Continuum limit of fishnet graphs and AdS sigma model

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15th Workshop on Non-Perturbative QCD,
IAP, Paris, June 2018

based on work done in collaboration with
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Motivation

Understand dynamics of planar graphs and its relation to sigma models

['t Hooft]

Best possible starting point: N=4 SYM [Maldacena'97]

String dual is believed to be known
Theory is believed to be integrable
(meaning we have methods for re-summing planar graphs)

Use solution to gain knowledge about other models by deforming / twisting the theory
Partial re-summation of planar graphs
(Reduce complexity, but maintain as many important properties as possible: conformal symmetry, integrability, etc.)

Fishnet theory

Baby version of N = 4 SYM

A theory for matrix scalar fields with quartic coupling

[Gurdogan,Kazakov'15]
[Caetano,Gurdogan,Kazakov'16]

$$\mathcal{L}_{\text{fishnet}} = N \text{tr} \left[\partial_\mu \phi_1 \partial_\mu \phi_1^* + \partial_\mu \phi_2 \partial_\mu \phi_2^* + (4\pi g)^2 \phi_1 \phi_2 \phi_1^* \phi_2^* \right]$$

It can be obtained by twisting N=4 SYM theory, so-called γ deformation,
sending the deformation parameter to i-infinity
while taking YM coupling to zero

[Frolov'05]
[Lunin,Maldacena'05]
[Grabner,Gromov,Kazakov,Korchemsky'17]
[Sieg,Wilhelm'16]

1. Gluons and gauginos decouple
2. Gauge group becomes a flavour group
3. Conformal symmetry is preserved for any coupling
   (at least in planar limit and for fine-tuned double-trace couplings)
4. Integrability is retained

Fishnet theory

Baby version of N = 4 SYM

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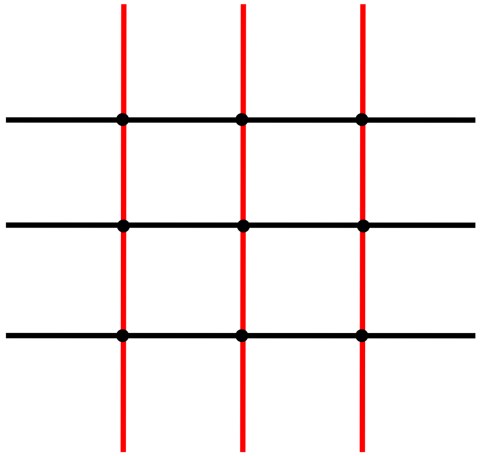
$$\mathcal{L}_{\text{fishnet}} = N \text{tr}\left[\partial_\mu \phi_1 \partial_\mu \phi_1^* + \partial_\mu \phi_2 \partial_\mu \phi_2^* + (4\pi g)^2 \phi_1 \phi_2 \phi_1^* \phi_2^*\right]$$

All planar graphs locally look the same



bulk graph

Integrability is not mysterious here and links directly to basic property of the ϕ^4 coupling in d=4

[Zamolodchikov'80]
[Isaev'03]
[Gromov,Kazakov,Korchemsky,Negro,Sizov'17]
[Chicherin,Kazakov,Loebbert,Muller,Zhong'16]

Win: simplicity, many fewer graphs

Lose: unitarity

Continuum limit & string?

What about duality to string in AdS?

Extremal twisting procedure forces the YM coupling to be small
String in highly curved AdS?

Question: what is the continuum limit of the fishnet graphs?

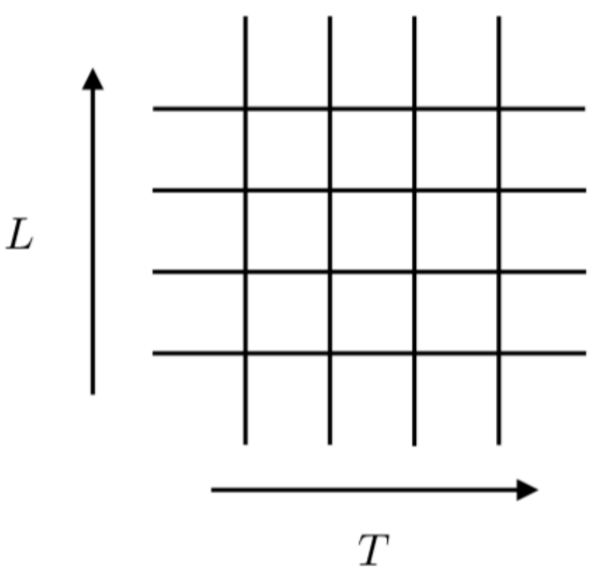
Important observation concerning large order behaviour

[Zamolodchikov'80]

Zamolodchikov's thermodynamical scaling

$$\log Z_{L,T} = -L \times T \log g_{cr}^2$$

$$g_{cr} = \frac{\Gamma(3/4)}{\sqrt{\pi}\Gamma(5/4)} = 0.7\ldots$$



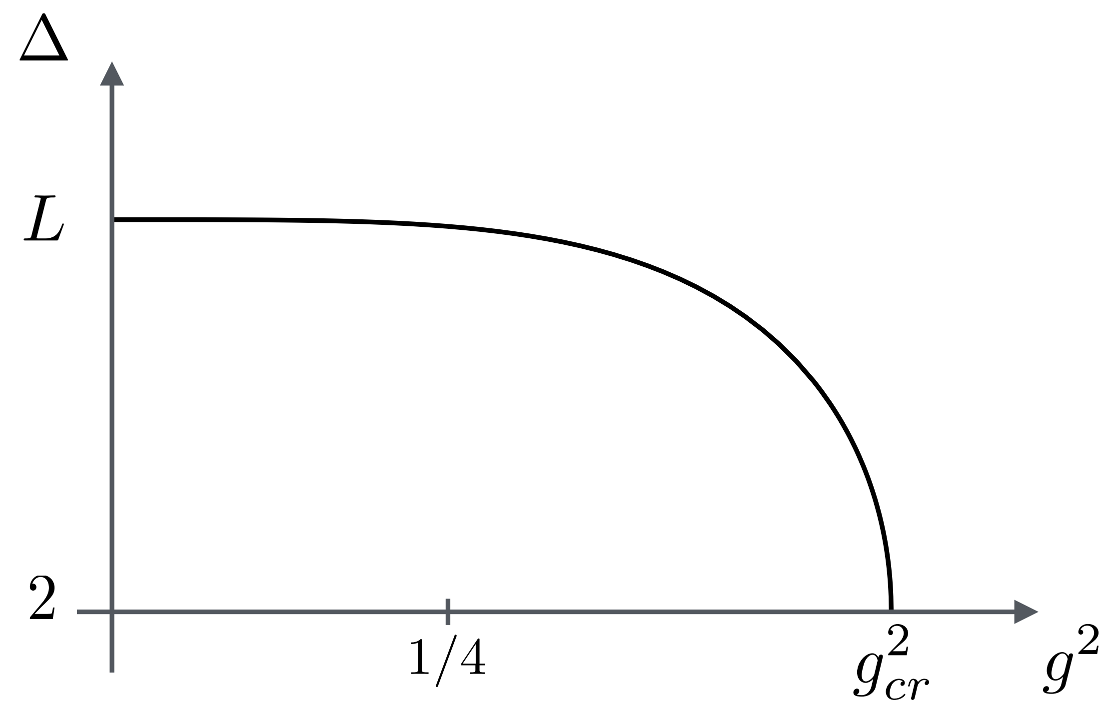
Determine a critical coupling where graphs become dense

Plan

Study continuum limit using integrable methods borrowed from N=4 SYM

Probe: scaling dimension Δ of BMN vacuum operator $\mathcal{O} = \mathrm{tr}\,\phi_1^L$

Qualitative picture of Δ as function of coupling g^2 when $L \to \infty$



Main claim : continuum limit is given by the 2d AdS5 sigma model

BMN operator maps to tachyon $\mathrm{tr}\,\phi_1^L \leftrightarrow V_\Delta \sim e^{-i\Delta t}$

Relation to sigma model energy $\log g^{2L} = \log g_{cr}^{2L} + E_{2d}(\Delta, L)$

Graphs versus integrability

Computation of anomalous dimension

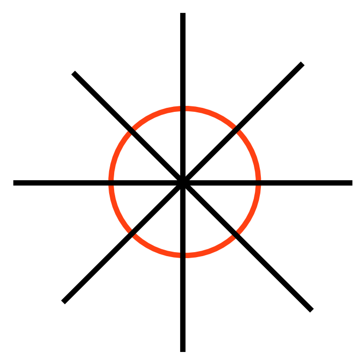
[Gurdogan,Kazakov'15]
[Caetano,Gurdogan,Kazakov'16]

BMN vacuum (not protected) $\mathrm{tr}\,\phi_1^L$

$$\Delta = L + \gamma$$

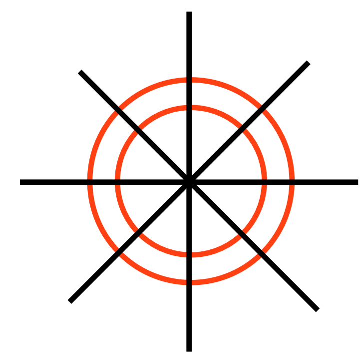
Graphs: loop corrections come from the "wheel" diagrams

$Z = 1 +$



1 wheel $\sim g^{2L}$

$+$



2 wheels $\sim g^{4L}$

$+ \cdots$

wave-function renormalization

Depends on cut off

$$R \sim \log \Lambda_{cut-off}$$

Anomalous dimension controls the logarithmic dependence on cut off

$$\log Z \sim -\gamma \times R$$

Graphs versus integrability

Computation of anomalous dimension

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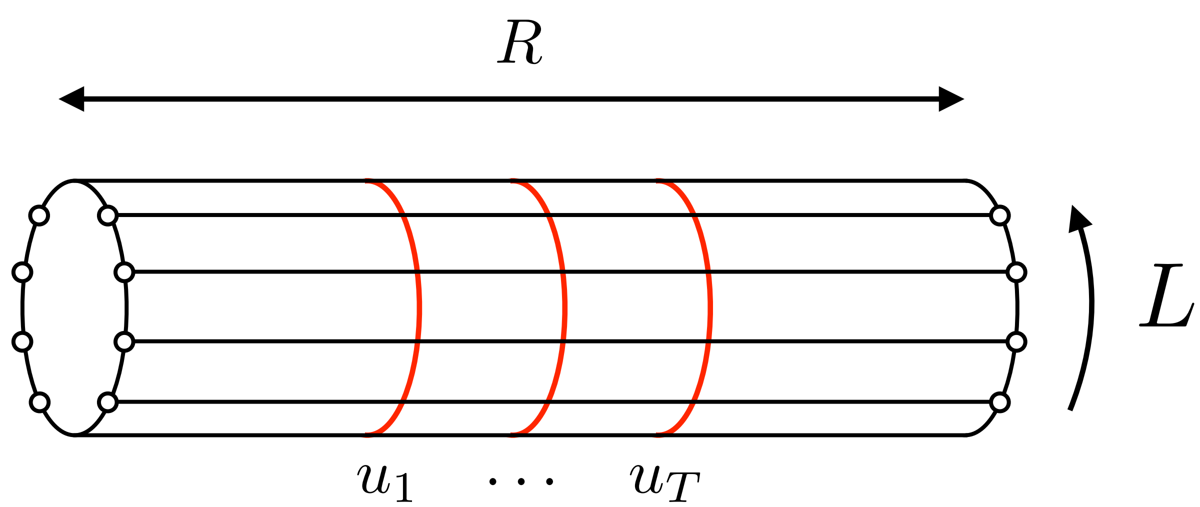
BMN vacuum (not protected) $\mathrm{tr}\,\phi_1^L$

$$\Delta = L + \gamma$$

Integrability: Free energy of a gas of *magnons* at temperature $1/L$

Partition function on $\mathbb{R} \times S_L$

$$\mathcal{Z}_{L,R} = \sum_{T \geqslant 0} g^{2LT} \times$$



Each magnon carries a rapidity "u", which is a momentum along euclidean time direction, and a discrete label "a", which enumerates harmonics on 3-sphere

$$\log \mathcal{Z}_{L,R} = -\Delta_L R + \dots$$

Weak coupling expansion: magnon = wheel

Integrability: direct quantum mechanical interpretation of the graphs

Thermodynamical Bethe Ansatz

Factorized scattering allows us to obtain free energy from TBA eqs

$$\Delta = L - 2\sum_{a\geqslant 1}\int \frac{du}{2\pi}\mathbf{Y}_a(u)$$

$$-\sum_{a\geqslant 1}\int \frac{du}{2\pi}\mathbf{Y}_a^2(u) - 2\sum_{a,b\geqslant 1}\int \frac{dudv}{(2\pi)^2}\mathbf{Y}_a(u)\mathcal{K}_{a,b}(u,v)\mathbf{Y}_b(v) + O(\mathbf{Y}^3)$$

Boltzmann weight: $\mathbf{Y}_a(u) = a^2 e^{Lh - L\epsilon_a(u)} \ll 1$

Mechanical energy: $\epsilon_a(u) = \log\left(u^2 + a^2/4\right)$

Scattering kernel: $\mathcal{K}_{a,b}(u,v) = \dfrac{\partial}{i\partial u}\log S_{a,b}(u,v) + \text{matrix part}$

Coupling constant is just a fugacity for the magnons (wheels)

$$g^2 = e^h$$

with h the chemical potential

Thermodynamical limit

Thermodynamical limit $L \to \infty$

Interesting when chemical potential gets bigger than mass of lightest magnon

$$h > \log \epsilon(u=0) = \log 1/4$$

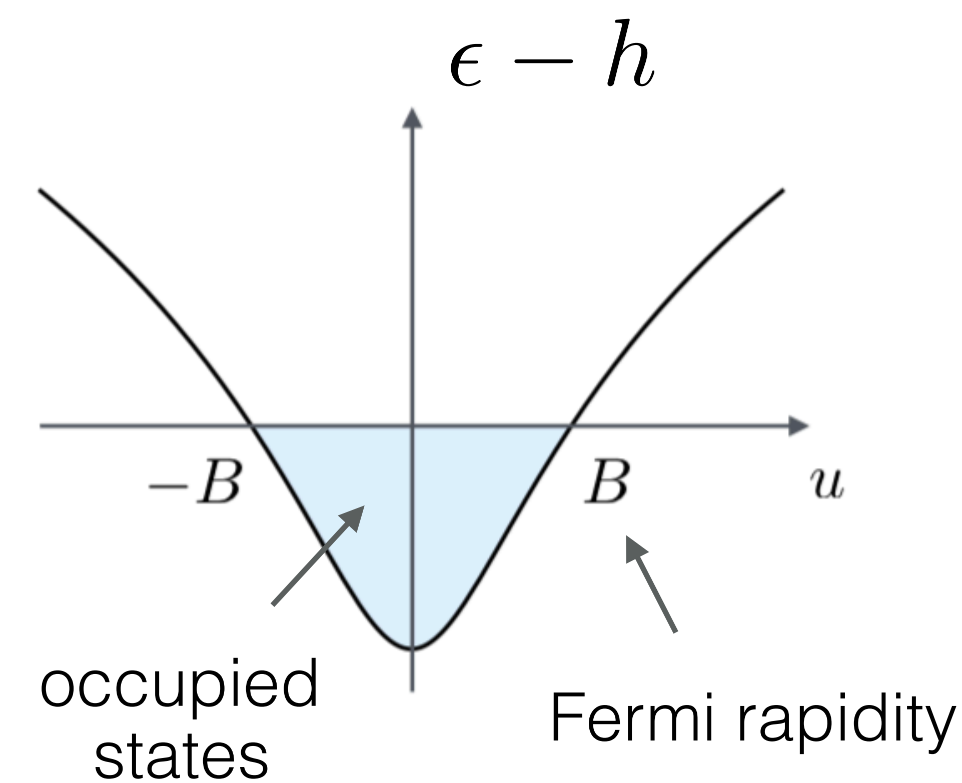
that is for

$$g > 1/2$$

A Fermi sea forms

All states below the Fermi rapidity are filled

Increasing coupling amounts to increasing B



Comment: only the s-wave (lightest) magnons condense
(higher Lorentz harmonics decouple)

Linear integral equation

In the thermodynamical regime the TBA eqs linearize

Integral equation for the rapidity distribution of energy levels

$$\chi(u) = C - \epsilon(u) + \int_{-B}^{B} \frac{du}{2\pi} \mathcal{K}(u-v)\chi(v)$$

BC: $\chi(u = \pm B) = 0$

Effective chemical potential: $C = \log g^2 - \int_{-B}^{B} \frac{du}{2\pi} k(u)\chi(u)$

Kernel: $\mathcal{K}(u) = 2\psi(1+iu) + 2\psi(1-iu) + \frac{2}{1+u^2}$

Scaling dimension: $\Delta/L = 1 - \int_{-B}^{B} \frac{du}{\pi} \chi(u)$

Critical regime

Small B : dilute gas, density of magnons is small

$$j = -df/dh \sim 0 \qquad \varepsilon = f + hj \sim 1$$

Critical regime relates to large magnon density $\quad B \to \infty$

$$\varepsilon \sim j \log g_{cr}^2 \qquad f \sim 0$$

All energy levels are filled

Equation is solved in Fourier space

$$\chi_{cr}(u) = C_{cr} - \epsilon(u) + \int_{-\infty}^{\infty} \frac{dv}{2\pi} \mathcal{K}(u-v)\chi_{cr}(v) \quad \Rightarrow \quad \chi_{cr} = \log \frac{\sqrt{2}\cosh\theta + 1}{\sqrt{2}\cosh\theta - 1}$$

with $\quad \theta = \pi u/2$

One verifies:

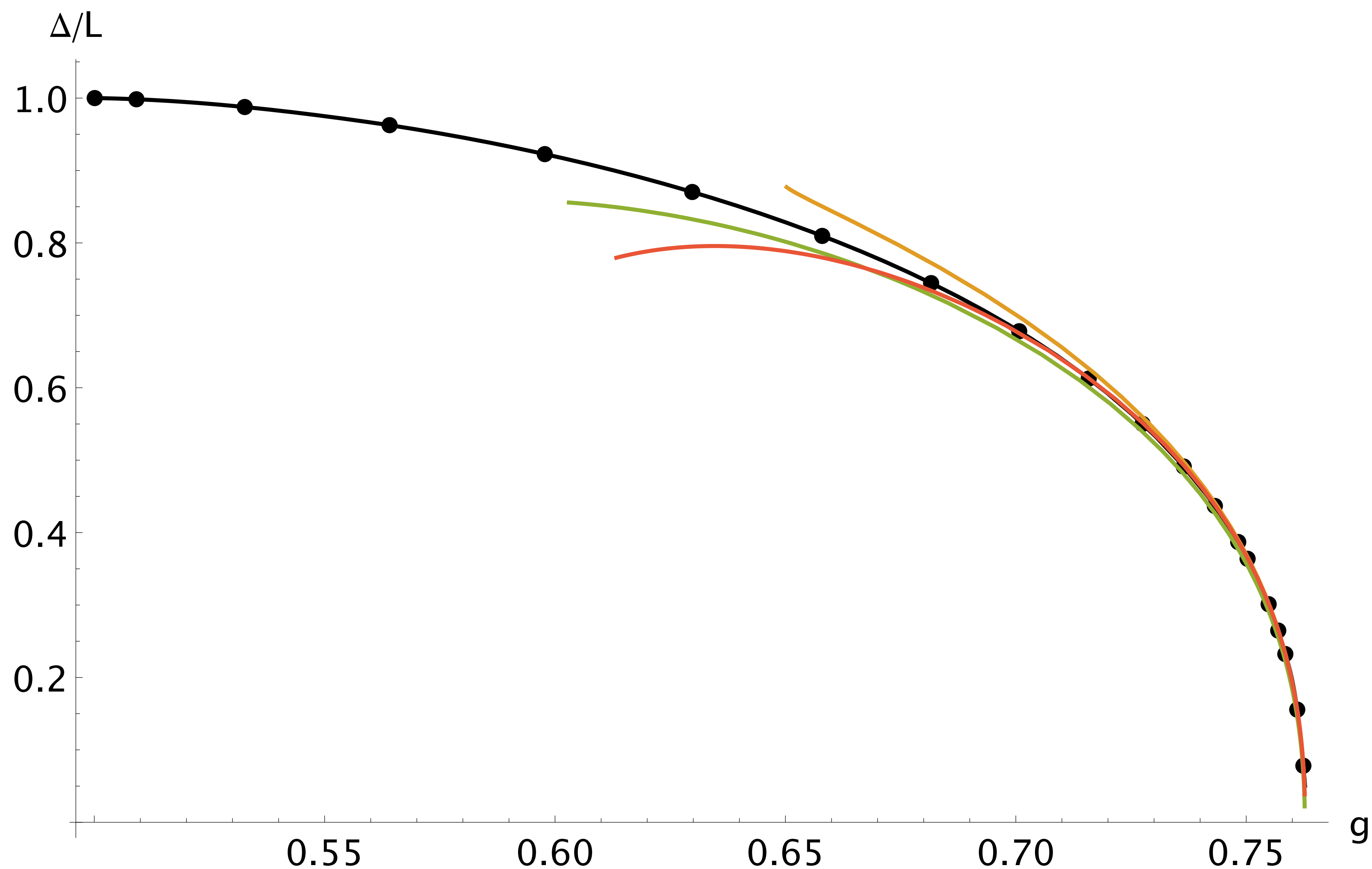
(i) Scaling dimension vanishes $\quad \Delta/L = 0$

(ii) Chemical potential (coupling) approaches predicted value

$$\chi_{cr} \sim e^{-|\theta|} \quad \Rightarrow \quad C_{cr} = 0 \quad \Rightarrow \quad g_{cr} = \Gamma(3/4)/\sqrt{\pi}\Gamma(5/4)$$

Numerical check

Integral equation can be solved numerically

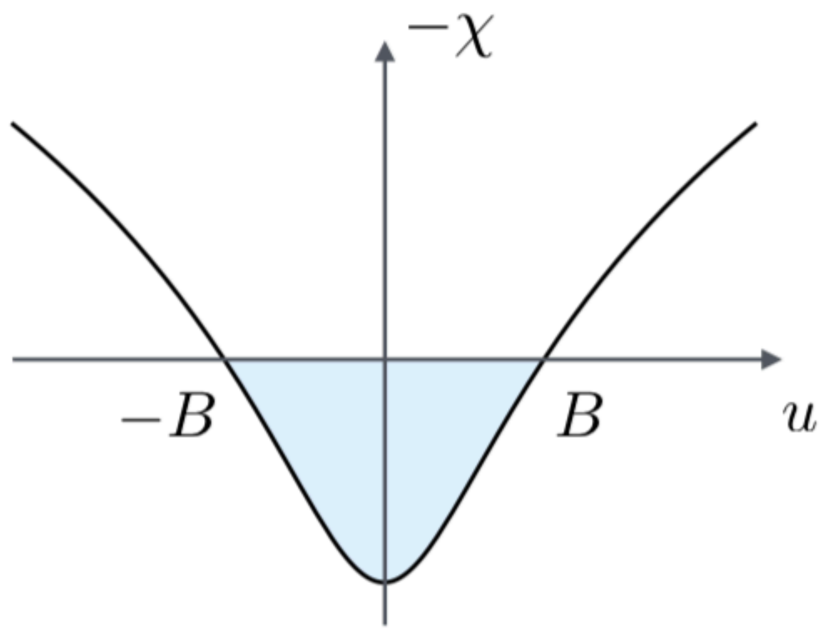


$g_{\rm thr} = 0.5$ $g_{\rm cr} = 0.76276$

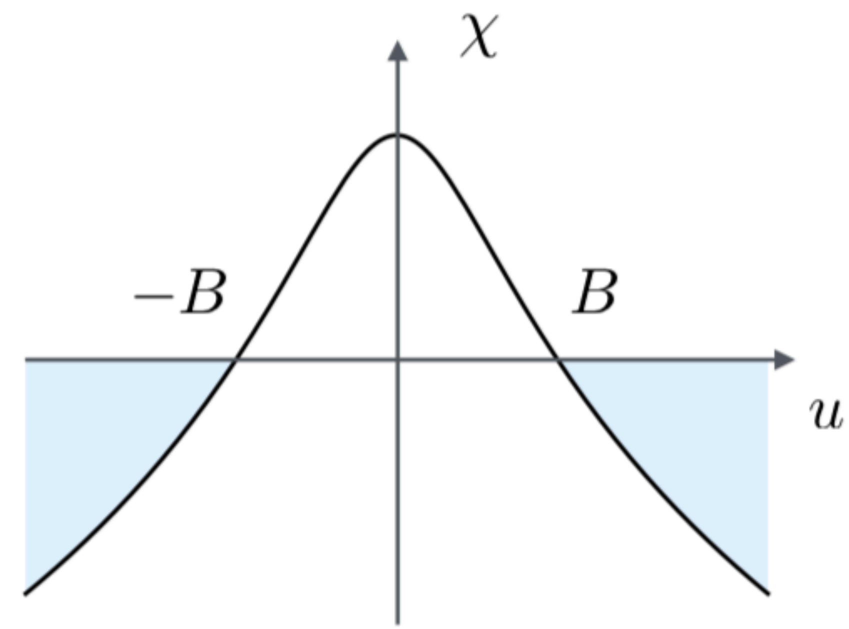
thermodynamical window

Near-critical regime

Particle-hole transformation:



Fermi sea of magnons



dual Fermi sea

Equation for dual excitations:

Dualize kernel by means of

$$K = -\frac{\mathcal{K}}{1-\mathcal{K}*} = -\mathcal{K} - \mathcal{K}*\mathcal{K} - \ldots$$

Act on both sides of the equation with $1 - K*$

Dual equations

Dual equation: $$\chi(\theta) = E(\theta) + \int\limits_{\theta^2 > B^2} \frac{d\theta'}{2\pi} K(\theta - \theta')\chi(\theta')$$

Dual energy formula: $$\log g^2 = \log g_{cr}^2 + \int\limits_{\theta^2 > B^2} \frac{d\theta}{2\pi} P'(\theta)\chi(\theta)$$

No chemical potential but extra BC: $\chi(\theta) \sim -2\rho \log\theta$ $\qquad \rho = \Delta/L =$ charge density

1) Dual kernel:

$$K(\theta) = \frac{\partial}{i\partial\theta} \log \frac{\Gamma(1+\frac{i\theta}{2\pi})\Gamma(\frac{1}{2}-\frac{i\theta}{2\pi})\Gamma(\frac{3}{4}+\frac{i\theta}{2\pi})\Gamma(\frac{1}{4}-\frac{i\theta}{2\pi})}{\Gamma(1-\frac{i\theta}{2\pi})\Gamma(\frac{1}{2}+\frac{i\theta}{2\pi})\Gamma(\frac{3}{4}-\frac{i\theta}{2\pi})\Gamma(\frac{1}{4}+\frac{i\theta}{2\pi})}$$

2) Dual dispersion relation:

$$E(\theta) = \chi_{cr}(\theta) = \log\left[\frac{\sqrt{2}\cosh\theta + 1}{\sqrt{2}\cosh\theta - 1}\right] \sim \frac{m}{2}e^{-|\theta|}$$

$$P(\theta) = -iE(\theta + i\frac{\pi}{2}) = i\log\left[\frac{\sqrt{2}\sinh\theta + i}{\sqrt{2}\sinh\theta - i}\right] \sim \mp\frac{m}{2}e^{-|\theta|}$$

where $m = 4\sqrt{2}$

Interpretation

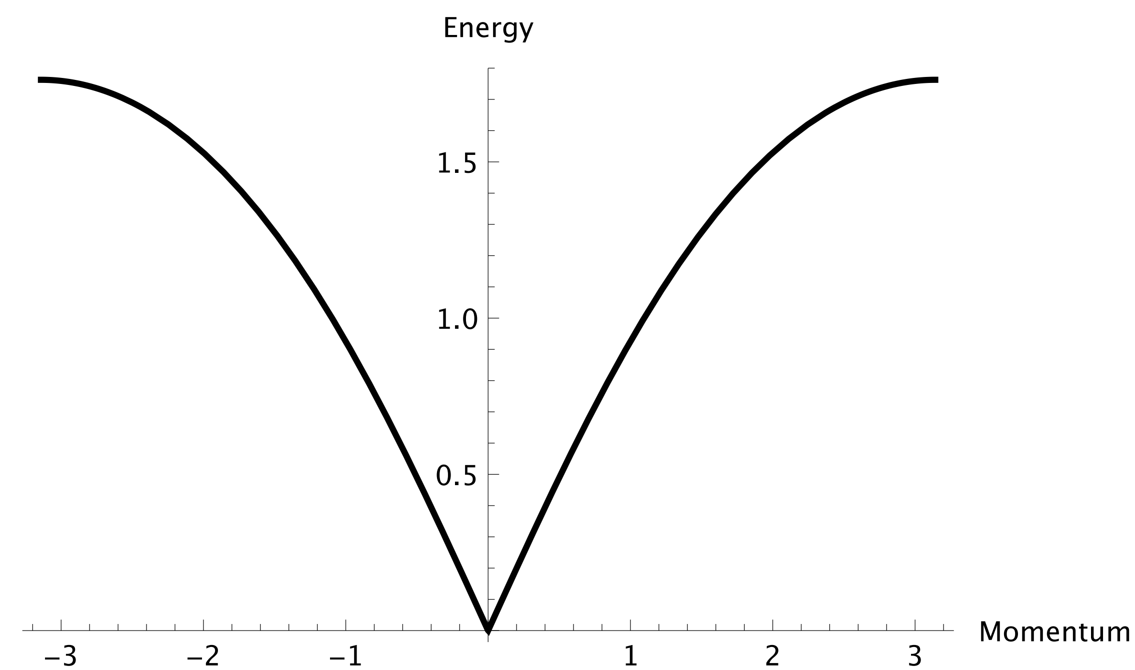
What is the dual system describing?

1) Kernel: $$K = -i\partial_\theta \log S_{O(6)}$$

Excitations scatter as particles in 2d O(6) non-linear sigma model
[Zamolodchikov&Zamolodchikov]

2) Dispersion relation:

$$\sinh^2\left(\frac{1}{2}E\right) = \sin^2\left(\frac{1}{2}P\right)$$



Gapless excitations (unlike O(6) model)

Besides, E decreases when θ increases
Support is non-compact and density is not normalizable (cannot count excitations)

No mass gap + continuous spectrum
Suggest: sigma model with ***non-compact*** target space
Proposal: integrable lattice completion of AdS_5 sigma model

2d non-linear sigma model

Sigma model with curved target space

$$\mathcal{L} = -\frac{1}{2e^2}\, G^{AB}\partial_a X_A \partial^a X_B$$

Weak coupling (large AdS radius) $\quad e^2 \ll 1$

Beta function related to Ricci scalar

For AdS_{d+1} : $\quad -Y_0^2 + Y_\perp^2 - Y_{d+1}^2 = -1$

One loop running: $\quad \mu\frac{\partial}{\partial\mu}e^2(\mu) = \frac{d}{2\pi}e^4(\mu) + \dots$

Alternatively: $\quad \frac{1}{e^2(\mu)} = \frac{d}{2\pi}\log\left(\Lambda/\mu\right)$

1. Theory is weakly coupled in IR
2. There is no mass gap
3. There is no isolated vacuum

Perturbative tachyon

Consider sigma model on cylinder of radius L

Interested in 2d “ground state” energy : tachyon

(best candidate for extremum of energy at given charge = global time energy)

$$V_\Delta \sim e^{-i\Delta t}$$

Classically, it corresponds to solution

$$Y^0 \pm iY^{d+1} = e^{\pm iH\tau}$$
$$Y_\perp = 0$$

Charge density $\quad \Delta/L = \frac{1}{e^2}(Y^0\dot{Y}_{d+1} - Y^{d+1}\dot{Y}_0) = -H/e^2$

Energy density $\quad E/L = \frac{1}{2e^2}(\dot{Y}^0\dot{Y}_0 + \dot{Y}^{d+1}\dot{Y}_{d+1}) = -H^2/(2e^2)$

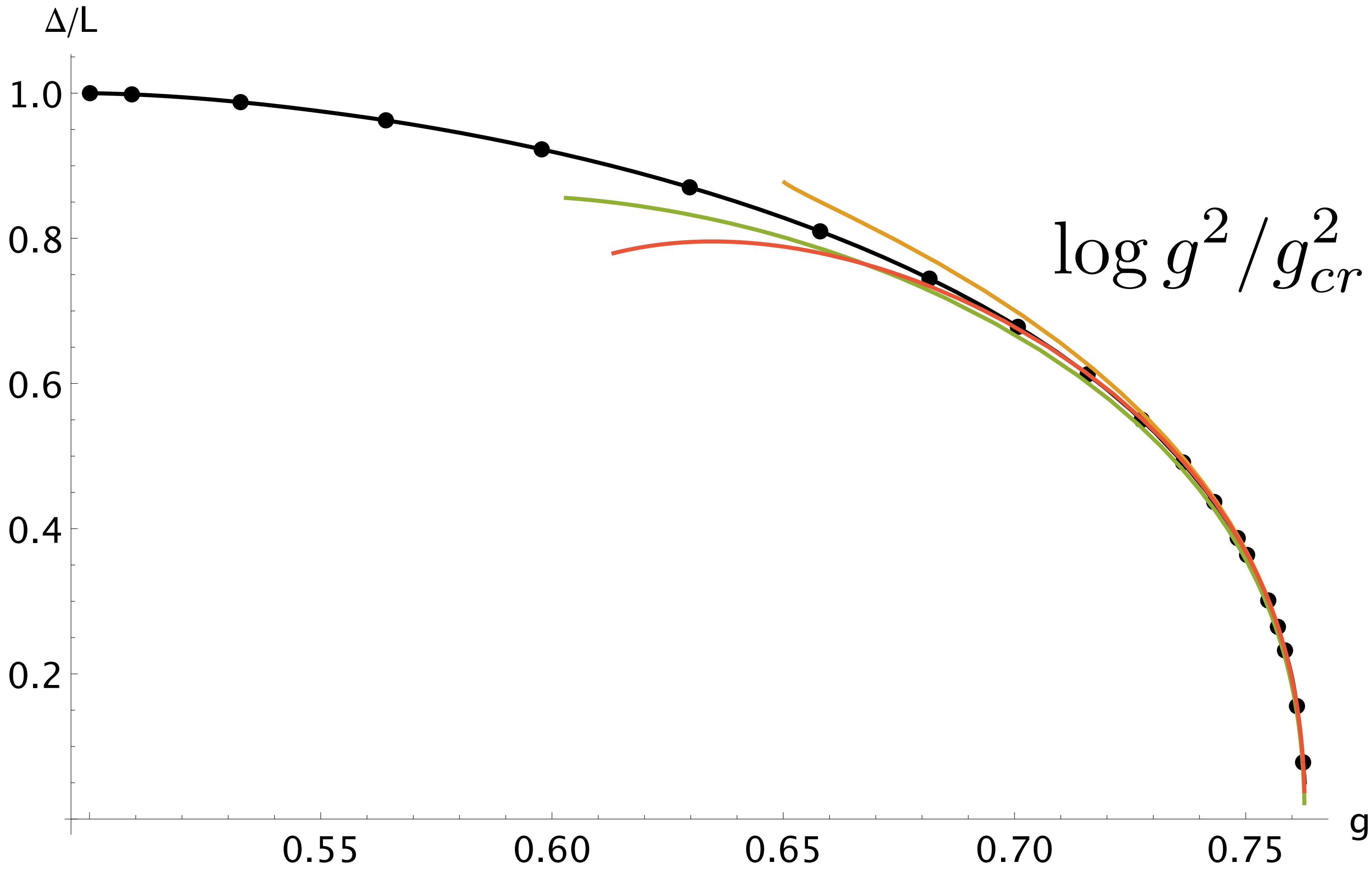
Classical result :
(c-o-m energy)

$$E = -\frac{e^2\Delta^2}{2L}$$

Same as in O(d+2) model if not for the sign of the coupling $\quad e^2 \leftrightarrow -e^2$

Back to numerics

"Quadratic Casimir" scaling near critical point fits numerics



$$\log g^2/g_{cr}^2 = -\frac{e^2\Delta^2}{2L^2}$$

$g_{\rm thr} = 0.5$

$g_{\rm cr} = 0.76276$

thermodynamical window

Back to equation

Compact case

$$\chi(\theta) = m\cosh\theta + \int_{-B}^{B} \frac{d\theta'}{2\pi} K_{O(6)}(\theta-\theta') * \chi(\theta')$$

Eq. describes gas of *massive* excitations at finite particle density ρ

Systematic expansion is known in the large density regime $B \gg 1$ [Volin'09]

Non-compact case

$$\chi(\theta) = \frac{m}{2} e^{-|\theta|} + \int_{(\theta')^2 > B^2} \frac{d\theta'}{2\pi} K_{O(6)}(\theta-\theta') * \chi(\theta')$$

Eq. describes gas of *massless* excitations at finite charge density ρ

Despite being different, two problems are identical, to any order in $1/B$

if we flip the sign of Fermi rapidity $B \to -B$

All-order result

Fermi parameter $B \sim 1/e^2 \sim \log(L/\Delta) \gg 1$

e^2 = coupling of sigma model at energy scale $\sim \rho = \Delta/L$

Flipping the sign of the coupling $e^2 \leftrightarrow -e^2$

Same as changing the sign of curvature

non-compact		compact
$\Delta \ll L$	$\leftrightarrow$	$\Delta \gg L$

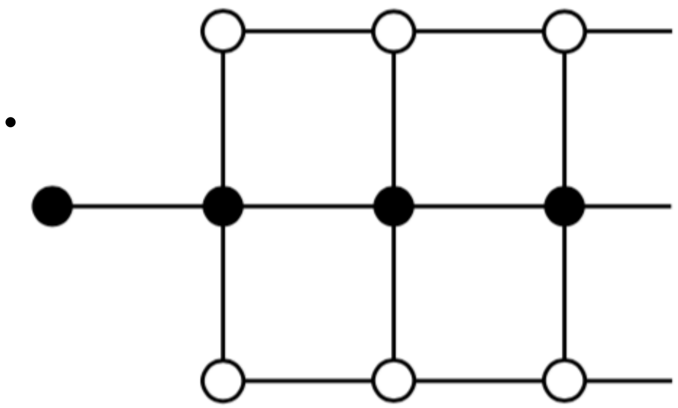
Same as changing "direction of RG flow"

Owing to this connection to the compact sigma model, the dual integral equation describes the tachyon of the AdS model to *all* orders in perturbation theory

TBA equations

Original magnon TBA (massive + chemical potential)

$$\log Y_1 = L \log g^2 - L\epsilon + \mathcal{K} * \log(1+Y_1) + \dots$$



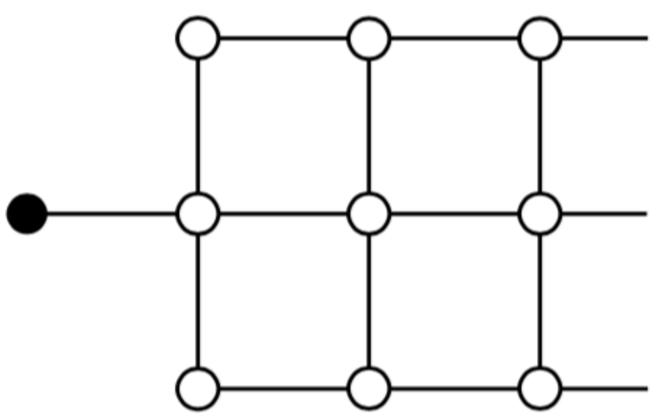
massive TBA

coupling = input

(all blacks nodes contribute to energy)

output :

$$\Delta = L - \sum_{a=1}^{\infty} \int \frac{du}{\pi} \log(1+Y_a)$$



massless TBA

(only 1 momentum carrier)

Dual TBA (massless + no chemical potential)

$$\log Y_1 = LE - K_{O(6)} * \log(1+1/Y_1) + \dots$$

Δ = input (tachyon rep)

coupling = output (sigma model energy)

$$\log g^{2L}/g_{cr}^{2L} = -\int \frac{d\theta}{2\pi} P'(\theta) \log(1+1/Y_1)$$

Central charge

Analysis in CFT limit (1/L effect aka Casimir energy)

[Zamolodchikov'90s]
[Klassen-Melzer'90s]

Split into left and right movers = scale-invariant (kink) solution

Kink is characterized by its asymptotic values on far left and far right

Standard analysis gives TBA central charge $c = c_0 - c_\infty$

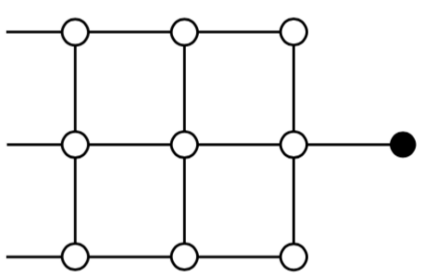
where $c_\star = \sum_i \mathcal{L}(\frac{Y_i^\star}{1+Y_i^\star})$

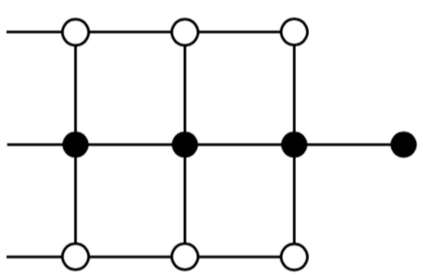
with Rogers dilogarithm $\mathcal{L}(x) = \frac{6}{\pi^2}(\mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x))$

Stationary solutions to Y-system are known

$c_0 = 7$

symmetric phase
O(6)





$c_\infty = 2$

broken phase
O(4)

Central charge is as it should be $c = 5$

CFT analysis

Central charge agrees with perturbative analysis in the AdS sigma model = count of the number of Goldstone bosons

Close to IR fixed point, i.e. large L, the 2d CFT gives information about the behaviour of the energy level

Operator-state correspondence: energy maps to 2d anomalous dimension of vertex operator

$$V_\Delta \sim e^{-i\Delta t} \qquad E_{2d} = -\frac{\pi c_{eff}(L)}{6L} - \frac{e^2 \Delta(\Delta - d)}{2L} + O(e^4)$$

Effective central charge at distance L: $c_{eff}(L) = 5 + O(e^2)$

Running coupling at distance L: $e^2 \sim \dfrac{2\pi}{d \log L}$

Conclusion

Conjecture : 4d planar fishnet graphs define an integrable lattice regularization of the 2d AdS5 sigma model

When all characteristic fishnet length scales are large, the sigma model description is weakly coupled

This is applicable at large L and “small” quantum numbers

Outlook

String worldsheet or not?

We found a marginality condition of sort $0 = L\mu + E_{2d}(L)$

with cosmological constant $\mu = \log g_{cr}^2 / g^2$

Non-critical strings with a tunable intercept exist in flat space, at least classically

Could it be that we are dealing with an AdS version of it?

What seems to be clear is that the on-shell condition comes from the sum over the wheels

$$\sum_{T \geqslant 0} (g/g_{cr})^{2LT} e^{-TE_{2d}(L,\Delta)} = \frac{1}{1-(g/g_{cr})^{2L} e^{-E_{2d}(L,\Delta)}}$$

with T acting as a discrete proper time (Schwinger parameter)

THANK YOU!