# NP from PT?

L Del Debbio

Higgs Centre for Theoretical Physics
University of Edinburgh

work in collaboration with

F Di Renzo, G Filaci

# large orders in perturbation theory

Bender & Wu (1969)

$$\left(-\frac{d^2}{dx^2}+\frac{1}{4}x^2+\frac{1}{4}\lambda x^4\right)\Phi(x)=E(\lambda)\Phi(x)$$

$$\lim_{x\to\pm\infty}\Phi(x)=0$$

perturbative solution

$$E_0(\lambda)=\frac{1}{2}+\sum_{n=1}^{\infty}A_n\lambda^n$$

$$A_1=\tfrac{3}{4}, \qquad (2.12a)$$

$$A_2=-21/8, \qquad (2.12b)$$

$$A_3=333/16, \qquad (2.12c)$$

$$A_4=-30\,885/128, \qquad (2.12d)$$

$$A_5=916\,731/256, \qquad (2.12e)$$

$$A_6=-65\,518\,401/1024, \qquad (2.12f)$$

$$A_7=2\,723\,294\,673/2048, \qquad (2.12g)$$

$$A_8=-1\,030\,495\,099\,053/32\,768, \qquad (2.12h)$$

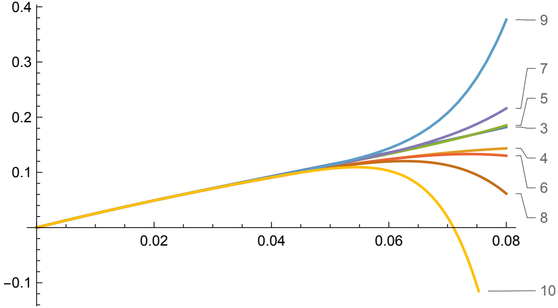
$$A_9=54\,626\,982\,511\,455/65\,536. \qquad (2.12i)$$

## factorial growth of the coefficients

$$A_n \sim (-)^{n+1} \left(6/\pi^3\right)^{1/2} \Gamma\left(n+\frac{1}{2}\right) 3^n$$

$\Longrightarrow$ asymptotic series

$$\int_0^\infty dw\, \frac{e^{w/t}}{1-w} = \sum_{n=0}^{\infty} n! t^{n+1}$$



# asymptotic series

$$\left| R(\lambda) - \sum_{n=1}^{N} p_n \lambda^n \right| < K_{N+1} \lambda^{N+1}$$

$$p_n \underset{n \to \infty}{\sim} K a^n n! n^b$$

minimized truncation error

$$K_N \propto a^N N! N^b \implies \bar{n}(\lambda) \sim \frac{1}{|a|\lambda}$$

size of the remainder

$$K_{\bar{n}} \lambda^{\bar{n}} \underset{\bar{n} \gg 1}{\sim} e^{-1/(|a|\lambda)}$$

# resummation

Borel transform

$$R(\lambda) = \sum_{n=1}^{\infty} r_n \lambda^n \implies B[R](t) = \sum_{n=0}^{\infty} r_n \frac{t^n}{n!}$$

Borel sum

$$\widetilde{R}(\lambda) = \int_0^{\infty} dt\, e^{-t/\lambda} B[R](t)$$

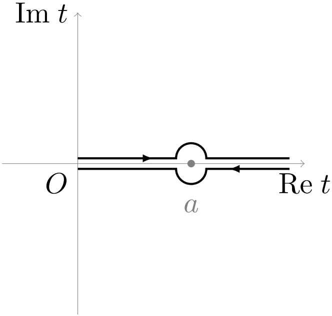
$$\boxed{r_n = K a^n \Gamma(n+1+b) \implies B[R](t) = \frac{K\Gamma(1+b)}{(1-at)^{1+b}}}$$

$\hookrightarrow$ singularities in the Borel plane

# ambiguities

deform the contour of the Borel integral

$$\mathrm{Im}\ \widetilde{R}(\lambda) = \mp \frac{\pi K}{a} e^{-1/(a\lambda)} \left(a\lambda\right)^{-b}$$



# nonperturbative contributions

$$\int d^4x\, e^{-iqx} \langle T j_\mu(x) j_\nu(0) \rangle = i \left( q_\mu q_\nu - q^2 g_{\mu\nu} \right) \Pi(Q^2)$$

$$D(Q^2) = 4\pi^2 \frac{d}{dQ^2} \Pi(Q^2) \simeq \frac{C_F}{\pi} \sum_n \alpha^{n+1} c_n \beta_0^n n!$$

for an asymptotic free theory

$$\alpha(Q) \sim -\frac{1}{\beta_0 \log Q^2/\Lambda^2}$$

ambiguity in the Adler function

$$\delta D(Q^2) \propto e^{2/(\beta_0 \alpha(Q))} \sim \left( \frac{\Lambda}{Q} \right)^4$$

a defect of perturbation theory? link to nonperturbative physics?

# check using NSPT

stochastic quantization for pure gauge theory

$$S_G\left[U\right] = -\frac{\beta}{2N_c}\sum_{\Box}\mathrm{Tr}\left(U_\Box + {U_\Box}^\dagger\right)$$

stochastic process

$$\frac{\partial}{\partial t}U_\mu(x;t) = i\Big[-\nabla_{x\mu}S_G[U_\mu(x;t)] - \eta_\mu(x;t)\Big]U_\mu(x;t)$$

$$= \tilde{F}_\mu(x;t)U_\mu(x;t)$$

$$\lim_{t\to\infty}\langle O[U(t)]\rangle = \frac{1}{Z}\int \mathcal{D}U\, e^{-S_G[U]}O[U]\,.$$

# perturbative expansion

NSPT is defined by the perturbative expansion of the stochastic process

$$U_\mu(x;t) = U_\mu^{(0)}(x;t) + \sum_{k=1} \beta^{-k/2} U_\mu^{(k)}(x;t)$$

$$F_\mu(x;t) = \sum_{k=1} \beta^{-k/2} F_\mu^{(k)}(x;t)$$

explicitly

$$U^{(1)}(t+\epsilon) = U^{(1)}(t) - F^{(1)}(t)$$

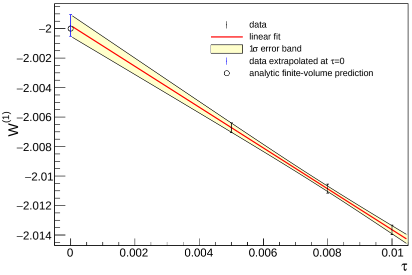
$$U^{(2)}(t+\epsilon) = U^{(2)}(t) - F^{(2)}(t) + \frac{1}{2}F^{(1)}(t)^2 - F^{(1)}(t)U^{(1)}(t)$$

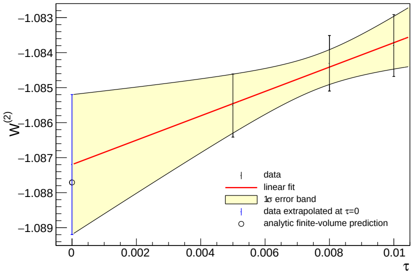
$$\ldots$$

# implementation in GRID

$\mathrm{SU}(3)$ gauge theory, $n_f = 2$, $8^4$ volume, TBC

Langevin dynamics



# understanding the algorithm

critical mass with Wilson fermions

$$\Gamma(ap = 0, am_c, \beta) = am_c + \Sigma(ap = 0, am_c, \beta) = 0$$

computed as a perturbative series

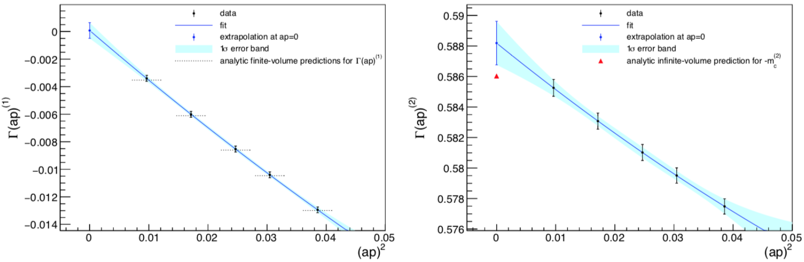
$$\Sigma(ap = 0, am, \beta) = \sum_n \gamma_n(am)\beta^{-n}$$
$$\Longrightarrow am_c = \sum_n m_n \beta^{-n}$$

iterative solution

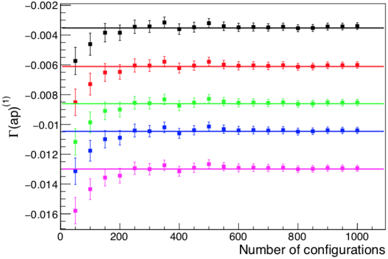
$$m_{K+1} = \gamma_{K+1}(M_K), \qquad M_K = \sum_{n=1}^{K} m_n \beta^{-n}$$

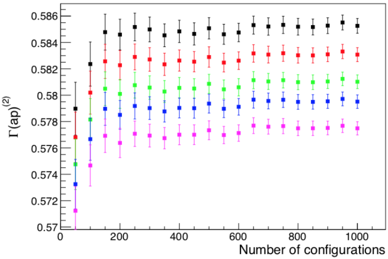
# critical mass

$\mathrm{SU}(2)$ gauge theory, $n_f = 2$, $12^4$ volume, TBC

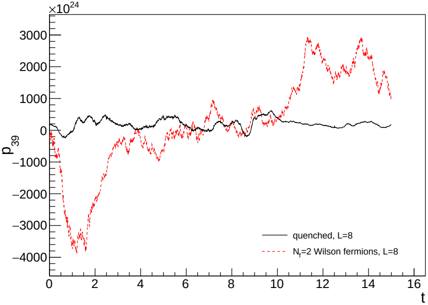


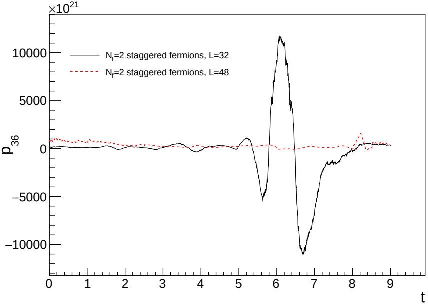
# convergence of the stochastic process



# numerical instabilities



# renormalons & condensates: plaquette

$$P = \frac{1}{6N_c L^4} \sum_{\Box} \mathrm{Re}\,\mathrm{Tr}\left(1 - U_{\Box}\right) \,.$$

$$\langle P \rangle_{\mathrm{MC}} = \sum_{n=0}^{\infty} c_n \alpha^{n+1} + \frac{\pi^2}{36} C_G(\alpha)\, a^4 \langle G^2 \rangle + \mathcal{O}\left(a^6\right)$$

$$C_G(\alpha)^{-1} = \frac{36}{\pi^2}\left(1 + \frac{\beta_1}{\beta_0}\frac{\alpha}{4\pi}\right)$$

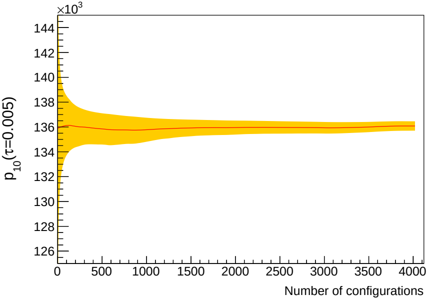
asymptotic behaviour

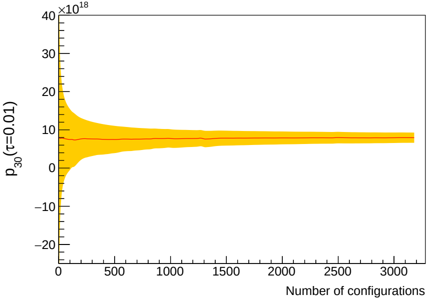
$$c_n \simeq N_P \left(\frac{\beta_0}{2\pi d_{F^2}}\right)^n \frac{\Gamma(n+1+d_{F^2}b)}{\Gamma(1+d_{F^2}b)} \left\{1 + \frac{d_{F^2}b}{n+d_{F^2}b} + \ldots\right\}$$

[Bali et al 14]

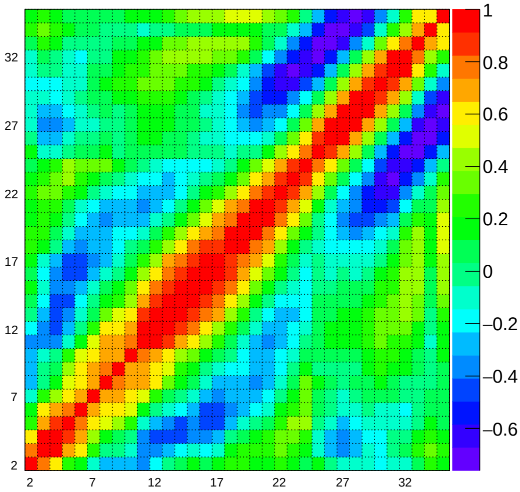
# measurements

$$\langle P \rangle_{\text{pert}} = \sum_{n=0}^{\infty} p_n \, \beta^{-n-1}$$

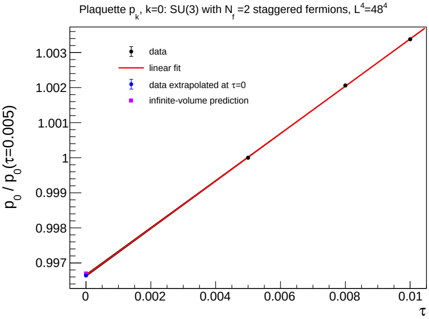


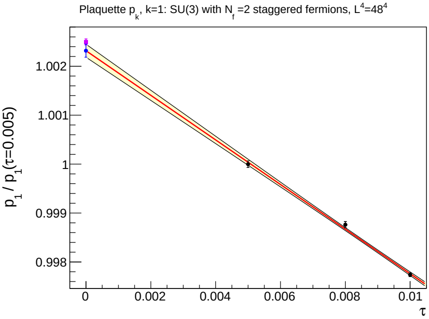



# correlations



# extrapolation



# renormalon behaviour

$$c_n \simeq N_P \left(\frac{\beta_0}{2\pi d_{F^2}}\right)^n \frac{\Gamma(n+1+d_{F^2}b)}{\Gamma(1+d_{F^2}b)} \left\{1 + \frac{d_{F^2}b}{n+d_{F^2}b} + \dots\right\}$$

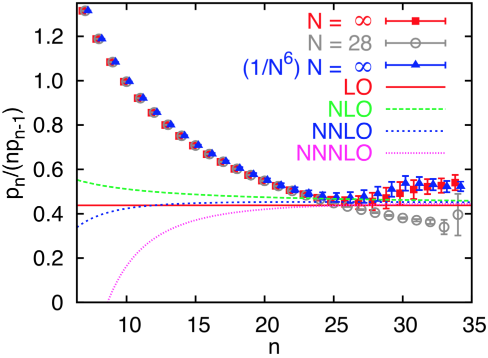
or equivalently

$$\frac{p_n}{np_{n-1}} = \frac{\beta_0}{8\pi}\left[1 + \frac{\beta_1}{2\beta_0^2}\frac{4}{n} + O\left(\frac{1}{n^2}\right)\right]$$

where

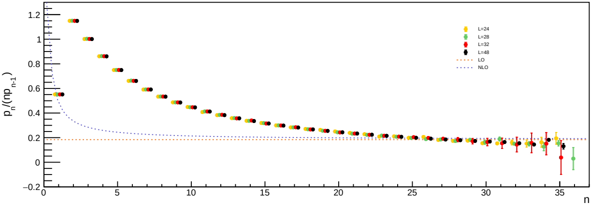
$$\beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f$$
$$\beta_1 = \frac{34}{3}N_c^2 - n_f\left(\frac{13}{3}N_c - \frac{1}{N_c}\right) .$$

# renormalon behaviour – pure gauge




[Bali et al 14]

# renormalon behaviour – with fermions



# minimal term and scaling of the condensate

$$(\bar{n} + d_{F^2} b) \frac{\beta_0 \alpha}{2\pi d_{F^2}} = \exp\left[-\frac{1}{2(\bar{n} + d_{F^2} b)} + \ldots\right]$$

and hence:

$$S_P(\alpha) = \sum_{n=0}^{\bar{n}} c_n \alpha^{n+1}$$

$$\frac{\pi^2}{36} C_G(\alpha) \langle G^2 \rangle = \frac{1}{a^4} \left[\langle P \rangle_{\mathrm{MC}} - S_P(\alpha)\right]$$

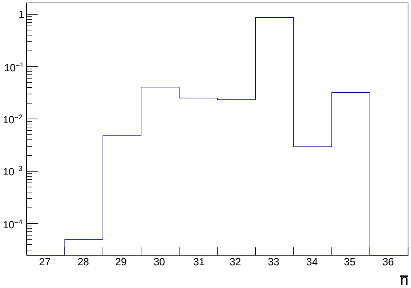
ambiguity

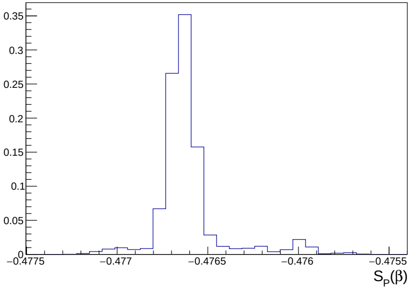
$$\boxed{\delta S_P(\alpha) = \sqrt{\bar{n}} c_{\bar{n}} \alpha^{\bar{n}+1}}$$

# determination of the minimal term

$$\boxed{p_n \beta^{-n-1} < p_{n+1} \beta^{-n-2}}$$

data for $L = 48$, $\beta = 5.3$

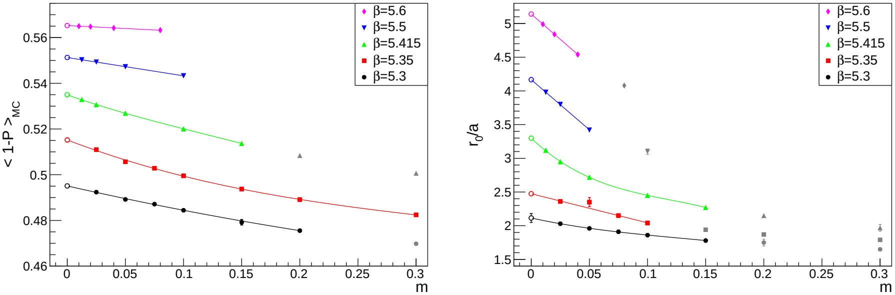


# Monte Carlo results

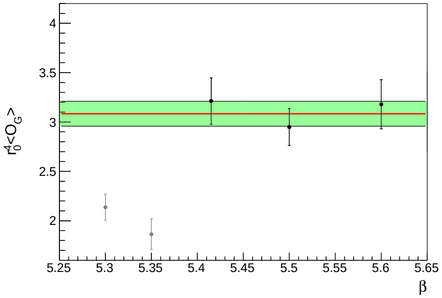
existing results for staggered fermions

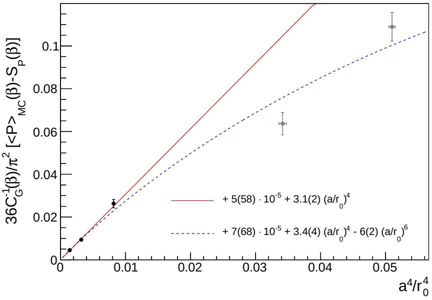
[Tamhankar et 99, Heller et al 94]



# condensate

$$a^4\langle G^2\rangle = \frac{36}{\pi^2} C_G(\alpha)^{-1}\left[\langle P\rangle_{\rm MC} - S_P(\alpha)\right]$$



# thoughts on IR conformality

- $p_n$ obey the expected *renormalon* behaviour
- study of lattice systematics
- relation with QCD condensates
- if IR conformality - then the condensates all vanish as $m \to 0$
- difference in the behaviour of the perturbative coefficients at large $p$?

# Euler integration

one step of numerical integration + one step of stochastic gauge fixing

$$U_\mu(x)' = e^{-F_\mu(x;t)}\, U_\mu(x;t)\,,$$
$$U_\mu(x;t+\tau) = e^{w[U'](x)} U_\mu(x)' e^{-w[U'](x+\hat{\mu})}\,,$$

where the force term is

$$F_\mu(x;t) = \frac{\tau}{\beta} \nabla_{x\mu} S_G[U(t)] + \sqrt{\frac{\tau}{\beta}}\, \eta_\mu(x;t)\,.$$

# TBC

periodicity up to gauge transformations

$$U_\mu(x + L\hat{\nu}) = \Omega_\nu U_\mu(x) \Omega_\nu^\dagger$$

consistency condition

$$\Omega_\mu \Omega_\nu = z_{\mu\nu}\, \Omega_\nu \Omega_\mu\,, \qquad z_{\mu\nu} \in Z_N$$
$$z_{\mu\nu} = \exp in_{\mu\nu} 2\pi/N$$

used TBC in the $(\hat{1}, \hat{2})$ plane

$$\Omega_1 = \begin{pmatrix} e^{-i\frac{2\pi}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\frac{2\pi}{3}} \end{pmatrix} \qquad \Omega_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\,,$$

corresponding to $z_{12} = \exp\left(i\frac{2\pi}{3}\right)$.

# NSPT for gauge + fermions

force for the Langevin evolution

$$F = \left(\epsilon \Phi^a + \sqrt{\epsilon}\eta^a\right) T^a$$
$$\Phi^a = \left[\nabla^a_{x\mu} S_G - \mathrm{Re}\ \left(\xi^\dagger \left(\nabla^a_{x\mu} M\right) M^{-1} \xi\right)\right]$$

expand the fermionic force in a perturbative series

$$U_\mu(x;t) = U_\mu^{(0)}(x;t) + \sum_{p>0} \beta^{-p/2} U_\mu^{(k)}(x;t)$$
$$\implies \quad M = M^{(0)} + \sum_{p>0} \beta^{-p/2} M^{(p)}$$
$$M^{-1} = M^{(0)\ -1} + \sum_{p>0} \beta^{-p/2} M^{(p)\ -1}$$

## adding smell

- adjoint fermions

$$\psi(x + L\hat{\nu}) = \Omega_\nu \psi(x) \Omega_\nu^\dagger$$

  same transformation as for gauge fields

- fundamental fermions
  add *smell* degrees of freedom

$$\psi(x + L\hat{\nu})_{ir} = \left(\Omega_\nu\right)_{ij} \psi(x)_{js} \left(\Lambda_\nu^\dagger\right)_{sr}$$

- correct number of degrees of freedom

$$\exp\left(n_f \mathrm{Tr} \log M\right) \mapsto \exp\left(\frac{n_f}{N} \mathrm{Tr} \log M\right)$$