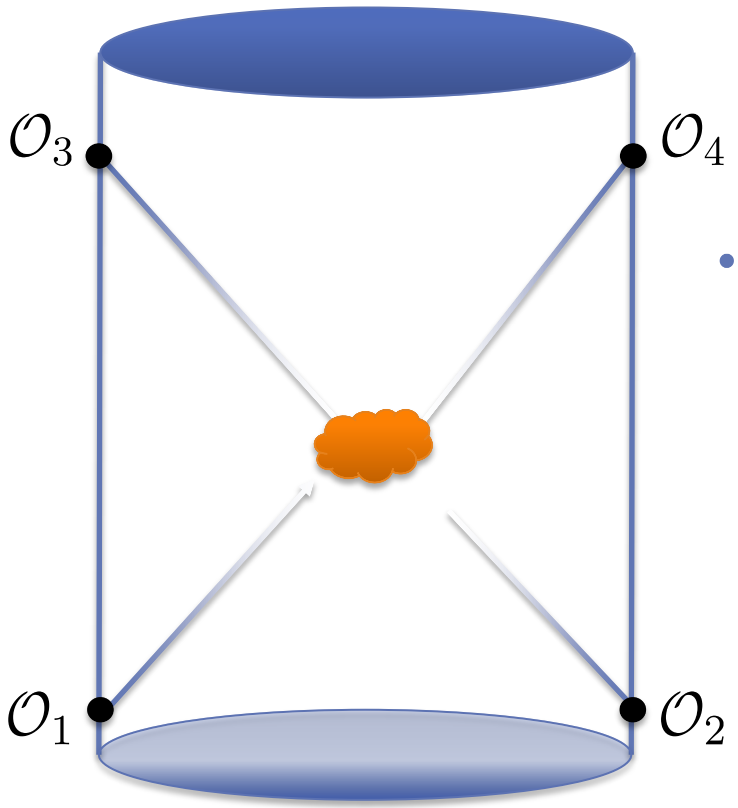


CFT to S-matrix

- Large AdS radius recovers flat space scattering.

$$\Delta_{\mathcal{O}} = m R_{\text{AdS}}$$

- CFT operators with large scaling dimension!



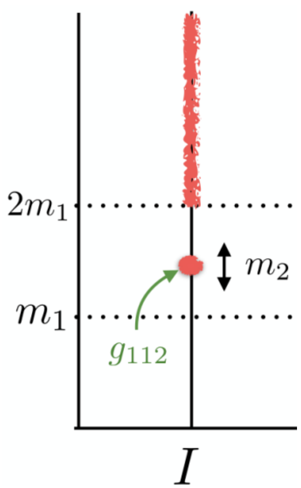
- Rest of the dictionary:

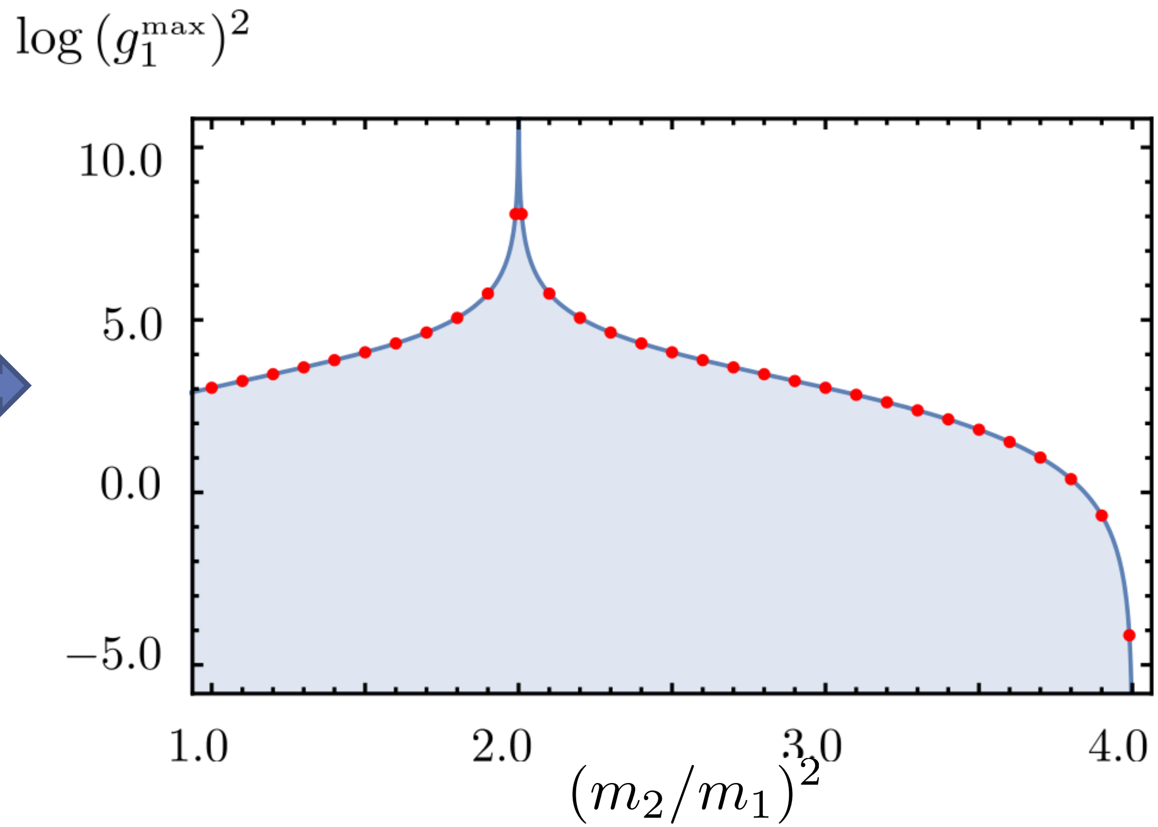
$$\langle \mathcal{O}(x_4)\mathcal{O}(x_3)\mathcal{O}(x_2)\mathcal{O}(x_1)\rangle$$

$$\downarrow$$

$$\langle k_3, k_4 | 1 + i\mathcal{T} | k_1, k_2 \rangle$$

Single particle exchange





- Universal bounds in 1+1d QFTs