SCATTERING IN CFT AND REGGE BEHAVIOR FOR SYK-LIKE MODELS

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Outline:

Scattering in CFT using OPE

Kinematics vs Dynamics

anomalous dimensions: $\Delta_\alpha(\ell) = \ell + \gamma_\alpha(\ell) + \tau_0$

Applications: N=4 SYM SYK-like Models

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INTRODUCTION

$$\gamma^*(1) + \gamma^*(3) \to \gamma^*(2) + \gamma^*(4)$$

$$\langle 0|T(\mathcal{J}_1(x_1)\mathcal{J}_2(x_2)\mathcal{J}_4(x_4)\mathcal{J}_3(x_3))|0\rangle = \frac{1}{(x_{12}^2)^{\Delta_1}(x_{34}^2)^{\Delta_3}} F^{(M)}(u,v)$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$$

$$F^{(M)}(u,v) = \sum_\alpha a_\alpha^{(12;34)} G_\alpha^{(M)}(u,v)$$

Minkowski setting: $u \to 0, \quad v \to 1$

$$F^{(M)}(u,v) \sim u^{-\lambda/2}$$

Why and how?

Lorentz Boost

leading to Singular behavior

Kinematics OF Double-Light-Cone Limit

$$\gamma^*(1) + \gamma^*(3) \to \gamma^*(2) + \gamma^*(4)$$

$$\langle 0|T(\mathcal{J}_1(x_1)\mathcal{J}_2(x_2)\mathcal{J}_4(x_4)\mathcal{J}_3(x_3))|0\rangle = \frac{1}{(x_{12}^2)^{\Delta_1}(x_{34}^2)^{\Delta_3}} F^{(M)}(u,v)$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2} \qquad u \to 0, \quad v \to 1$$

"Regge Limit", or, "Double-Light-Cone"

For both Euclidean and Minkowski settings, the limit corresponds to $x_{12}^2 \to 0$ and $x_{34}^2 \to 0$ and $x_i^2 \to 0$, $i = 1, 2, 3, 4$, with other invariants between left- and right-movers fixed:

$$L^2 \simeq x_{13}^2 \simeq x_{23}^2 \simeq x_{24}^2 \simeq x_{14}^2 = O(1)$$

In Euclidean, a single scale, L, corresponding dilatation under $O(5,1)$

$$SO(1,1) \subset SO(5,1)$$

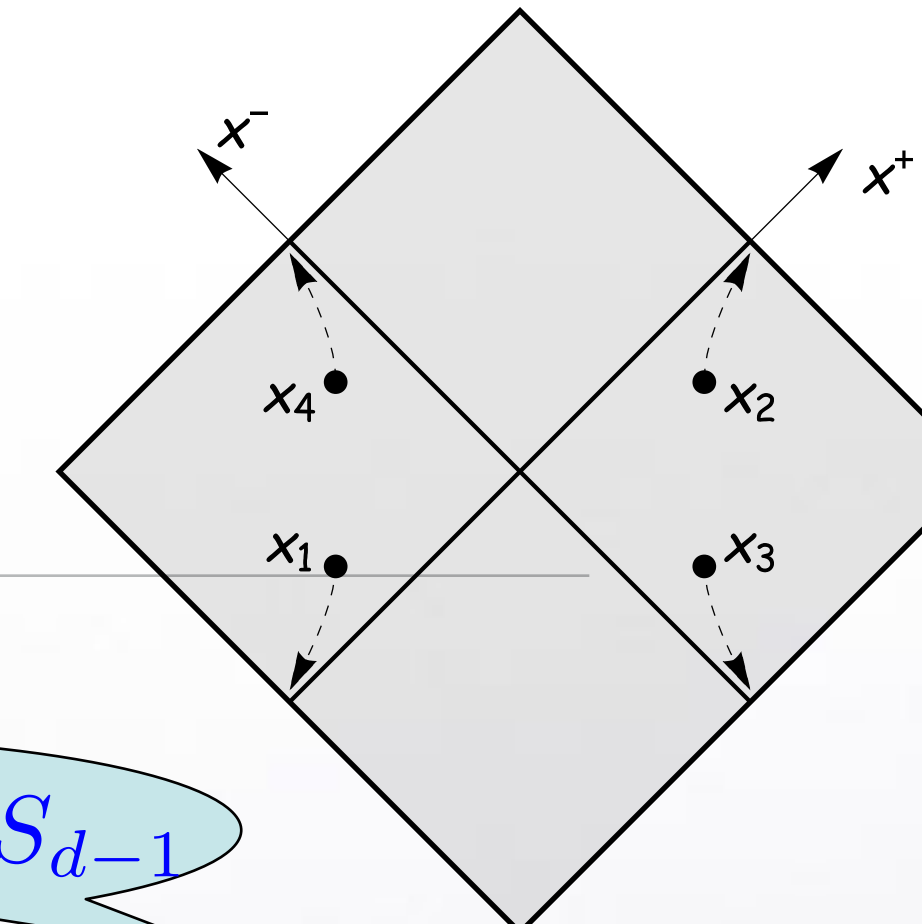
Minkowski: Lorentz Boost, Dilatation and Kinematics of Double Light-Cone Limit:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$$

$$u \to 0, \quad v \to 1$$

$$SO(1,1) \times SO(1,1) \subset SO(4,2)$$

We shall keep all x_i spacelike, $x^2 = -x^+x^- + x_\perp^2 = (-x_0^2 + x_L^2) + x_\perp^2 > 0$



Rindler-like parametrization

Dilatation: $r_i = \sqrt{-x_i^+ x_i^-} = \mu_0\, e^{-\eta_i} > 0$

Boost: $x_i^\pm = \pm\epsilon_i r_i e^{\pm y_i}, \quad i = 1, 2, \quad \epsilon_1 = -1, \epsilon_2 = +$

$x_j^\pm = \mp\epsilon_j r_j e^{\mp y_j}, \quad j = 3, 4, \quad \epsilon_3 = -1, \epsilon_4 = +$

geodesics in AdS_{d-1}

$$u = \frac{16}{(e^{2y} + 2R(1,3) + e^{-2y})^2} \qquad v = \frac{(e^{2y} - 2R(1,3) + e^{-2y})^2}{(e^{2y} + 2R(1,3) + e^{-2y})^2}$$

$$R(i,j) = \frac{r_i^2 + r_j^2 + b_\perp^2}{2r_i r_j}$$

$$w_0^{-1} \equiv \sqrt{u} \simeq \frac{(r_1 + r_2)(r_3 + r_4)}{z_{12} z_{34}}\, e^{-2y}$$

geodesics in AdS_{d-1}

$z_{12} = \sqrt{r_1 r_2}\,, \quad$ *and* $\quad z_{34} = \sqrt{r_3 r_4}.$

$$\sigma_0 \equiv \frac{1 - v + u}{2\sqrt{u}} \simeq \frac{b_\perp^2 + z_{12}^2 + z_{34}^2}{2 z_{12} z_{34}} + O(e^{-2y})$$

$$\sqrt{u}^{-1} \simeq w \Leftrightarrow (z_{12} z_{34} s)/\mu_0^2$$

$$SO(1,1) \times SO(1,1) \subset SO(4,2)$$

$$\gamma^*(1) + \gamma^*(3) \to \gamma^*(2) + \gamma^*(4)$$

$$\langle 0|T(\mathcal{J}_1(x_1)\mathcal{J}_2(x_2)\mathcal{J}_4(x_4)\mathcal{J}_3(x_3))|0\rangle = \frac{1}{(x_{12}^2)^{\Delta_1}(x_{34}^2)^{\Delta_3}} F^{(M)}(u,v)$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$$

t-channel OPE

$$F^{(M)}(u,v) = \sum_\alpha a_\alpha^{(12;34)} G_\alpha^{(M)}(u,v)$$

Minkowski setting: $u \to 0, \quad v \to 1$

Dilatation:

Boost:

HE scattering

$$\Delta_\alpha(\ell) = \ell + \gamma_\alpha(\ell) + \tau_0$$

New Variables:

$$u = x\bar{x}, \quad v = (1-x)(1-\bar{x})$$

$$q \equiv \frac{2-x}{x}, \quad \text{and} \quad \bar{q} \equiv \frac{2-\bar{x}}{\bar{x}}$$

$$w = \sqrt{q\bar{q}} \simeq \sqrt{u}^{-1} \to \infty \qquad \sigma = (\sqrt{q/\bar{q}} + \sqrt{\bar{q}/q})/2 \to \infty$$

Near-Forward Scattering and Boundary Conditions for MCB

$$T(s,t;z_{12},z_{34}) \sim -iw \int d^2\vec{b}\, e^{i\vec{b}\cdot\vec{q}}[e^{i\chi(s,b_\perp,z_{12},z_{34})} - 1]$$

$$T(s,t;z_{12},z_{34}) \simeq w \int d^2\vec{b}\, e^{i\vec{b}\cdot\vec{q}}\chi(s,b_\perp,z_{12},z_{34}) + O(\chi^2).$$

$$\chi(s,b_\perp,z_{12},z_{34}) \leftrightarrow F^{(M)}_{conn}(u,v)$$

Illustration: Contribution from the stress-energy tensor, $\mathcal{T}^{\mu\nu}$, $\Delta = d$ and $\ell = 2$.

Spin factor, s^2, large coupling terms, $\partial_{x_i^-}\partial_{x_j^+}$, $i = 1,2$ and $j = 3,4$

Scalar propagator, $\langle\phi(x)\phi(0)\rangle = 1/(x^2)^\Delta$

$$\chi(s,\vec{b}) \sim s^{\ell-1}\int dx^+dx^-\langle\phi(x)\phi(0)\rangle \sim w^{\ell-1}\,(\frac{b^2}{2z_{12}z_{34}})^{1-\Delta} \sim w^{\ell-1}\,\sigma^{1-\Delta}$$

geodesics in AdS_{d-1}

$$\sqrt{u}^{-1} \simeq w \Leftrightarrow (z_{12}z_{34}s)/\mu_0^2$$

$$\sigma_0 \equiv \frac{1-v+u}{2\sqrt{u}} \simeq \frac{b_\perp^2 + z_{12}^2 + z_{34}^2}{2z_{12}z_{34}} + O(e^{-2y})$$

CFT, OPE, and Regge Limit

Minkowski OPE and Scattering

$$F(w,\sigma) = \sum_\alpha \sum_\ell a^{(12),(34)}_{\ell,\alpha}\, G(w,\sigma;\ell,\Delta_{\ell,\alpha})$$

$$\mathcal{D}\, G_{\Delta,\ell}(u,v) = C_{\Delta,\ell}\, G_{\Delta,\ell}(u,v)$$

$\mathcal{D} = (1-u-v)\partial_v(v\partial_v) + u\partial_u(2u\partial_u - d) - (1+u-v)(u\partial_u + v\partial_v)(u\partial_u + v\partial_v)\,,$
with $\Delta_{12}=0$, $\Delta_{34}=0$, and $\Delta_{ij} = \Delta_i - \Delta_j$

$$C_{\Delta,\ell} = \Delta(\Delta-d)/2 + \ell(\ell+d-2)/2$$

$$C_{\Delta,\ell} = (\widetilde{\Delta}^2 + \widetilde{\ell}^2)/2 - (\epsilon^2+\epsilon+1/2) \qquad \epsilon = (d-2)/2$$

$$\widetilde{\Delta} = \Delta - (\epsilon+1)\,,\ \widetilde{\ell} = \ell+\epsilon$$

$$C_{\Delta,\ell} = \lambda_+(\lambda_+ - 1) + (\lambda_-(\lambda_- - 1) + 2\epsilon\lambda_-) \qquad \lambda_\pm = (\Delta \pm \ell)/2$$

Unitary Representation of $O(5,1)$

$$F(w,\sigma) = \sum_{\ell} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi}\, a(\ell,\nu)\, \mathcal{G}(\ell,\nu;w,\sigma)$$

$$a(\ell,\nu) = \sum_{\alpha} \frac{r_\alpha(\ell)}{\nu^2 + \widetilde{\Delta}_\alpha(\ell)^2} = \sum_{\alpha} \frac{r_\alpha(\ell)}{2\nu} \Big(\frac{1}{\nu + i\widetilde{\Delta}_\alpha(\ell)} + \frac{1}{\nu - i\widetilde{\Delta}_\alpha(\ell)} \Big)$$

$$\widetilde{\Delta} \equiv i\nu = \Delta - d/2$$

$$F(w,\sigma) = \sum_{\alpha} \sum_{\ell} a^{(12),(34)}_{\ell,\alpha}\, G(w,\sigma;\ell,\Delta_{\ell,\alpha})$$

$\mathcal{G}(\ell,\nu;w,\sigma) = \mathcal{G}^{(+)}(\ell,\nu;w,\sigma) + \mathcal{G}^{(-)}(\ell,\nu;w,\sigma)$, where $\mathcal{G}^{(+)}(\ell,\nu;w,\sigma) = \mathcal{G}^{(-)}(\ell,-\nu;w,\sigma)$, with $\mathcal{G}^{(+)}$ leading to convergence in the lower ν-plane and $\mathcal{G}^{(-)}$ in the upper ν-plane

Euclidean CFT

$$SO(5,1) = SO(1,1) \times SO(4)$$

$$\mathcal{A}(u,v) \leftrightarrow \int_{d/2-i\infty}^{d/2+i\infty} \frac{d\Delta}{2\pi i} \sum_j \quad a_j(\Delta)\, G_{\Delta,j}(u,v)$$

Minkowski CFT:

$$SO(4,2) = SO(1,1) \times SO(3,1)$$

Unitary Representation of $O(4,2)$

$$\mathcal{A}(u,v) = \int_{d/2-i\infty}^{d/2+i\infty} \frac{d\Delta}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{d\ell}{2\pi i}\, a(\Delta,\ell)\, \mathcal{G}(u,v,\Delta,\ell)$$

Conformal Regge theory $\Leftrightarrow$ meromorphic representation in the $\nu - \ell$ plane

$$a(\ell,\nu) = \sum_\alpha \frac{r_\alpha(\ell)}{\nu^2 + \tilde{\Delta}_\alpha(\ell)^2} = \sum_\alpha \frac{r_\alpha(\ell)}{2\nu}\Big(\frac{1}{\nu + i\tilde{\Delta}_\alpha(\ell)} + \frac{1}{\nu - i\tilde{\Delta}_\alpha(\ell)}\Big)$$

Minkowski OPE and Scattering

$$F(w,\sigma) = \sum_\alpha \sum_\ell a^{(12),(34)}_{\ell,\alpha}\, G(w,\sigma;\ell,\Delta_{\ell,\alpha})$$

Sommerfeld-Watson Transform:

$$\sum_{\ell=2n} \to \sum_{\ell=2n<L_0} - \int_{L_0-i\infty}^{L_0+i\infty} \frac{d\ell}{2i} \frac{1-e^{i\pi(1-\ell)}}{\sin\pi\ell}$$

$$F(w,\sigma) = F_{Regge}(w,\sigma) - \int_{-i\infty}^{i\infty} \frac{d\tilde{\ell}}{2i} \frac{1+e^{-i\pi\ell}}{\sin\pi\ell} \int_{-i\infty}^{i\infty} \frac{d\widetilde{\Delta}}{2\pi i}\, a(\ell,\nu)\, \mathcal{G}(\ell,\nu;w,\sigma)$$

$$Im\, F(w,\sigma) = \pm \sum_\alpha \int_{L_0-i\infty}^{L_0+i\infty} \frac{d\ell}{2i} a^{(12),(34)}(\ell,\Delta_\alpha(\ell)) G(w,\sigma;\ell,\Delta_\alpha(\ell))$$

$F_{Regge}(w, \sigma)$ has two types of contributions:

$ImF(w, \sigma)$ corresponds to "Double Commutator": $\langle 0|[R(2), R(1)][L(4)L(3)]\rangle|($

$a(\ell, \nu) = \text{inverse transform}: \quad \int_1^\infty d\mu(\sigma) \int_1^\infty d\mu(w)\, \mathrm{Im}F(w, \sigma)\, \tilde{\mathcal{G}}(\ell, \nu, w, \sigma)$

Dynamics

$$a_j(\Delta) \sim \frac{1}{\Delta - \Delta_j} \quad \to \quad \frac{1}{\Delta - \Delta(j)}$$

Single Trace Gauge Invariant Operators of $\mathcal{N} = 4$ SYM,

$$Tr[F^2], \quad Tr[F_{\mu\rho}F_{\rho\nu}], \quad Tr[F_{\mu\rho}D_{\pm}^{S}F_{\rho\nu}], \quad Tr[Z^{\tau}], \quad Tr[D_{\pm}^{S}Z^{\tau}], \cdots$$

Super-gravity in the $\lambda \to \infty$:

$$Tr[F^2] \leftrightarrow \phi, \quad Tr[F_{\mu\rho}F_{\rho\nu}] \leftrightarrow G_{\mu\nu}, \quad \cdots$$

Symmetry of Spectral Curve:

$$\Delta(j) \leftrightarrow 4 - \Delta(j)$$

Minkowski Conformal Blocks

Quadratic Cassimir: $$\mathcal{D}\, G_{\Delta,\ell}(u,v) = C_{\Delta,\ell}\, G_{\Delta,\ell}(u,v)$$

$\mathcal{D} = (1-u-v)\partial_v(v\partial_v) + u\partial_u(2u\partial_u - d) - (1+u-v)(u\partial_u + v\partial_v)(u\partial_u + v\partial_v)\,,$
with $\Delta_{12} = 0$, $\Delta_{34} = 0$, and $\Delta_{ij} = \Delta_i - \Delta_j$

$$C_{\Delta,\ell} = \Delta(\Delta - d)/2 + \ell(\ell + d - 2)/2$$

$$C_{\Delta,\ell} = (\widetilde{\Delta}^2 + \widetilde{\ell}^2)/2 - (\epsilon^2 + \epsilon + 1/2)$$

$\widetilde{\Delta} = \Delta - (\epsilon + 1)\,,\ \widetilde{\ell} = \ell + \epsilon \qquad \epsilon = (d-2)/2$

$$C_{\Delta,\ell} = \lambda_+(\lambda_+ - 1) + (\lambda_-(\lambda_- - 1) + 2\epsilon\lambda_-)$$

$\lambda_\pm = (\Delta \pm \ell)/2$

$$F^{(M)}(u,v) = \sum_\alpha a_\alpha^{(12;34)} G_\alpha^{(M)}(u,v)$$

$$u \to 0, \quad v \to 1$$

$$Dilatation: \qquad \sigma \to \infty, \qquad \sigma_0 \simeq \frac{1-v+u}{2\sqrt{u}} \simeq \frac{1-v}{2\sqrt{u}}$$

$$Boost: \qquad w \to \infty. \qquad w \sim e^{2y} \sim \sqrt{u}^{-1}.$$

$$G^{(M)}_{(\Delta,\ell)}(u,v) \sim \sqrt{u}^{\Delta-\ell}\,(1-v)^{1-\Delta} = \sqrt{u}^{1-\ell}\left(\frac{1-v}{\sqrt{u}}\right)^{1-\Delta}.$$

$$SO(1,1)\times SO(1,1) \subset SO(4,2)$$

$$G^{(E)}_{(\Delta,\ell)}(u,v) \sim \sqrt{u}^{\Delta-\ell}\,(1-v)^{\ell} = \sqrt{u}^{\Delta}\left(\frac{1-v}{\sqrt{u}}\right)^{\ell}$$

$$SO(1,1) \subset SO(5,1)$$

DIS: $\quad M_\ell = \int_0^1 x^{\ell-2} F_2(x,Q^2)\,dx \sim Q^{-\gamma_\ell}$

$F_2(x,Q^2) \sim x^{1-\ell_{eff}} \qquad 1 < \ell_{eff} \leq 2$

SYK: $\quad \langle \chi_R^\dagger(t)\chi_L^\dagger(0)\chi_L(t)\chi_R(0)\rangle_\beta \sim e^{\lambda t}$

Lyapunov exponent - $0 < \lambda \leq 1$

Explicit Construction of MCB:

$$\widetilde{k}_{2\lambda}(q) = q^{-\lambda}\ {}_2F_1(\lambda/2+1/2, \lambda/2; \lambda+3/2; q^{-2}) = 2^{\lambda}\frac{\Gamma(\lambda+1/2)}{\pi^{1/2}\Gamma(\lambda)} Q_{\lambda-1}(q)$$

$d=2$: $$G^{M}_{(\Delta,\ell)}(u,v) = \widetilde{k}_{2(1-\lambda_+)}(q_<)\,\widetilde{k}_{2\lambda_-}(q_>) = \frac{\Gamma(3/2-\lambda_+)\Gamma(\lambda_-+1/2)}{2^{\ell-1}\Gamma(1-\lambda_+)\Gamma(\lambda_-)} Q_{-\lambda_+}(q_<)\, Q_{\lambda_--1}(q_>)$$

$d=4$: $$G^{(M)}_{(\Delta,\ell)}(u,v) = 2^{2-\ell}\frac{\Gamma(3/2-\lambda_+)\Gamma(\lambda_--1/2)}{\Gamma(1-\lambda_+)\Gamma(\lambda_--1)}\,\mathrm{sgn}(\bar{q}-q)\left(\frac{1}{q_>-q_<}\right) Q_{-\lambda_+}(q_<)\, Q_{\lambda_--2}(q_>)$$

$$d=2: \quad G^{(E)}_{(\Delta,\ell)}(u,v) = \widetilde{k}_{2\lambda_+}(q)\widetilde{k}_{2\lambda_-}(\bar{q}) + \widetilde{k}_{2\lambda_+}(\bar{q})\widetilde{k}_{2\lambda_-}(q)$$

$$d=4: \quad G^{(E)}_{(\Delta,\ell)}(u,v) = \frac{1}{q-\bar{q}}\Big(\widetilde{k}_{2\lambda_+}(q)\widetilde{k}_{2(\lambda_--1)}(\bar{q}) - \widetilde{k}_{2\lambda_+}(\bar{q})\widetilde{k}_{2(\lambda_--1)}(q)\Big).$$

Crossing

$$(u,v) \rightarrow (u/v, 1/v), \quad \text{same as} \quad (q,\bar{q}) \rightarrow (-q,-\bar{q}),$$

$$G^{(M)}_{(\Delta,\ell)}(u,v) = (-1)^{1-\ell} G^{M}_{(\Delta,\ell)}(u/v, 1/v)$$

Symmetric Treatment

$$\mathcal{D}\, G_{\Delta,\ell}(u,v) = C_{\Delta,\ell}\, G_{\Delta,\ell}(u,v)$$

Adopt (w,σ) as two independent variables with the physical region specified by $1 < w < \infty$ and $1 < \sigma < \infty$

Boundary Conditions at: $w = \sqrt{q\bar{q}} \simeq \sqrt{u}^{-1} \to \infty$ $\quad \sigma = (\sqrt{q/\bar{q}} + \sqrt{\bar{q}/q})/2 \to \infty$

$$\mathcal{D} = (\mathcal{L}_{0,w} + \mathcal{L}_{0,\sigma} + w^{-2}\mathcal{L}_2)/2$$

where $\mathcal{L}_{0,w}(w\partial_w)$, $\mathcal{L}_{0,\sigma}(\partial_\sigma,\sigma)$ and $\mathcal{L}_2(w\partial_w,\partial_\sigma,\sigma)$ are homogeneous in w

$$G^{(M)}_{(\Delta,\ell)}(u,v) = w^s\Big(g_0(\sigma) + w^{-2}g_1(\sigma) + w^{-4}g_2(\sigma) + \cdots\Big) = \sum_{n=0}^{\infty} w^{s-2n} g_n(\sigma).$$

Indicial equqtion leads to $\quad s = \ell - 1$

Leading Order:

$$G^{(M)}_{(\Delta,\ell)}(w,\sigma) = w^{\ell-1} g_0(\sigma,\Delta,d)$$

$$\mathcal{L}_{0,\sigma} = (\sigma^2-1)\partial_\sigma^2 + (d-1)\sigma\partial_\sigma$$

$$\Big(\mathcal{L}_{0,\sigma} - (\Delta-1)(\Delta-d+1)\Big)\, g_0(\sigma) = 0$$

Again, differential equation for associated Legendre functions, with solution $Q^\mu_\nu(\sigma)$, vanishing at $\sigma \to \infty$

$$d=4 \quad g_0(\sigma;\Delta,4) = \frac{e^{-(\Delta-2)\xi}}{\sinh\xi}$$

$$d=3: \quad g_0(\sigma;\Delta,3) = Q_{\Delta-2}(\sigma)$$

$$d=2: \quad g_0(\sigma;\Delta,2) = e^{-(\Delta-1)\xi}$$

$$d=1: \quad g_0(\sigma;\Delta,1) = \sinh\sigma\, Q^{(-1)}_{\Delta-1}(\sigma) = \frac{dQ_{\Delta-1}(\sigma)}{d\xi}$$

$\sigma = \cosh\xi$, and ξ is the geodesics in AdS_{d-1}

$g_0(\sigma;\Delta,d)$ corresponds precisely to a scalar, Euclidean bulk-to-bulk propagator in AdS_{d-1}, or more precisely H_{d-1}, with conformal dimension $\Delta-1$

Higher Order Expansion:

$$[-\mathcal{L}_{0,\sigma} + m^2(\ell,\Delta)]g_n(\sigma) = J_n(\sigma)$$

$$G^{(M)}_{(\Delta,\ell)}(-w,\sigma) = (-)^{\ell-1} G^{(M)}_{(\Delta,\ell)}(w,\sigma)$$

The Case of $d = 1$

There exists a kinematical relation $\sqrt{v} = 1 + \sqrt{u}$ for $d = 1$.
In terms of q and $\bar{q}$, it leads to $q = \bar{q}$ and $\sigma = \cosh\xi = 1.$

$$G^{(M)}_{\Delta=0,\ell}(w, \sigma = 1) = w^{\ell-1} \sum_{n=0}^{\infty} g_n(\sigma = 1) w^{-2n}$$

$$[(w^2 - 1)\frac{d^2}{dw^2} + 2w\frac{d}{dw}]\, G^{(M)}_{\ell}(w) = \ell(\ell - 1) G^{(M)}_{\ell}(w)$$

$$G^{(M)}_{\ell}(w) = 2^{1-\ell} \frac{\Gamma(3/2 - \ell)}{\pi^{1/2}\Gamma(1 - \ell)}\, Q_{-\ell}(w) \equiv c_\ell\, Q_{-\ell}(w) \equiv \bar{Q}_{-\ell}(w)$$

$G^{(M)}_{\ell}(w) \simeq w^{\ell-1}$, with unit coefficient as $w \to \infty$

Kinematics of Scattering for CFT at $d=1$:

$$t_1 + t_3 \to t_2 + t_4$$

$$\tau = \frac{t_{21}t_{43}}{t_{23}t_{41}} \quad \text{or} \quad \tau_c = \frac{t_{13}t_{42}}{t_{23}t_{41}} \qquad |\tau_c| + |\tau| = 1$$

$$w \equiv (2-\tau)/\tau, \qquad 1 < w < \infty$$

$$\Gamma(w) = \sum_{0<\ell<L_0,\,\text{even}} a(\ell)\,\bar{Q}_{-\ell}(w) - \int_{L_0-i\infty}^{L_0+i\infty} \frac{d\ell}{2i}\,\frac{1+e^{-i\pi\ell}}{\sin\pi\ell}\, a(\ell)\,\bar{Q}_{-\ell}(w)$$

$$\mathrm{Im}\,\Gamma(w) = -\int_{L_0-i\infty}^{L_0+i\infty} \frac{d\ell}{2i}\, a(\ell)\,\bar{Q}_{-\ell}(w) = -\int_{L_0-i\infty}^{L_0+i\infty} \frac{d\ell}{2\pi i}\,(2\ell-1)\,A(\ell)\,\frac{\tan\ell\pi}{\pi}\,Q_{-\ell}(w)$$

$$A(\ell) = \int_1^\infty dw\,\mathrm{Im}\,\Gamma(w)\,Q_{\ell-1}(w)$$

pole of $A(\ell)$, e.g., $A(\ell) \sim \frac{1}{\ell-\ell^*}$ $\Leftrightarrow$ power growth i.e., $\mathrm{Im}\Gamma(w) \sim w^{\ell^*-1}$

1-d Scattering: Hilbert Space Treatment

$$\langle f|g\rangle = \int_1^\infty dw\, f(w)^* g(w)$$

$$D_{w,0}P(w) = \left[-\frac{d}{dw}(w^2-1)\frac{d}{dw}\right]P(w) = \lambda P(w),$$

$$\lambda = k^2 + 1/4$$

$$\int_1^\infty dw\, P_{-1/2-ik}(w) P_{-1/2+ik'}(w) = \frac{1}{k\tanh \pi k}\delta(k-k'),$$

$$\int_0^\infty dk\, k \tanh \pi k\, P_{-1/2-ik}(w) P_{-1/2+ik}(w') = \delta(w-w').$$

$$F(w) = \int_{-\infty}^\infty \frac{dk}{2\pi}\; k\, f(k)\; P_{-1/2+ik}(w)$$

$$f(k) = \pi \tanh k \int_1^\infty dw\;\; F(w)\, P_{-1/2+ik}(w)$$

$$[-\frac{d}{dw}(w^2-1)\frac{d}{dw} + m^2]G(w,w') = \delta(w-w'),$$

$$G(w,w') = \frac{1}{2}\int_{-\infty}^\infty dk\;\; k\;\tanh \pi k \frac{P_{-1/2+ik}(w) P_{-1/2+ik}(w')}{k^2+1/4+m^2}$$

$$\frac{\pi\, P_\ell(z)}{\tan \ell\pi} = Q_\ell(z) - Q_{-\ell-1}(z),$$

$$ik = \tilde{\ell} = \ell - 1/2$$

Minkowski CFT in $d = 1$ and Scattering for SYK-Like Models

$$\Gamma(w) \simeq w^{\ell^* - 1}$$

$$\Gamma = \Gamma_1 + K_0 \otimes \Gamma \qquad \Gamma_n = K_0 \otimes \Gamma_{n-1},$$

$$\Gamma = \sum_{n=1}^{\infty} \Gamma_n,$$

$$Im\,\Gamma = \mathrm{Im}\,\Gamma_1 + \widetilde{K}_0 \otimes' Im\,\Gamma \qquad (\mathrm{Im}\,\Gamma)_n = \widetilde{K}_0 \otimes' (\mathrm{Im}\,\Gamma)_{n-1}.$$

$$\mathrm{Im}\,\Gamma = \sum_{n=1}^{\infty} (\mathrm{Im}\,\Gamma)_n,$$

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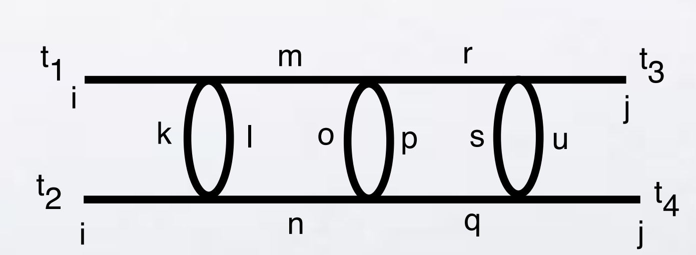

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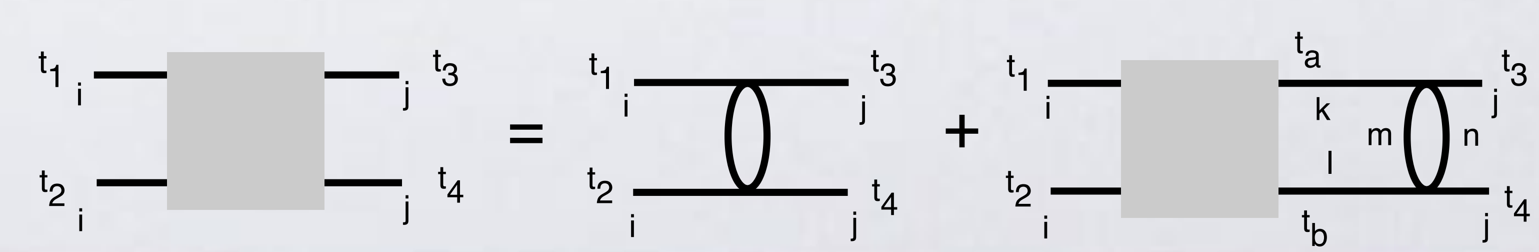


Scattering for SYK-Like Models

$$\Gamma(t_2,t_1;t_4,t_3)_n = \int dt_5 dt_6 K_0(t_2,t_1;t_6,t_5)\ \Gamma(t_6,t_5;t_4,t_3)_{(n-1)}$$

$$K_0 = (1/\alpha_0)(\frac{t_{21}t_{65}}{t_{25}t_{61}}/\frac{t_{15}t_{62}}{t_{25}t_{61}})^{2\delta} = (1/\alpha_0)\left(\frac{\tau_k}{1-\tau_k}\right)^{2\delta}$$

$$w = \frac{2-\tau}{\tau} = \frac{t_{12}^2+t_{34}^2-(\bar t_{12}-\bar t_{34})^2}{2t_{12}t_{34}}$$
$$w_k = \frac{2-\tau_k}{\tau_k} = \frac{t_{12}^2+t_{56}^2-(\bar t_{12}-\bar t_{56})^2}{2t_{12}t_{56}}$$
$$w' = \frac{2-\tau'}{\tau'} = \frac{t_{56}^2+t_{34}^2-(\bar t_{56}-\bar t_{34})^2}{2t_{56}t_{34}}$$
where $\bar t_{ij} = (t_i + t_j)$.

$$\Gamma(w)_n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\,w_k d\,w' \frac{1}{\sqrt{D}} K_0(w_k)\,\Gamma(w')_{n-1}$$

$$D(x,y,z) = x^2+y^2+z^2-1-2xyz$$

$$\mathrm{Im}\Gamma(w)_n = \int_1^w\int_1^w d\,w_k d\,w' \frac{\theta^+(D)}{\sqrt{D}}\,\widetilde K_0(w_k)\,\mathrm{Im}\Gamma(w')_{n-1}$$

$$\widetilde K_0(w_k) = \frac{2^{1+\delta}(1-\delta)(1-2\delta)}{\Gamma(1+2\delta)\Gamma(1-2\delta)}(w_k-1)^{-2\delta}\theta(w_k-1)$$

Diagonalization:

$$\mathrm{Im}\,\Gamma(w) = \mathrm{Im}\,\Gamma_1(w) + \int_1^w \int_1^w d\,w_k d\,w' \frac{\theta^+(D)}{\sqrt{D}}\,\widetilde{K}_0(w_k)\,\mathrm{Im}\Gamma(w')$$

$$A(\ell) = \int_1^\infty dw\,Q_{\ell-1}(w)\,\mathrm{Im}\,\Gamma(w) \qquad A_1(\ell) = \int_1^\infty dw\,Q_{\ell-1}(w)\,\mathrm{Im}\,\Gamma_1(w) \qquad k(\ell) = \int_1^\infty dw\,Q_{\ell-1}(w)\,\widetilde{K}_0(w)$$

$$\int_1^\infty \frac{d\,w}{\sqrt{D(w, w_k, w')}}\theta^{(+)}(D) Q_\ell(w) = Q_\ell(w_k) Q_\ell(w')$$

$$A(\ell) = A_1(\ell) + k(\ell, \delta)\,A(\ell).$$

$$A(\ell) = \frac{A_1(\ell)}{1 - k(\ell)}$$

Identifying the Leading Intercept ℓ^* for SYK-like Models:

$$k(\ell,\delta) = \frac{2^{2\delta+1}(1-\delta)(1-2\delta)}{\Gamma(1+2\delta)\Gamma(1-2\delta)} \int_1^\infty dw\,(w-1)^{-2\delta}\,Q_{\ell-1}(w)$$

$$k(\ell,\delta) = \frac{\Gamma(3-2\delta)}{\Gamma(1+2\delta)} \frac{\Gamma(\ell+2\delta-1)}{\Gamma(\ell-2\delta+1)}$$

$$k(\ell^*) = 1$$

$$\ell^* = 2$$

$$\delta = 1/q = 1/4$$

$$A(\ell) = \frac{3(\ell-1/2)^2}{\ell-2}$$

$$\mathrm{Im}\,\Gamma(w) \to \gamma'\,w + O(w^{1-2\delta})$$

$$\Gamma(w) \simeq -\pi^{-1}\gamma' w[\log(1-w)+\log(w-1)] + \gamma'' w + O(w^{1-2\delta}),$$

Out-of-time-order Thermal 4-point

$$\langle \chi_R^\dagger(t)\chi_L^\dagger(0)\chi_L(t)\chi_R(0)\rangle_\beta \sim e^{\kappa\, t}$$

Lyapunov exponent – κ

$$t_1 \to \phi(t_1) = e^{(2\pi/\beta)i\epsilon}, \quad t_3 \to \phi(t_3) = e^{(2\pi/\beta)2i\epsilon} e^{2\pi t/\beta},$$
$$t_2 \to \phi(t_2) = e^{(2\pi/\beta)3i\epsilon}, \quad t_4 \to \phi(t_4) = e^{(2\pi/\beta)4i\epsilon} e^{2\pi t/\beta}.$$

boosting $t \to e^{2\pi i t/\beta}$ to finite temperture

Chaos bound: $\kappa = \ell^* - 1 \leq 1$

Application of Minkowski $d > 1$ CFT for Scattering:

Lorentz boost and dilatation consist of $O(1,1) \times O(1,1)$ subgroup of the full conformal transformations, $O(4,2)$.

It has long been known that approximate $O(2,2)$ symmetry is an important feature of QCD near-forward scattering at high energies

Treat the case of deep inelastic scattering (DIS) as a realization of $O(2,2)$ invariance for near forward scattering.

$$\gamma(1) + \text{proton}(2) \to \gamma(3) + \text{proton}(4)$$

$$T^{\mu\nu}(p,q;p',q') = \langle p' | \mathbf{T}\{J^\mu(x) J^\nu(0)\} | p \rangle$$

$$\text{At } t = 0; \quad T^{\mu\nu} = W_1(x,Q^2)\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + W_2(x,Q^2)\left(p_\mu + \frac{q_\mu}{2x}\right)\left(p_\nu + \frac{q_\nu}{2x}\right).$$

$$\text{DIS}: \quad \langle p | [J^\mu(x), J^\nu(0)] | p \rangle = \mathcal{F}_1(x,Q^2)\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \mathcal{F}_2(x,Q^2)\left(p_\mu + \frac{q_\mu}{2x}\right)\left(p_\nu + \frac{q_\nu}{2x}\right)$$

$$\mathcal{F}_\alpha(x,q) = 2\pi \,\mathrm{Im}\, W_\alpha(x,q)$$

$$\mathcal{F}_2(x,q) = (q^2/4\pi^2\alpha_{em})(\sigma_T + \sigma_L)$$

Deep-Inelastic Scattering as Minkowski CFT

Reduction to $d = 2$:
Discontinuity:
Mellon Representation:

$$W_2(w, \sigma_2) = \sum_\alpha \sum_{\ell\ even} a_\alpha(\ell) \mathcal{K}_\alpha(w, \sigma_2; \ell)$$

$$W(w, \sigma_2) = W_0(w, \sigma_2) - \sum_\alpha \int_{L_0 - i\infty}^{L_0 + i\infty} \frac{d\ell}{2\pi i} \frac{1 + e^{-i\pi\ell}}{\sin \pi\ell} a_\alpha^{(12),(34)}(\ell) K_\alpha(w, \sigma_0; \ell)$$

$$Im W(w, \sigma_2) = \sum_\alpha \int_{L_0 - i\infty}^{L_0 + i\infty} \frac{d\ell}{2i} a_\alpha^{(12),(34)}(\ell) K_\alpha(w, \sigma_2; \ell)$$

Dilatation: $\rightarrow \frac{dM(\sigma_2, 2n)}{d \log \sigma_2} \simeq -(\Delta(2n) - 2) A(\sigma_2, 2n) \rightarrow$ **DGLAP**

Lorentz Boost $\rightarrow \mathcal{F}_2(w, \sigma) \simeq w^{\ell_{eff} - 1} \rightarrow$ **Effective Spin**, ℓ_{eff}

Spectral Curve:

$$Im\, F(w,\sigma) = \pm \sum_{\alpha} \int_{L_0 - i\infty}^{L_0 + i\infty} \frac{d\ell}{2i} a^{(12),(34)}(\ell, \Delta_\alpha(\ell)) G(w,\sigma;\ell,\Delta_\alpha(\ell))$$

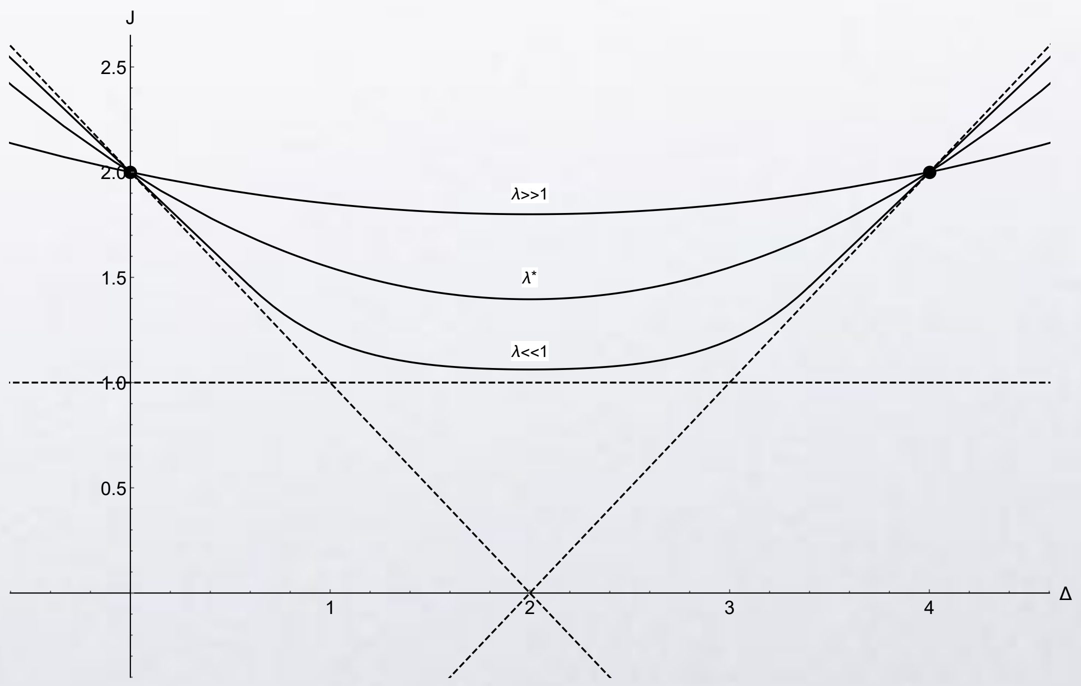
$$\Delta(\ell)(\Delta(\ell) - d) = m^2_{AdS}(\ell)$$

$$d = 4, \quad m^2_{AdS}(\ell) = \sum_{n=1} \beta_n (\ell - 2)^n$$

B.Basso, 1109.3154v2

$$\Delta_P(\ell) \simeq 2 + B(\ell)\sqrt{\ell - \ell_{eff}}$$

$$\mathrm{Im}\, F(w,\sigma) \sim \frac{w^{\ell_{eff}-1}}{|\ln w|^{3/2}}$$



POMERON AND ODDERON IN STRONG COUPLING:

$$\widetilde{\Delta}(S)^2 = \tau^2 + a_1(\tau,\lambda)S + a_2(\tau,\lambda)S^2 + \cdots$$

B.Basso, 1109.3154v2

POMERON

$$\alpha_p = 2 - \frac{2}{\lambda^{1/2}} - \frac{1}{\lambda} + \frac{1}{4\lambda^{3/2}} + \frac{6\zeta(3)+2}{\lambda^2} + \frac{18\zeta(3)+\frac{361}{64}}{\lambda^{5/2}} + \frac{39\zeta(3)+\frac{447}{32}}{\lambda^3} + \cdots$$

Brower, Polchinski, Strassler, Tan

Costa, Goncalves, Penedones (1209.4355)

Kotikov, Lipatov (1301.0882)

Gromov et al.

ODDERON

Kotikov, Lipatov, et al.

Solution-a:

$$\alpha_O = 1 - \frac{8}{\lambda^{1/2}} - \frac{4}{\lambda} + \frac{13}{\lambda^{3/2}} + \frac{96\zeta(3)+41}{\lambda^2} + \frac{288\zeta(3)+\frac{1823}{16}}{\lambda^{5/2}} + \frac{720\zeta(5)+1344\zeta(3)-\frac{3585}{4}}{\lambda^3}$$

Solution-b:

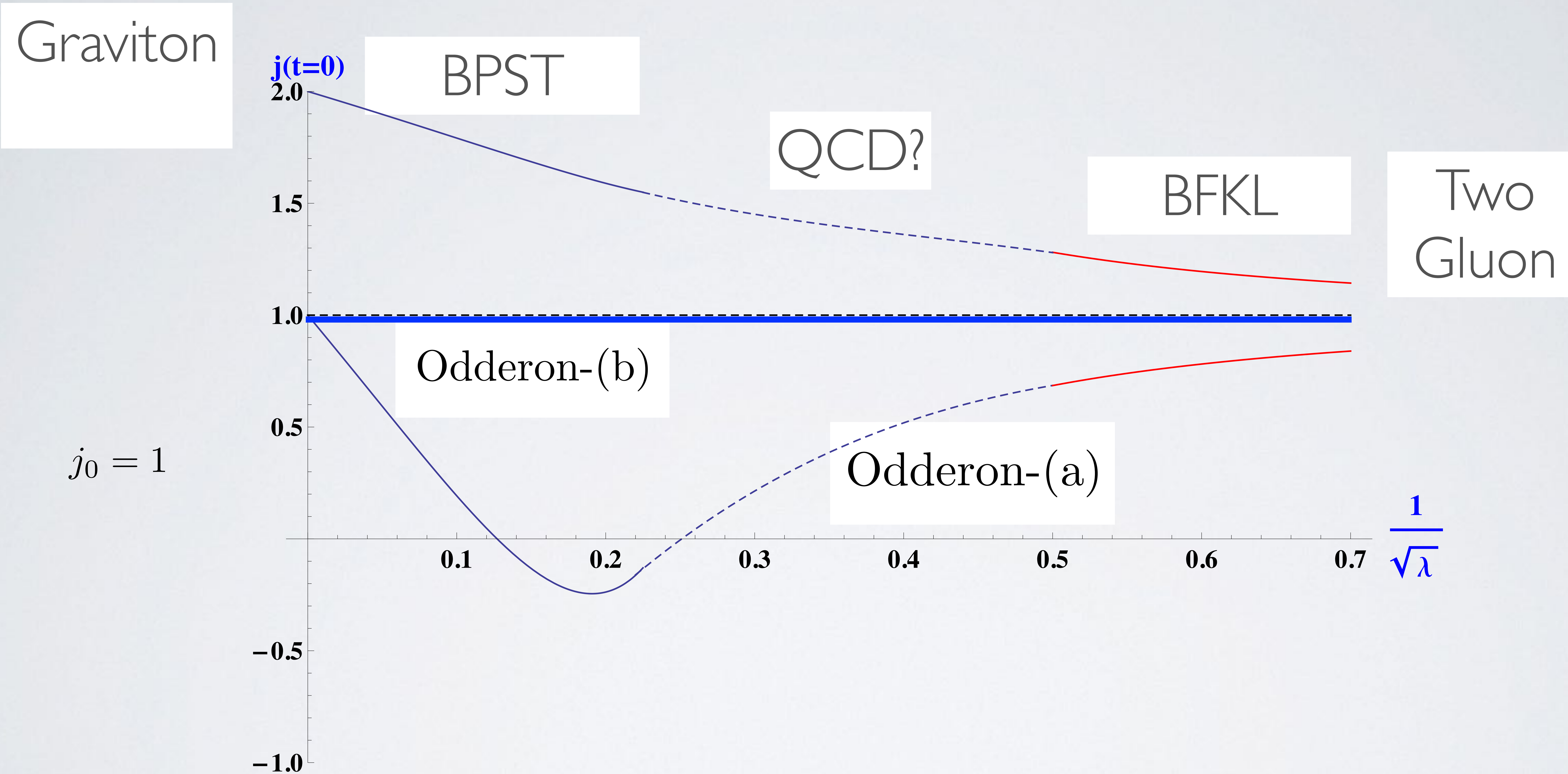
$$\alpha_O = 1 - \frac{0}{\lambda^{1/2}} - \frac{0}{\lambda} + \frac{0}{\lambda^{3/2}} + \frac{0}{\lambda^2} + \frac{0}{\lambda^{5/2}} + \frac{0}{\lambda^3} + \cdots$$

Brower, Djuric, Tan

Avsar, Hatta, Matsuo

Brower, Costa, Djuric, Raben, Tan

$\mathcal{N} = 4$ Strong vs Weak $g^2 N_c$



$j_0 = 1$

Summary and Outlook for Scattering in AdS-CFT

- Provide meaning for Pomeron non-perturbatively from first principles.
- Realization of conformal invariance beyond perturbative QCD
- New starting point for unitarization, saturation, etc.
- First principle description of elastic/total cross sections, DIS at small-x, Central Diffractive Glueball production at LHC, etc.
- HIgher point functions.
- Inclusive Production and Dimensional Scalings.
- "non-perturbative" (e.g., blackhole physics, locality in the bulk).