Structure constants of twist-2 light-ray operators in the triple Regge limit

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Outline

1. Setting the stage: correlator of 3 light-ray operators
   - Three-point correlator of local operators:
   - Light-ray operators as analytic continuation of local operators
   - Three-point correlator of LR operators: what to expect
2. CF of two light-ray operators in the BFKL limit
   - Correlation function of two “Wilson frames” in the BFKL approximation.
   - CF of two light-ray operators.
3. 3-point CF in the BK limit
   - 3-point CF of “Wilson frames” from the BK equation
   - Structure constants at $\omega_1 = \omega_2 + \omega_3$
4. 3-point CF in the triple Regge limit
   - BFKL kernel in a triple Regge limit
   - Structure constants at the “triple BFKL point”
5. Conclusions and outlook

Supermultiplet of twist-2 operators

SU_4 singlet operators. (Korchemsky et al)

$$\tilde{S}^l_{1n}(z) = \tilde{F}^l_n(z) + \frac{l-1}{24}\tilde{\Lambda}^l_n(z) + \frac{l(l-1)}{24}\tilde{\Phi}^l_n(z)$$

$$\tilde{S}^l_{2n}(z) = \tilde{F}^l_n(z) - \frac{1}{24}\tilde{\Lambda}^l_n(z) - \frac{l(l+1)}{72}\tilde{\Phi}^l_n(z)$$

$$\tilde{S}^l_{3n}(z) = \tilde{F}^l_n(z) - \frac{l+2}{12}\tilde{\Lambda}^l_n(z) + \frac{(l+1)(l+2)}{24}\tilde{\Phi}^l_n(z)$$

$$\tilde{F}^l_n(z) \equiv i^{l-2}\mathrm{tr}\, F_{\mu n}\partial_n^{l-2} C^{\frac{5}{2}}_{l-2}\Big(\frac{\overleftarrow{\nabla}_n + \overrightarrow{\nabla}_n}{\partial_n}\Big) F^\mu_{\ n} \; + \; O(g^2)$$

$$\tilde{\Lambda}^l_n(z) \equiv i^{l-1}\mathrm{tr}\, \bar{\lambda}\partial_n^{l-1} C^{\frac{3}{2}}_{l-1}\Big(\frac{\overleftarrow{\nabla}_n + \overrightarrow{\nabla}_n}{\partial_n}\Big)\lambda(z) \; + \; O(g^2)$$

$$\tilde{\Phi}^l_n(z) \equiv i^{l}\mathrm{tr}\, \bar{\phi}^I\partial_n^{l-1} C^{\frac{3}{2}}_{l}\Big(\frac{\overleftarrow{\nabla}_n + \overrightarrow{\nabla}_n}{\partial_n}\Big)\phi^I(z) \; + \; O(g^2)$$

$C^\lambda_l(x)$ - Gegenbauer polynomials, $n^2 = 0$, and $F^\mu_{\ n} \equiv F^{\mu\nu}n_\nu$ etc.

All operators have the same anomalous dimension

$$\gamma^{S_1}_l(\alpha_s) \equiv \gamma_l(\alpha_s) = \frac{2\alpha_s}{\pi}N_c[\psi(l-1) + C] + O(\alpha_s^2), \qquad \gamma^{S_2}_l = \gamma^{S_1}_{l+1}, \qquad \gamma^{S_3}_l = \gamma^{S_1}_{l+2}$$

3-point CF of local operators

Rychkov et al: CF of 3 operators with spin $(n_1^2 = n_2^2 = n_3^2 = 0)$

$$< \mathcal{O}_{n_1}^{l_1}(x)\mathcal{O}_{n_2}^{l_2}(y)\mathcal{O}_{n_3}^{l_3}(z) >= \sum_{m_{12},m_{13},m_{23}\geq 0} \lambda_{m_{12},m_{23},m_{13}} \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix}$$

The sum runs over

$$m_1 = l_1 - m_{12} - m_{13} \geq 0\,, \quad m_2 = l_2 - m_{12} - m_{23} \geq 0\,, \quad m_3 = l_3 - m_{13} - m_{23} \geq 0$$

where Δ_i is dimension and l_i is Lorentz spin.

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ m_{23} & m_{13} & m_{12} \end{bmatrix} = \frac{(V_{1,23})^{l_1-m_{12}-m_{13}}(V_{2,31})^{l_2-m_{12}-m_{23}}(V_{3,12})^{l_3-m_{13}-m_{23}}(H_{12})^{m_{12}}(H_{13})^{m_{13}}(H_{23})^{m_{23}}}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}|x-z|^{\Delta_1+\Delta_3-\Delta_2}|y-z|^{\Delta_2+\Delta_3-\Delta_1}}$$

V and H - some tensor structures

3s

If we define "forward" operators

$$\begin{aligned}
\Phi_n^l(x_\perp) &= \int du\ \bar{\phi}^a_{AB}\nabla_n^l\phi^{ABa}(un+x_\perp),\\
\Lambda_n^l(x_\perp) &= \int du\ i\bar{\lambda}^a_A\nabla_n^{l-1}\sigma_n\lambda^a_A(un+x_\perp)\\
F^l(x_\perp) &= \int du\ F^a_{ni}\nabla_n^{l-2}F^{ai}_n(un+x_\perp),
\end{aligned}$$

the renorm-invariant operators are

$$\begin{aligned}
S^l_{1n} &= F^l_n + \frac{1}{4}\Lambda^l_n - \frac{1}{2}\Phi^l_n, \quad S^l_{2n} = F^l_n - \frac{1}{4(l-1)}\Lambda^l_n + \frac{(l+1)}{6(l-1)}\Phi^l_n\\
S^l_{3n} &= F^l_n - \frac{l+2}{2(l-1)}\Lambda^l_n - \frac{(l+1)(l+2)}{2l(l-1)}\Phi^l_n
\end{aligned}$$

and Rychkov's structures reduce to one ($x_\perp\cdot n_1 = x_\perp\cdot n_2 = x_\perp\cdot n_3 = 0$)

$$\begin{aligned}
&\langle S^{l_1}_{n_1}(x_{1_\perp})S^{l_2}_{n_2}(x_{2_\perp})S^{k_3}_{n_3}(x_{3_\perp})\rangle =\\
&= C(g^2,l_i)\frac{(2n_1\cdot n_2)^{\frac{l_1+l_2-l_3-1}{2}}(2n_1\cdot n_3)^{\frac{l_1+l_3-l_2-1}{2}}(2n_2\cdot n_3)^{\frac{l_2+l_3-l_1-1}{2}}}{|x_{12_\perp}|^{\Delta_1+\Delta_2-\Delta_3-1}|x_{13_\perp}|^{\Delta_1+\Delta_3-\Delta_2-1}|x_{23_\perp}|^{\Delta_2+\Delta_3-\Delta_1-1}}
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$$F^l(x_\perp) = \int du\ F^a_{ni} \nabla_n^{l-2} F^{ai}_n(un + x_\perp),$$

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$$S^l_{3n} = F^l_n - \frac{l+2}{2(l-1)}\Lambda^l_n - \frac{(l+1)(l+2)}{2l(l-1)}\Phi^l_n$$

and Rychkov's structures reduce to one $(x_\perp \cdot n_1 = x_\perp \cdot n_2 = x_\perp \cdot n_3 = 0)$

$$\langle S^{l_1}_{n_1}(x_{1_\perp}) S^{l_2}_{n_2}(x_{2_\perp}) S^{k_3}_{n_3}(x_{3_\perp})\rangle =$$
$$= C(g^2, l_i) \frac{(2n_1 \cdot n_2)^{\frac{l_1+l_2-l_3-1}{2}} (2n_1 \cdot n_3)^{\frac{l_1+l_3-l_2-1}{2}} (2n_2 \cdot n_3)^{\frac{l_2+l_3-l_1-1}{2}}}{|x_{12_\perp}|^{\Delta_1+\Delta_2-\Delta_3-1} |x_{13_\perp}|^{\Delta_1+\Delta_3-\Delta_2-1} |x_{23_\perp}|^{\Delta_2+\Delta_3-\Delta_1-1}}$$

Our aim is to find the structure constants $C(g^2, l_i)$ in the "BFKL limit" $l_i \to 1$

Light-ray operators

Gluon light-ray (LR) operator of twist 2

$$F^a_{-i}(x'_+ + x_\perp)[x'_+, x_+]^{ab} F^{b\ i}_{-}(x_+ + x_\perp)$$

Forward matrix element - gluon parton density

$$z^\mu z^\nu \langle p|F^a_{\mu\xi}(z)[z,0]^{ab}F^{b\xi}_{\nu}(0)|p\rangle^\mu \stackrel{z^2=0}{=} 2(pz)^2 \int_0^1 dx_B\, x_B D_g(x_B,\mu)\cos(pz)x_B$$

Evolution equation (in gluodynamics)

$$\mu^2 \frac{d}{d\mu^2} F^a_{-i}(x'_+ + x_\perp)[x'_+, x_+]^{ab} F^{b\ i}_{-}(x_+ + x_\perp)$$
$$= \int_{x_+}^{x'_+} dz'_+ \int_{x_+}^{z'_+} dz_+\, K(x'_+, x_+; z'_+, z_+; \alpha_s) F^a_{-i}(z'_+ + x_\perp)[z'_+, z_+]^{ab} F^{b\ i}_{-}(z_+ + x_\perp)$$

"Forward" LR operator

$$F(L_+, x_\perp) = \int dx_+\, F^a_{-i}(L_+ + x_+ + x_\perp)[L_+ + x_+, x_+]^{ab} F^{bi}_{-}(x_+ + x_\perp)$$

Light-ray operators

Conformal LR operator ($j = \frac{3}{2} + i\nu$)

$$F^{\mu}_{j}(x_\perp) \ = \int_0^\infty dL_+ \ L_+^{1-j} F^\mu(L_+, x_\perp)$$

Evolution equation for “forward” conformal light-ray operators

$$\Rightarrow \ \mu^2 \frac{d}{d\mu^2} F_j(z_\perp) \ = \int_0^1 du \ K_{gg}(u, \alpha_s) u^{j-2} F_j(z_\perp)$$

$\Rightarrow \gamma_j(\alpha_s)$ is an analytical continuation of $\gamma_n(\alpha_s)$

Supermultiplet of LR operators

Gluino and scalar LR operators

$$\Lambda(L_+,x_\perp) \;=\; \frac{i}{2}\int dx'_+\,[\bar\lambda^a(L_+ + x_+ + x_\perp)[x'_+ + x_+, x_+]^{ab}\sigma_-\nabla_-\lambda^b(x_+ + x_\perp) + \mathrm{c.c}]$$

$$\Phi(L_+,x_\perp) \;=\; \int dx'_+\,\phi^{a,I}(L_+ + x_+ + x_\perp)[x'_+ + x_+, x_+]^{ab}\nabla_-^2\phi^{b,I}(x_+ + x_\perp)$$

$$\Lambda_j(x_\perp) \;=\; \int_0^\infty dL_+\,L_+^{-j+1}\Lambda(L_+,x_\perp), \qquad \Phi_j(x_\perp) \;=\; \int_0^\infty dL_+\,L_+^{-j+1}\Phi(L_+,x_\perp)$$

SU_4 singlet LR operators.

$$S_{1j}(x_\perp) \;=\; F_j(x_\perp) + \frac{j-1}{8}\Lambda_j(x_\perp) - \frac{j(j-1)}{8}\Phi_j(x_\perp)$$

$$S_{2j}(x_\perp) \;=\; F_j(x_\perp) - \frac{1}{8}\Lambda_j(x_\perp) + \frac{j(j+1)}{24}\Phi_j(x_\perp)$$

$$S_{3j}(x_\perp) \;=\; F_j(x_\perp) - \frac{j+2}{4}\Lambda_j(x_\perp) - \frac{(j+1)(j+2)}{8}\Phi_j(x_\perp)$$

All operators have the same anomalous dimension

$$\gamma_j^{S_1}(\alpha_s) \equiv \gamma_j(\alpha_s) = \frac{2\alpha_s}{\pi}N_c[\psi(j-1) + C] + O(\alpha_s^2), \qquad \gamma_j^{S_2} = \gamma_{j+1}^{S_1}, \qquad \gamma_j^{S_3} = \gamma_{j+2}^{S_1}$$

Correlators of LR operators

Since LR operators are "analytic continuation" of local operators, we expect ($j_1 = \frac{3}{2} + i\nu_1, j_2 = \frac{3}{2} + i\nu_2$)

$$\langle S^{j_1}_{n_1}(x_{1_\perp}) S^{j_2}_{n_2}(x_{2_\perp})\rangle \;=\; \delta(\nu_1 - \nu_2) f(\alpha_s, j) \frac{(2n_1 \cdot n_2)^{\omega_1} (\mu^2)^{-\gamma(j_1,\alpha_s)}}{|x_{12_\perp}|^{\Delta(\alpha_s, j_1)}}$$

for 2-point CF and similarly ($j_i \equiv 1 + \omega_i$)

$$\langle S^{j_1}_{n_1}(x_{1_\perp})\; S^{j_2}_{n_2}(x_{2_\perp})\; S^{j_2}_{n_3}(x_{3_\perp})\rangle \;=\; \frac{F(\alpha_s, \omega_1, \omega_2, \omega_3)}{(\omega_1 + \omega_2 - \omega_3)(\omega_1 + \omega_3 - \omega_2)(\omega_2 + \omega_3 - \omega_1)}$$
$$\times\; \frac{(2n_1 \cdot n_2)^{\frac{\omega_1 + \omega_2 - \omega_3}{2}}}{|x_{12_\perp}|^{\Delta_1 + \Delta_2 - \Delta_3}} \frac{(2n_1 \cdot n_3)^{\frac{\omega_1 + \omega_3 - \omega_2}{2}}}{|x_{13_\perp}|^{\Delta_1 + \Delta_3 - \Delta_2}} \frac{(2n_2 \cdot n_3)^{\frac{\omega_2 + \omega_3 - \omega_1}{2}}}{|x_{23_\perp}|^{\Delta_2 + \Delta_3 - \Delta_1}} \mu^{-\gamma(j_1) - \gamma(j_2) - \gamma(j_3)}$$

for the 3-point CF ($\Delta \;=\; j + \gamma(j) = 1 + \omega + \gamma_\omega$ - dimension).

Correlators of LR operators

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$$\langle S^{j_1}_{n_1}(x_{1_\perp}) S^{j_2}_{n_2}(x_{2_\perp})\rangle \;=\; \delta(\nu_1 - \nu_2) f(\alpha_s, j) \frac{(2n_1\cdot n_2)^{\omega_1}(\mu^2)^{-\gamma(j_1,\alpha_s)}}{|x_{12_\perp}|^{\Delta(\alpha_s, j_1)}}$$

for 2-point CF and similarly ($j_i \equiv 1 + \omega_i$)

$$\langle S^{j_1}_{n_1}(x_{1_\perp})\, S^{j_2}_{n_2}(x_{2_\perp})\, S^{j_2}_{n_3}(x_{3_\perp})\rangle \;=\; \frac{F(\alpha_s, \omega_1, \omega_2, \omega_3)}{(\omega_1 + \omega_2 - \omega_3)(\omega_1 + \omega_3 - \omega_2)(\omega_2 + \omega_3 - \omega_1)}$$
$$\times \frac{(2n_1\cdot n_2)^{\frac{\omega_1+\omega_2-\omega_3}{2}}}{|x_{12_\perp}|^{\Delta_1+\Delta_2-\Delta_3}} \frac{(2n_1\cdot n_3)^{\frac{\omega_1+\omega_3-\omega_2}{2}}}{|x_{13_\perp}|^{\Delta_1+\Delta_3-\Delta_2}} \frac{(2n_2\cdot n_3)^{\frac{\omega_2+\omega_3-\omega_1}{2}}}{|x_{23_\perp}|^{\Delta_2+\Delta_3-\Delta_1}} \mu^{-\gamma(j_1)-\gamma(j_2)-\gamma(j_3)}$$

for the 3-point CF ($\Delta \;=\; j + \gamma(j) = 1 + \omega + \gamma_\omega$ - dimension).

The goal is to calculate $f(\alpha_s, j)$ and $F(\alpha_s, j_1, j_2, j_3)$ at $j_i = 1 + \omega_i$ in the "BFKL limit" $g^2 \to 0$, $\omega \to 0$, and $\frac{g^2}{\omega}$ = fixed

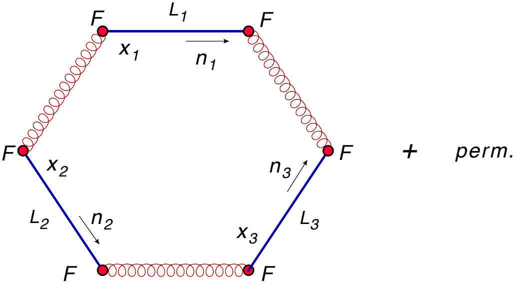
Warm-up exercise: LO

Since LR operators are “analytic continuation” of local operators, we expect

$$\langle S^{j_1}_{n_1}(x_{1_\perp})\, S^{j_2}_{n_2}(x_{2_\perp})\, S^{j_2}_{n_3}(x_{3_\perp})\rangle \;=\; \frac{F(\alpha_s,\omega_1,\omega_2,\omega_3)}{(\omega_1+\omega_2-\omega_3)(\omega_1+\omega_3-\omega_2)(\omega_2+\omega_3-\omega_1)}$$
$$\times\ \frac{(2n_1\cdot n_2)^{\frac{\omega_1+\omega_2-\omega_3}{2}}}{|x_{12_\perp}|^{\Delta_1+\Delta_2-\Delta_3}}\,\frac{(2n_1\cdot n_3)^{\frac{\omega_1+\omega_3-\omega_2}{2}}}{|x_{13_\perp}|^{\Delta_1+\Delta_3-\Delta_2}}\,\frac{(2n_2\cdot n_3)^{\frac{\omega_2+\omega_3-\omega_1}{2}}}{|x_{23_\perp}|^{\Delta_2+\Delta_3-\Delta_1}}$$

$\Delta \;=\; j+\gamma(j)$ - dimension

Warm-up exercise: LO



Warm-up exercise: LO

$$\langle \mathcal{S}_{n_1}^{1+\omega_1}(x_{1_\perp})\mathcal{S}_{n_2}^{1+\omega_2}(x_{2_\perp})\mathcal{S}_{n_3}^{1+\omega_3}(z_{3_\perp})\rangle = -\frac{(N_c^2-1)F(\alpha_s,\omega_1,\omega_2,\omega_3)}{4\pi^6(\omega_1+\omega_2-\omega_3)(\omega_1+\omega_3-\omega_2)(\omega_2+\omega_3-\omega_1)}$$
$$\times \frac{1}{x_{12}^2x_{13}^2x_{23}^2}\Big(\frac{2n_1\cdot n_2}{x_{12}^2}\Big)^{\frac{\omega_1+\omega_2-\omega_3}{2}}\Big(\frac{2n_1\cdot n_3}{x_{13}^2}\Big)^{\frac{\omega_1+\omega_3-\omega_2}{2}}\Big(\frac{2n_2\cdot n_3}{x_{23}^2}\Big)^{\frac{\omega_2+\omega_3-\omega_1}{2}}$$

$$F(\alpha_s,\omega_1,\omega_2,\omega_3) = \Gamma\Big(1+\frac{\omega_1+\omega_2-\omega_3}{2}\Big)\Gamma\Big(1+\frac{\omega_2+\omega_3-\omega_1}{2}\Big)\Gamma\Big(1+\frac{\omega_1+\omega_3-\omega_2}{2}\Big)$$
$$\times \prod_i \Gamma(1-\omega_i)\Gamma\Big(\frac{\omega_1+\omega_2-\omega_3}{2}+2\Big)\Gamma\Big(\frac{\omega_2+\omega_3-\omega_1}{2}+2\Big)\Gamma\Big(\frac{\omega_1+\omega_3-\omega_2}{2}+2\Big)$$
$$\times \Big\{\Big(e^{i\pi\omega_3}-1\Big)\Big[e^{i\pi(\omega_1-\omega_2)}+e^{i\pi(\omega_2-\omega_1)}-2e^{-i\pi\omega_3}\Big]+\Big(e^{i\pi\omega_1}-1\Big)\Big[e^{i\pi(\omega_2-\omega_3)}+e^{i\pi(\omega_3-\omega_2)}$$
$$-2e^{-i\pi\omega_1}\Big]+\Big(e^{i\pi\omega_2}-1\Big)\Big[e^{i\pi(\omega_3-\omega_1)}+e^{i\pi(\omega_1-\omega_3)}-2e^{-i\pi\omega_2}\Big]$$
$$+e^{i\pi(\omega_1+\omega_2-\omega_3)}+e^{i\pi(\omega_2+\omega_3-\omega_1)}+e^{i\pi(\omega_1+\omega_3-\omega_2)}-e^{i\pi(\omega_1+\omega_2+\omega_3)}-2\Big\}$$

At small ω's

$$F(\omega_1,\omega_2,\omega_3) \simeq -2\pi^2(\omega_1^2+\omega_2^2+\omega_3^2)-\pi^2(\omega_1^2+\omega_2^2+\omega_3^2-2\omega_1\omega_2-2\omega_1\omega_3-2\omega_2\omega_3)$$

In higher orders one should expect

$$\langle \mathcal{S}_{n_1}^{1+\omega_1}(x_{1_\perp})\mathcal{S}_{n_2}^{1+\omega_2}(x_{2_\perp})\mathcal{S}_{n_3}^{1+\omega_3}(z_{3_\perp})\rangle =$$
$$= -\frac{N_c^2-1}{4\pi^6 x_{12}^2 x_{13}^2 x_{23}^2}\frac{1}{(\omega_1+\omega_2-\omega_3)(\omega_1+\omega_3-\omega_2)(\omega_2+\omega_3-\omega_1)}$$
$$\times \frac{(2n_1\cdot n_2)^{\frac{\omega_1+\omega_2-\omega_3}{2}}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}}\frac{(2n_1\cdot n_3)^{\frac{\omega_1+\omega_3-\omega_2}{2}}}{(x_{13}^2)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}}\frac{(2n_2\cdot n_3)^{\frac{\omega_2+\omega_3-\omega_1}{2}}}{(x_{23}^2)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}}F(\omega_1,\omega_2,\omega_3;g^2),$$
$$F(\omega_1,\omega_2,\omega_3;g^2) \simeq F(\omega_1,\omega_2,\omega_3)\Big[1+\sum c_n\big(\frac{g^2}{\omega_i}\big)^n\Big]$$

It could be obtained from the CF of three color dipoles with long sides collinear to n_1, n_2, and n_3 and transverse short sides.

This means analyzing QCD (or N=4 SYM) in the triple Regge limit.
Triple Regge limit: scattering of 3 particles moving with speed $\sim c$ in x, y, and z directions.

In higher orders one should expect

$$
\begin{aligned}
&\langle \mathcal{S}^{1+\omega_1}_{n_1}(x_{1_\perp})\mathcal{S}^{1+\omega_2}_{n_2}(x_{2_\perp})\mathcal{S}^{1+\omega_3}_{n_3}(z_{3_\perp})\rangle = \\
&= -\frac{N_c^2-1}{4\pi^6 x_{12}^2 x_{13}^2 x_{23}^2}\frac{1}{(\omega_1+\omega_2-\omega_3)(\omega_1+\omega_3-\omega_2)(\omega_2+\omega_3-\omega_1)} \\
&\times \frac{(2n_1\cdot n_2)^{\frac{\omega_1+\omega_2-\omega_3}{2}}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}}\frac{(2n_1\cdot n_3)^{\frac{\omega_1+\omega_3-\omega_2}{2}}}{(x_{13}^2)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}}\frac{(2n_2\cdot n_3)^{\frac{\omega_2+\omega_3-\omega_1}{2}}}{(x_{23}^2)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}}F(\omega_1,\omega_2,\omega_3;g^2), \\
&F(\omega_1,\omega_2,\omega_3;g^2) \simeq F(\omega_1,\omega_2,\omega_3)\Big[1+\sum c_n\Big(\frac{g^2}{\omega_i}\Big)^n\Big]
\end{aligned}
$$

It could be obtained from the CF of three color dipoles with long sides collinear to n_1, n_2, and n_3 and transverse short sides.

This means analyzing QCD (or N=4 SYM) in the triple Regge limit.
Triple Regge limit: scattering of 3 particles moving with speed $\sim c$ in x, y, and z directions.

What we can do in a meantime is to take $n_3 \to n_2$ and consider the CF of a dipole in $n_1 = n_+$ direction and two dipoes in $n_2 = n_3 = n_-$ directions which can be obtained using the BK evolution.

CF of two light-ray operators

$$S_j^+(x_\perp) \;=\; \int dx_+ \int_0^\infty dL_+\, L_+^{1-j} F^a_{-i}(L_+ + x_+ + x_\perp)[L_+ + x_+, x_+]^{ab} F_-^{bi}(x_+ + x_\perp) \\ +\ \text{gluinos}\ +\ \text{scalars}$$

$$S_{j'}^-(y_\perp) \;=\; \int dy_- \int_0^\infty dL_-\, L_-^{1-j'} F^a_{-i}(L_- + y_- + y_\perp)[L_- + y_-, y_-]^{ab} F_-^{bi}(y_- + y_\perp) \\ +\ \text{gluinos}\ +\ \text{scalars}$$

A general formula for the correlation function of two LR operators reads

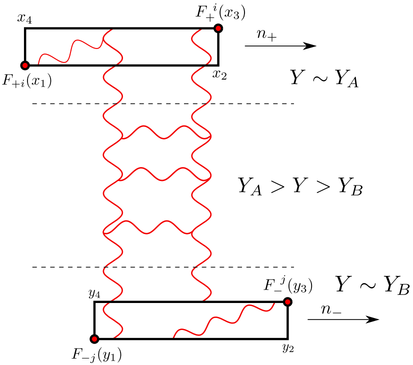
$$\langle S^+_{j=\frac{3}{2}+i\nu}(x_\perp) S^-_{j'=\frac{3}{2}+i\nu'}(y_\perp)\rangle = \frac{\delta(\nu-\nu')a(j,\alpha_s)(\mu^2)^{-\gamma(j,\alpha_s)}}{((x-y)^2_\perp)^{j+1+\gamma(j,\alpha_s)}}$$

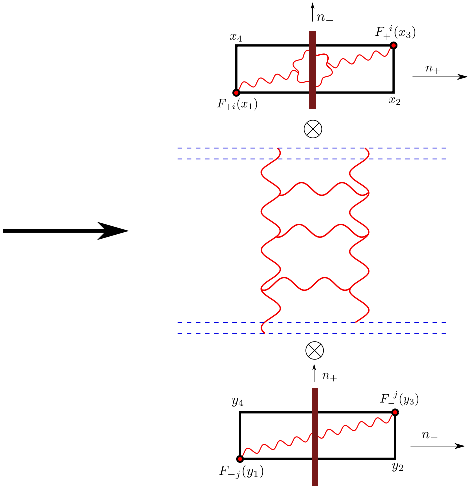
$\delta(\nu-\nu')$ reflects boost invariance: $x_+ \to \lambda x_+$, $x_- \to \frac{1}{\lambda}x_-$ does not change the correlation functions which depend on $x_+ y_-$.

We need to reproduce it and find $\gamma(j,\alpha_s)$ and $a(j,\alpha_s)$ as $j \to 1$.

Correlator of two "Wilson frames"

"Wilson frame": light-ray operator with point-splitting in the transverse direction





Correlator of two color dipoles

$$\langle \mathbf{U}^{\sigma_-}(\mathbf{x_{1\perp}},\mathbf{z_\perp})\mathbf{V}^{\sigma_+}(\mathbf{y_{1\perp}},\mathbf{w_\perp})\rangle =$$

$$= -\frac{8g^4}{N^2}\int\int \frac{d\nu\,\nu^2\ d^2z_0}{(\frac{1}{4}+\nu^2)^2}\left(\frac{(x_1-z)^2_\perp}{(x_1-z_0)^2_\perp(z-z_0)^2_\perp}\right)^{\frac{1}{2}+i\nu}$$

$$\left(\frac{(y_1-w)^2_\perp}{(y_1-z_0)^2_\perp(w-z_0)^2_\perp}\right)^{\frac{1}{2}-i\nu}\left(\frac{\sigma_+\sigma_-}{\sigma_{+0}\sigma_{-0}}\right)^{\aleph(\nu)}.$$

where $\aleph(\nu)=\frac{\alpha_s}{\pi}N_c\big[2\psi(1)-\psi\big(\frac{1}{2}+i\nu_1\big)-\psi\big(\frac{1}{2}-i\nu_1\big)\big]$.

From experience with 4-point CFs in the BFKL limit

$$\left(\frac{\sigma_+\sigma_-}{\sigma_{+0}\sigma_{-0}}\right)^{\aleph(\nu)}\rightarrow$$

$$\rightarrow\frac{i}{\sin\pi\aleph(\nu)}\left(\frac{((x_1-y_3)^2)^{\frac{\aleph(\nu)}{2}}((x_3-y_1)^2)^{\frac{\aleph(\nu)}{2}}}{(x_{13}^2)^{\frac{\aleph(\nu)}{2}}(y_{13}^2)^{\frac{\aleph(\nu)}{2}}}-\frac{((x_1-y_1)^2)^{\frac{\aleph(\nu)}{2}}((x_3-y_3)^2)^{\frac{\aleph(\nu)}{2}}}{(x_{13}^2)^{\frac{\aleph(\nu)}{2}}(y_{13}^2)^{\frac{\aleph(\nu)}{2}}}\right).$$

(For small $x^\perp_{13}$ and $y^\perp_{13}$ Wilson frames are approximately conformally invariant)

$$\langle \breve{S}^{2+\omega_1}_{gl+} \breve{S}^{2+\omega_2}_{gl-} \rangle = -i\frac{N^2 g^4}{4\pi^3} \int d\nu (\Delta_\perp^2)^{\aleph(\nu)-\omega} B(-\omega, \omega - \aleph(\nu)) \frac{1 - e^{i\pi(2\aleph(\nu)-\omega)}}{\sin \pi\aleph(\nu)} \cdot$$

$$\times \frac{\nu^2}{(\frac{1}{4}+\nu^2)^2} \frac{\delta(\omega_1 - \omega_2)}{(|x_{13}|_\perp^2 |y_{13}|_\perp^2)^{1+\frac{\aleph(\nu)}{2}}} \left(\frac{(|x_{13}|_\perp^2)^{\frac{1}{2}+i\nu} (|y_{13}|_\perp^2)^{\frac{1}{2}+i\nu}}{(|x-y|_\perp^2)^{1+2i\nu}} G(\nu) + (\nu \to -\nu) \right)$$

where

$$G(\nu) = -i \frac{4^{-1-2i\nu} \pi^3 (i - 2\nu)^2}{\Gamma^2(\frac{3}{2} - i\nu)\Gamma^2(1 + i\nu) \sinh(2\pi\nu)}.$$

Now we can carry out the last integration over ν as the pole contribution at $\omega = \aleph(\nu)$.

We pick here the first pole Ψ-functions in pomeron intercept which corresponds to the operator with the lowest possible twist=2.

Correlator of two "Wilson frames" V. Kazakov, E. Sobko, I.B.

$$\langle S_+^{1+\omega_1}(x_{1\perp},x_{3\perp})S_-^{1+\omega_2}(y_{1\perp},y_{3\perp})\rangle \xrightarrow[x_{13\perp},\,y_{13\perp}\to 0]{} \delta(\omega_1-\omega_2)\Upsilon(\tilde\gamma)\frac{(x_{13\perp}^2)^{\frac{\tilde\gamma}{2}-\frac{\omega}{2}}(y_{13\perp}^2)^{\frac{\tilde\gamma}{2}-\frac{\omega}{2}}}{((x-y)_\perp^2)^{2+\tilde\gamma}},$$

$$\Upsilon(\tilde\gamma) = -N^2g^4\frac{2^{-1-2\tilde\gamma}\pi}{\tilde\gamma^2\Gamma^2(1-\frac{\tilde\gamma}{2})\Gamma^2(\frac{1}{2}+\frac{\tilde\gamma}{2})\sin(\pi\tilde\gamma)\hat\aleph'(\tilde\gamma)}$$

$\tilde\gamma=-1+2i\nu$ is the solution of $\omega=\hat\aleph(\tilde\gamma)$
We use the point-splitting regularization in the orthogonal direction for our light-ray operators $\Rightarrow$ cutoffs are defined as $\Lambda_x=\frac{1}{|x_{13\perp}|}$ and $\Lambda_y=\frac{1}{|y_{13\perp}|}$
Rewrite

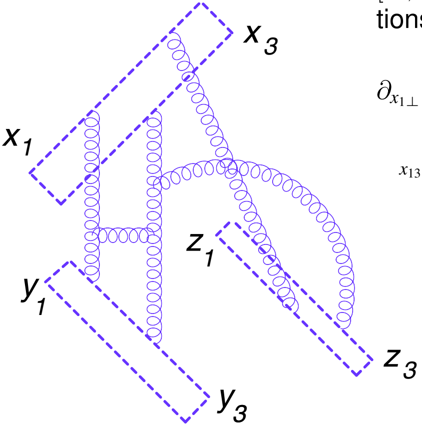
$$\langle S_+^{1+\omega_1}(x_{1\perp},x_{3\perp})S_-^{1+\omega_2}(y_{1\perp},y_{3\perp})\rangle \xrightarrow[x_{13\perp},\,y_{13\perp}\to 0]{} \delta(\omega_1-\omega_2)\Upsilon(\gamma+\omega)\frac{(x_{13\perp}^2)^{\frac{\gamma}{2}}(y_{13\perp}^2)^{\frac{\gamma}{2}}}{((x-y)_\perp^2)^{2+\omega+\gamma}}.$$

The anomalous dimension $\gamma=\tilde\gamma-\omega$ satisfies

$$\omega=\hat\aleph(\gamma+\omega)=\hat\aleph(\gamma)+\hat\aleph'(\gamma)\hat\aleph(\gamma)+o(g^4)\,.$$

- Lipatov-Fadin formula

CF of three Wilson frames: one in “+” direction and two in “-”



$[x_1, x_3]_\Box$ ≡ Wilson frame (without F insertions)

$$\partial_{x_{1\perp}} \cdot \partial_{x_{3\perp}} \int dx_{1-} dx_{3-} (x_{1-} - x_{3-})^{-2-\omega} [x_1, x_3]_\Box \to$$

$$\underset{x_{13\perp} \to 0,\ \omega \to 0}{\to} |x_{13\perp}|^{\gamma_J} c(g^2_{YM}, N_c, \omega) \breve{\mathcal{S}}^{2+\omega}(x_{1\perp}).$$

$$[x_1, x_3]_\Box \equiv \mathbf{U}^{\sigma_{1-}}(x_{1\perp}, x_{3\perp})$$

with some cutoff σ_{1-}

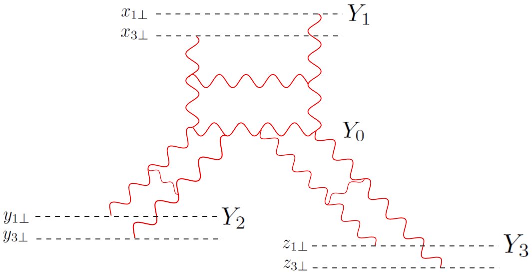
Using decomposition over Wilson lines we get:

$$\langle S_+^{2+\omega_1}(x_{1\perp},x_{3\perp})S_-^{2+\omega_2}(y_{1\perp},y_{3\perp})S_-^{2+\omega_3}(z_{1\perp},z_{3\perp})\rangle =$$

$$= \mathcal{D}_\perp \int_{-\infty}^{\infty} dx_{1-} \int_{x_{1-}}^{\infty} dx_{3-}\; x_{31-}^{-2-\omega_1} \int_{-\infty}^{\infty} dy_{1+} \int_{y_{1+}}^{\infty} dy_{3+}\; y_{31+}^{-2-\omega_2} \int_{-\infty}^{\infty} dz_{1+} \int_{z_{1+}}^{\infty} dz_{3+}\; z_{31+}^{-2-\omega_3} \times$$

$$\times \langle \mathbf{U}^{\sigma_{1-}}(x_{1\perp},x_{3\perp})\mathbf{V}^{\sigma_{2+}}(y_{1\perp},y_{3\perp})\mathbf{W}^{\sigma_{3+}}(z_{1\perp},z_{3\perp})\rangle,$$

where $\mathcal{D}_\perp = -\frac{N^3}{c(\omega_1)c(\omega_2)c(\omega_3)}(\partial_{x_{1\perp}}\cdot\partial_{x_{3\perp}})(\partial_{y_{1\perp}}\cdot\partial_{y_{3\perp}})(\partial_{z_{1\perp}}\cdot\partial_{z_{3\perp}})$.



- BK equation:

$$\sigma \frac{d}{d\sigma} \mathbf{U}^{\sigma}(z_1, z_2) = \mathcal{K}_{\mathrm{BK}} * \mathbf{U}^{\sigma}(z_1, z_2),$$

where $\mathcal{K}_{\mathrm{BK}}$ in LO approximation:

$$\mathcal{K}_{\mathrm{LO\,BK}} * \mathbf{U}(z_1, z_2) =$$

$$= \frac{2g^2}{\pi} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[\mathbf{U}(z_1, z_3) + \mathbf{U}(z_3, z_2) - \mathbf{U}(z_1, z_2) - \mathbf{U}(z_1, z_3)\mathbf{U}(z_3, z_2)\right].$$

- Schematically calculation of correlation function of 3 dipoles can be wrote as:

$$\int dY_0 (\mathbf{U}^{Y_1} \to \mathbf{U}^{Y_0}) \otimes (\text{BK vertex at } Y_0) \otimes \begin{pmatrix} \langle \mathbf{U}^{Y_0} \mathbf{V}^{Y_2} \rangle \\ \langle \mathbf{U}^{Y_0} \mathbf{W}^{Y_3} \rangle \end{pmatrix}$$

where we introduced rapidity $Y_i = \ln \sigma_i$

The structure of 3-point correlator in 2d - $\perp$ space

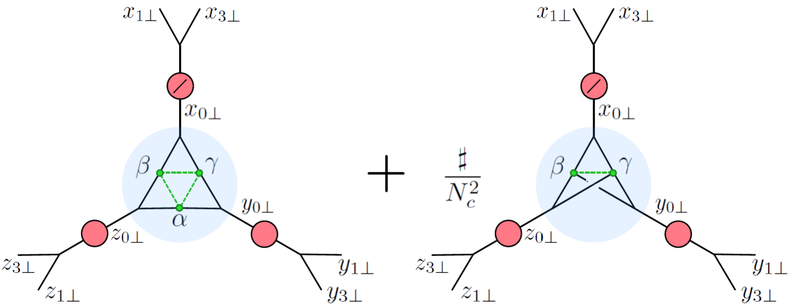


Figure : The structure of 3-point correlator. Red circles correspond to BFKL propagators (the crossed one has extra multiplier $(\frac{1}{4}+\nu_1^2)^2$). The blue blob corresponds to the 3-point functions of 2-dimensional BFKL CFT. The triple veritces correspond to E-functions. The $\alpha\beta\gamma$-triangle in the first, planar, term and $\beta\gamma$-link in the second, nonplanar, term correspond to triple pomeron vertex.

Result:

$$\langle \mathcal{S}^{1+\omega_1}_{n_1}(x_{1\perp},x_{3\perp})\mathcal{S}^{1+\omega_2}_{n_2}(y_{1\perp},y_{3\perp})\mathcal{S}^{1+\omega_3}_{n_2}(z_{1\perp},z_{3\perp})\rangle =$$
$$= -ig^{10}\frac{\delta(\omega_1-\omega_2-\omega_3)}{c(\omega_1)c(\omega_2)c(\omega_3)}H\frac{\Psi(\nu_1^*,\nu_2^*,\nu_3^*)|x_{13}|^{\gamma_1}|y_{13}|^{\gamma_2}|z_{13}|^{\gamma_3}}{|x-y|^{2+\gamma_1+\gamma_2-\gamma_3}|x-z|^{2+\gamma_1+\gamma_3-\gamma_2}|y-z|^{2+\gamma_2+\gamma_3-\gamma_1}}$$

where ν_i^* is a solution of BFKL equation for anomalous dimensions $\omega_i = \aleph(\nu_i^*)$

$$H = \frac{2^{10}(N_c^2-1)^2}{\pi^2 N_c^5}\gamma_1^2(2+\gamma_1)^4(2+\gamma_2)^2(2+\gamma_3)^2\frac{G(\nu_1^*)}{\aleph'(\nu_1^*)}\frac{G(\nu_2^*)}{\aleph'(\nu_2^*)}\frac{G(\nu_3^*)}{\aleph'(\nu_3^*)},$$

$\gamma_i = \gamma(j_i)$ - anomalous dimension ($j_i = 1+\omega_i$) and

$$G(\nu) = \frac{\nu^2}{(\frac{1}{4}+\nu^2)^2}\frac{\pi\Gamma^2(\frac{1}{2}+i\nu)\Gamma(-2i\nu)}{\Gamma^2(\frac{1}{2}-i\nu)\Gamma(1+2i\nu)},$$
$$\Psi(\nu_1,\nu_2,\nu_3) = \Omega(h_1,h_2,h_3) - \frac{2\pi}{N_c^2}\Lambda(h_1,h_2,h_3)\mathrm{Re}(\psi(1)-\psi(h_1)-\psi(h_2)-\psi(h_3)),$$

where $h_i = \frac{1}{2}+i\nu_i = 1+\frac{\gamma_i}{2}$.

Expression for Ω and Λ was obtained by G.Korchemsky in terms of higher hypergeometric and Meijer G-functions.

$n_2 \to n_3$ limit

To identify the function $\Psi(\nu_1^*, \nu_2^*, \nu_3^*)$ with structure constants of CF of three LR operators we need to consider limit $n_2 \to n_3$ in the formula

$$\langle S^{j_1}_{n_1}(x_{1_\perp})\, S^{j_2}_{n_2}(x_{2_\perp})\, S^{j_2}_{n_3}(x_{3_\perp})\rangle \;=\; \frac{C(\alpha_s, \omega_1, \omega_2, \omega_3)}{(\omega_1+\omega_2-\omega_3)(\omega_1+\omega_3-\omega_2)(\omega_2+\omega_3-\omega_1)}$$
$$\times\; \frac{(n_1\cdot n_2)^{\frac{\omega_1+\omega_2-\omega_3}{2}}}{|x_{12_\perp}|^{\Delta_1+\Delta_2-\Delta_3-1}} \frac{(n_1\cdot n_3)^{\frac{\omega_1+\omega_3-\omega_2}{2}}}{|x_{13_\perp}|^{\Delta_1+\Delta_3-\Delta_2-1}} \frac{(n_2\cdot n_3)^{\frac{\omega_2+\omega_3-\omega_1}{2}}}{|x_{23_\perp}|^{\Delta_2+\Delta_3-\Delta_1-1}}$$

The limit $n_2 \to n_3$ is tricky:
in the limit $n_2 \to n_3$ we get a “zero mode” coming from boost invariance at $n_2 = n_3$

$$\frac{1}{\omega_1-\omega_2-\omega_3}\Big(\frac{(n_2,n_3)}{s}\Big)^{\frac{\omega_2+\omega_3-\omega_1}{2}} \overset{n_2\to n_3}{\to} \int d\xi e^{-\xi(\omega_1-\omega_2-\omega_3)} \;=\; \delta(\omega_1-\omega_2-\omega_3)$$

Limit $n_2 \to n_3$

Rapidity integral at $n_2 = n_3$

$$\int dY_1 dY_2 dY_3 \int dY_0\, \theta(Y_1 - Y_0)\theta(Y_0 + Y_2)\theta(Y_0 + Y_3) e^{-\omega_1 Y_1 - \omega_2 Y_2 - \omega_3 Y_3}$$

$$e^{\aleph_1(Y_1 - Y_0) + \aleph_2(Y_0 + Y_2) + \aleph_3(Y_0 + Y_3)} = \frac{\delta(\omega_1 - \omega_2 - \omega_3)}{(\omega_1 - \aleph_1)(\omega_2 - \aleph_2)(\omega_3 - \aleph_3)}$$

Let us take $n_2 \neq n_3$ but $n_1 \cdot n_2 \simeq n_1 \cdot n_3$. We can use our formulas for $n_2 = n_3$ case until longitudinal distances between frames "2" and "3" are smaller than typical transverse separation $\Delta_\perp^2$, i.e. when $(y_1 - z_1)^2 \leq \Delta_\perp^2 \;\Leftrightarrow\; l_2 l_3 n^{23} \leq \Delta_\perp^2$.
In terms of rapidities $Y_2 = \ln l_2 \frac{\sqrt{n^{12}}}{\Delta_\perp},\; Y_3 = \ln l_3 \frac{\sqrt{n^{12}}}{\Delta_\perp}$ this restriction means $Y_2 + Y_3 \;\leq\; r,\; r \equiv \;\ln \frac{n_1 \cdot n_2}{n_2 \cdot n_3}$.

Limit $n_2 \to n_3$

Rapidity integral with restriction $Y_2 + Y_3 \ \leq \ r, \ r \equiv \ \ln\frac{n_1 \cdot n_2}{n_2 \cdot n_3}$.

$$\int dY_1 dY_2 dY_3 \int dY_0 \ \theta(Y_1 - Y_0)\theta(Y_0 + Y_2)\theta(Y_0 + Y_3)\theta\big(Y_2 + Y_3 < r\big)$$

$$e^{-\omega_1 Y_1 - \omega_2 Y_2 - \omega_3 Y_3 + \aleph_1(Y_1 - Y_0) + \aleph_2(Y_0 + Y_2) + \aleph_3(Y_0 + Y_3)}$$

$$= \frac{e^{-\frac{r}{2}(\omega_2 + \omega_3 - \omega_1)}}{(\omega_1 - \omega_2 - \omega_3)(\omega_1 - \aleph_1)\big(\omega_2 - \aleph_2 + \frac{\omega_1 - \omega_2 - \omega_3}{2}\big)\big(\omega_3 - \aleph_3 + \frac{\omega_1 - \omega_2 - \omega_3}{2}\big)}$$

$$\stackrel{\omega_2 + \omega_3 \to \omega_1}{\to} \quad \frac{\left(\frac{n_2 \cdot n_3}{n_1 \cdot n_2}\right)^{\omega_2 + \omega_3 - \omega_1}}{(\omega_1 - \omega_2 - \omega_3)(\omega_1 - \aleph_1)(\omega_2 - \aleph_2)(\omega_3 - \aleph_3)}$$

$$\Rightarrow \ \frac{1}{\omega_1 - \omega_2 - \omega_3}\left(\frac{(n_2, n_3)}{s}\right)^{\frac{\omega_2 + \omega_3 - \omega_1}{2}} \ \stackrel{n_2 \to n_3}{\to} \ \delta(\omega_1 - \omega_2 - \omega_3)$$

Finally for normalized structure constant $c_{\omega_1,\omega_2,\omega_3} = \frac{C_{+--}(\{\Delta_i\},\{1+\omega_i\})}{\sqrt{b_{1+\omega_1}b_{1+\omega_2}b_{1+\omega_3}}}$ we get:

$$C_{\omega_1,\omega_2,\omega_3} = i^{3/2}g^4\frac{2}{\pi^5}\frac{\sqrt{N_c^2-1}}{N_c^2}\gamma_1^2(2+\gamma_1)^2\sqrt{\frac{G(\nu_1^*)}{\aleph'(\nu_1^*)}\frac{G(\nu_2^*)}{\aleph'(\nu_2^*)}\frac{G(\nu_3^*)}{\aleph'(\nu_3^*)}}\Psi(\nu_1^*,\nu_2^*,\nu_3^*),$$

where $\omega_i = \aleph(\nu_i^*)$ and

$$G(\nu) = \frac{\nu^2}{(\frac{1}{4}+\nu^2)^2}\frac{\pi\Gamma^2(\frac{1}{2}+i\nu)\Gamma(-2i\nu)}{\Gamma^2(\frac{1}{2}-i\nu)\Gamma(1+2i\nu)},$$

$$\Psi(\nu_1,\nu_2,\nu_3) = \Omega(h_1,h_2,h_3) - \frac{2\pi}{N_c^2}\Lambda(h_1,h_2,h_3)\mathrm{Re}(\psi(1)-\psi(h_1)-\psi(h_2)-\psi(h_3))$$

with notation $h_i = \frac{1}{2}+i\nu_i = 1+\frac{\gamma_i}{2}$, $\omega_i = \aleph(\nu_i)$

The structure of the formula is $C_{\omega_1,\omega_2,\omega_3} = g\frac{\sqrt{N_c^2-1}}{N_c^2}f(\frac{g^2}{\omega_1},\frac{g^2}{\omega_2},\frac{g^2}{\omega_3})$

Structure constant in the BFKL limit

In the limit $\frac{g^2}{\omega_i} \to 0$ we get the asymptotics:

$$\Omega(h_1^*, h_2^*, h_3^*) \to -\frac{16\pi^3}{\gamma_1^2\gamma_2^2\gamma_3^2} \cdot [\gamma_1^2(\gamma_2 + \gamma_3) + \gamma_2^2(\gamma_1 + \gamma_3) +$$

$$+\gamma_3^2(\gamma_1 + \gamma_2) + \gamma_1\gamma_2\gamma_3)(1 + O(g^2/\omega_i))$$

$$\Lambda(h_1^*, h_2^*, h_3^*) \to \frac{8\pi^2(\gamma_1 + \gamma_2 + \gamma_3)}{\gamma_1\gamma_2\gamma_3}(1 + O(g^2/\omega_i))$$

$\gamma_i = -\frac{8g^2}{\omega_i} + o(\frac{g^2}{\omega_i})$

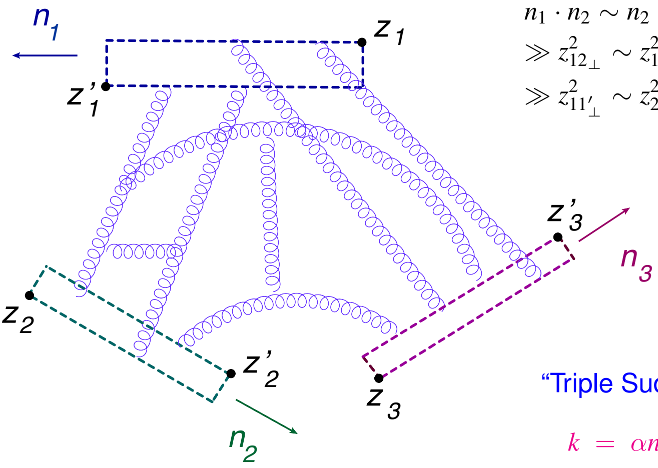
$$C_{\omega_1,\omega_2,\omega_3} = -ig^2\frac{\sqrt{N_c^2 - 1}}{\sqrt{2\pi}N_c^2}\frac{1}{\omega_1^{\frac{5}{2}}\omega_2^{\frac{1}{2}}\omega_3^{\frac{1}{2}}}(\omega_1^2(\omega_2 + \omega_3) +$$

$$\omega_2^2(\omega_1 + \omega_3) + \omega_3^2(\omega_1 + \omega_2) + \omega_1\omega_2\omega_3)(1 + O(g^2))$$

Wilson frames in the triple Regge limit

$$S^{\omega_1}_{n_1}(z_{1_\perp}, z'_{1_\perp}) \;=\; \int du_1 \int_0^\infty dl\; l^{-\omega_1} \mathrm{Tr}\{F_{n_1 i}(u_1 n_1 + l n_1 + z_{1_\perp}) F_{n_1}^{\;i}(u n_1 + z'_{1_\perp})\}$$

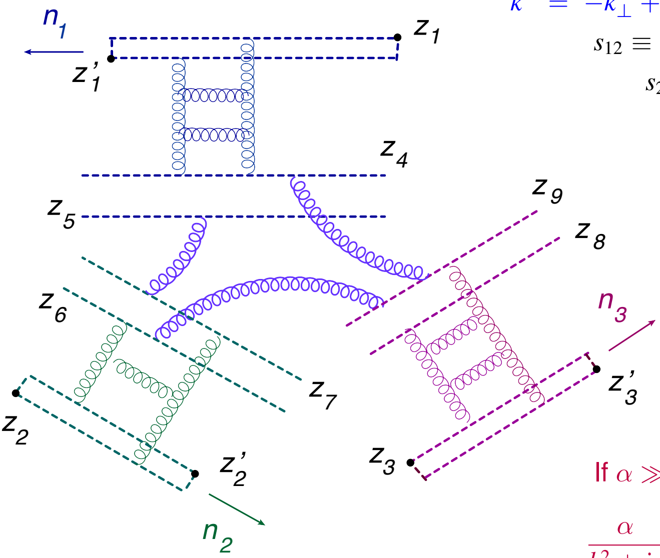
$$n_1^2 = n_2^2 = n_3^2 = 0, \quad z_\perp \cdot n_i = 0,$$
$$n_1 \cdot n_2 \sim n_2 \cdot n_3 \sim n_1 \cdot n_3 \;\gg$$
$$\gg z^2_{12_\perp} \sim z^2_{13_\perp} \sim z^2_{23_\perp} \sim \Delta^2_\perp \;\gg$$
$$\gg z^2_{11'_\perp} \sim z^2_{22'_\perp} \sim z^2_{33'_\perp}$$



“Triple Sudakov variables”:

$$k \;=\; \alpha n_1 + \beta n_2 + \gamma n_3 + k_\perp$$

Triple BFKL evolution



$$k^2 = -k_\perp^2 + \alpha\beta s_{12} + \alpha\gamma s_{13} + \beta\gamma s_{23}$$

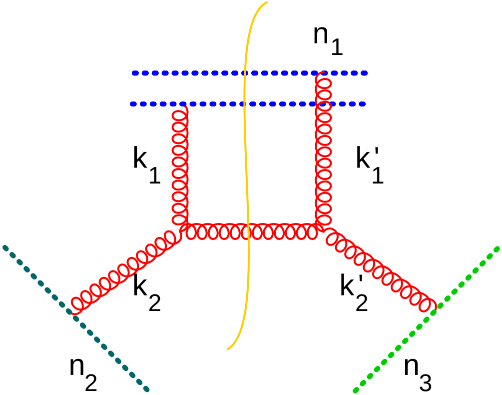
$$s_{12} \equiv 2n_1 \cdot n_2, \quad s_{13} \equiv 2n_1 \cdot n_3,$$

$$s_{23} \equiv 2n_2 \cdot n_3$$

If $\alpha \gg \beta, \gamma \Rightarrow$ eikonal

$$\frac{\alpha}{k^2 + i\epsilon} \to \frac{1}{\beta + \gamma + i\epsilon\alpha}$$

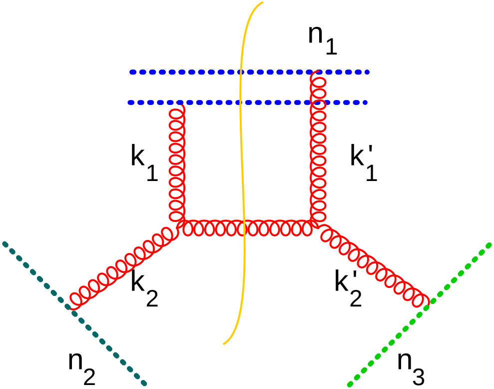
Check: BFKL kernel in the triple Regge regime



$$L^\mu(k_2,k_1)L_\mu(k_2',k_1') \;=\; (\tilde{k}_1-\tilde{k}_1')^2+\frac{\tilde{k}_1^2\tilde{k}'^2_2+\tilde{k}_2^2\tilde{k}'^2_1}{(\tilde{k}_1+\tilde{k}_2)^2}$$

BFKL kernel with $\tilde{k}^2 \;=\; k_\perp^2+\frac{s_{12}s_{23}}{s_{13}}\beta_k^2, \qquad s_{ik} \;=\; 2n_i\cdot n_k$

BFKL evolution in triple Regge limit



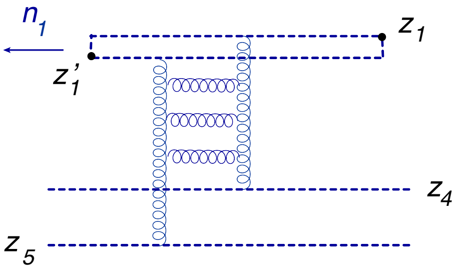
$$k_1 = \alpha_1 n_1 + \beta_1 n_2 + \gamma_1 n_3 + k_{1_\perp}$$
$$k_2 = \alpha_2 n_1 + \beta_2 n_2 + \gamma_2 n_3 + k_{2_\perp}$$
$$k'_2 = \alpha'_2 n_1 + \beta'_2 n_2 + \gamma'_2 n_3 + k'_{2_\perp}$$

The BFKL evolution is logarithmic until $\alpha_1\beta_2 s_{12},\ \alpha_1\gamma'_2 s_{13} \geq m_\perp^2 \sim \Delta_\perp^{-2}$.
In terms of rapidities $\eta_1 \equiv \ln\alpha$, $\eta_2 \equiv \ln\beta$, $\eta_3 \equiv \ln\gamma$

$$\eta_1 + \eta_2 \geq \ln\frac{\Delta_\perp^2}{s_{12}}, \quad \eta_1 + \eta_3 \geq \ln\frac{\Delta_\perp^2}{s_{13}},$$

BFKL evolution of Wilson frames

Evolution from $\alpha_{\max} \sim \frac{l_1\sqrt{s}}{|z_{11'}|}$ to $\alpha_{\min}$



$$\int du_1 \mathrm{Tr}\{F_{n_1 i}(u_1 n_1 + l_1 n_1 + z_{1_\perp})F_{n_1}^{\ i}(un_1 + z'_{1_\perp})$$

$$\overset{z_{11'}\to 0}{=} \frac{s^2 N_c}{16 l_1 z_{11'}^2}\int \frac{d\nu}{\pi^4} 2^{4i\nu}\frac{\Gamma\left(\frac{3}{2}-i\nu\right)\Gamma(1+i\nu)}{\Gamma\left(\frac{3}{2}+i\nu\right)\Gamma(-i\nu)}\left[\frac{z_{11'}^2 z_{45}^2}{z_{14}^2 z_{15}^2}\right]^{\frac{1}{2}+i\nu} e^{-\aleph(\nu)(\ln l_1 - y_1)}\mathrm{Tr}\{U_{z_4}U_{z_5}^\dagger\}^{y_1}$$

$$\eta_1 = \ln\frac{l_1\sqrt{s}}{|z_{11'}|},\ y_1 = \ln\alpha_{\min}$$

At small $z_{11'}$ we can close the contour over $\frac{1}{2}+i\nu$ in the right half-plane and the leading contribution comes from the pole of $\aleph(\nu)$ at $i\nu = \frac{1}{2} - \frac{\alpha_s N_c}{\pi\omega_1}$

Result = longitudinal integral × transverse integral

Longitudinal integral

$$\int d\eta_1 d\eta_2 d\eta_3 \int_{-\infty}^{\eta_1} dy_1 \int_{-\infty}^{\eta_2} dy_2 \int_{-\infty}^{\eta_3} dy_3$$

$$\times\, e^{-\omega_1\eta_1-\omega_2\eta_2-\omega_3\eta_3+\aleph_1(\eta_1-y_1)+\aleph_2(\eta_2-y_2)+\aleph_3(\eta_3-y_3)}$$

$$\theta\Big(y_1+y_2\geq \ln\frac{\Delta_\perp^{-2}}{s_{12}}\Big)\theta\Big(y_2+y_3\geq \ln\frac{\Delta_\perp^{-2}}{s_{23}}\Big)\theta\Big(y_1+y_3\geq \ln\frac{\Delta_\perp^{-2}}{s_{13}}\Big)$$

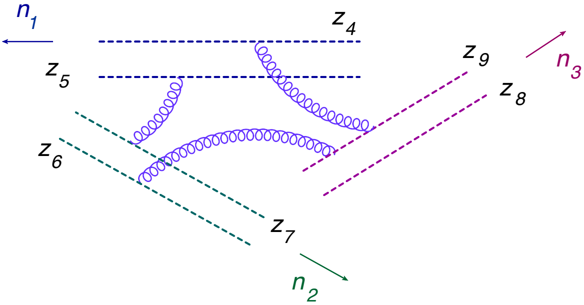
$$= \frac{4}{(\omega_1-\aleph_1)(\omega_2-\aleph_2)(\omega_3-\aleph_3)}$$

$$\times\, \frac{(s_{12}z_{12_\perp}^2)^{\frac{\omega_1+\omega_2-\omega_3}{2}}(s_{13}z_{13_\perp}^2)^{\frac{\omega_1+\omega_3-\omega_2}{2}}(s_{23}z_{23_\perp}^2)^{\frac{\omega_2+\omega_3-\omega_1}{2}}}{(\omega_1+\omega_2-\omega_3)(\omega_1+\omega_3-\omega_2)(\omega_2+\omega_3-\omega_1)}$$

$$\eta_1\equiv \ln l_1\sqrt{\frac{s_{12}s_{13}}{s_{23}z_{11'}^2}},\ \ \eta_2\equiv \ln l_2\sqrt{\frac{s_{12}s_{23}}{s_{13}z_{22'}^2}},\ \ \eta_3\equiv \ln l_3\sqrt{\frac{s_{13}s_{23}}{s_{12}z_{33'}^2}}$$

At the end of evolution

As usual, the end of evolution means $\frac{s}{m^2} \gg 1$ but $\alpha_s \ln \frac{s}{m^2} \ll 1$
$\Rightarrow$ leading-order tree diagrams

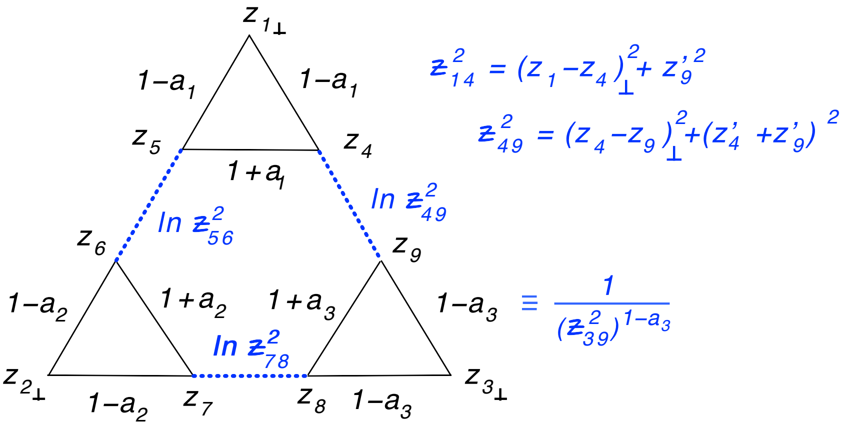


$$\ln \frac{\mathcal{Z}_{46}^{(12)} \mathcal{Z}_{57}^{(12)}}{\mathcal{Z}_{47}^{(12)} \mathcal{Z}_{56}^{(12)}} \ln \frac{\mathcal{Z}_{68}^{(23)} \mathcal{Z}_{79}^{(23)}}{\mathcal{Z}_{69}^{(23)} \mathcal{Z}_{78}^{(23)}} \ln \frac{\mathcal{Z}_{48}^{(23)} \mathcal{Z}_{59}^{(13)}}{\mathcal{Z}_{49}^{(23)} \mathcal{Z}_{58}^{(13)}}$$

$$\mathcal{Z}_{ij}^{(kl)} \equiv z_{ij}^2 - 2\frac{(z_{ij} \cdot n_k)(z_{ij} \cdot n_l)}{n_k \cdot n_l}$$

“Transverse” integral

The transverse planes of the evolution of Wilson frames are different
$\Rightarrow$ the resulting integral is two-dimensional but not translation invariant



$$= \; (z^2_{12_\perp})^{-1+a_1+a_2-a_3}(z^2_{13_\perp})^{-1+a_1+a_3-a_2}(z^2_{23_\perp})^{-1+a_2+a_3-a_1}\Phi(a_1,a_2,a_3)$$

$a_1=\frac{1}{2}-i\nu_1$ etc

Result = longitudinal integral × transverse integral

$$\text{Result} = \int d\nu_1 d\nu_2 d\nu_3 \, z_{11'}^{i\nu_1 - \frac{1}{2}} z_{22'}^{i\nu_2 - \frac{1}{2}} z_{33'}^{i\nu_3 - \frac{1}{2}}$$
$$\times \text{ Longitudinal integral} \times \text{Transverse integral}$$

$$\text{Longitudinal integral} = \frac{4}{(\omega_1 - \aleph(\nu_1))(\omega_2 - \aleph(\nu_2))(\omega_3 - \aleph(\nu_3))}$$
$$\times \frac{(s_{12} z_{12_\perp}^2)^{\frac{\omega_1+\omega_2-\omega_3}{2}} (s_{13} z_{13_\perp}^2)^{\frac{\omega_1+\omega_3-\omega_2}{2}} (s_{23} z_{23_\perp}^2)^{\frac{\omega_2+\omega_3-\omega_1}{2}}}{(\omega_1+\omega_2-\omega_3)(\omega_1+\omega_3-\omega_2)(\omega_2+\omega_3-\omega_1)}$$

$$\text{Transverse integral} = (z_{12_\perp}^2)^{-\frac{1}{2} - i\nu_1 - i\nu_2 + i\nu_3} (z_{13_\perp}^2)^{-\frac{1}{2} - i\nu_1 - i\nu_3 + i\nu_2}$$
$$\times (z_{23_\perp}^2)^{-\frac{1}{2} - i\nu_2 - i\nu_3 + i\nu_1} \Phi\left(\frac{1}{2} - i\nu_1, \frac{1}{2} - i\nu_2, \frac{1}{2} - i\nu_3\right)$$

Three-point CF in the triple Regge limit

Taking residues at $\nu_1 = \aleph^{-1}(\omega_1),\ \nu_2 = \aleph^{-1}(\omega_2),\ \nu_3 = \aleph^{-1}(\omega_3)$ we get

$$\langle S^{j_1}_{n_1}(z_{1_\perp})\ S^{j_2}_{n_2}(z_{2_\perp})\ S^{j_2}_{n_3}(z_{3_\perp})\rangle$$
$$= \frac{\Phi(-\gamma_1, -\gamma_2, -\gamma_3)}{(\omega_1+\omega_2-\omega_3)(\omega_1+\omega_3-\omega_2)(\omega_2+\omega_3-\omega_1)}$$
$$\times \frac{(s_{12})^{\frac{\omega_1+\omega_2-\omega_3}{2}}}{|z_{12_\perp}|^{\Delta_1+\Delta_2-\Delta_3}} \frac{(s_{13})^{\frac{\omega_1+\omega_3-\omega_2}{2}}}{|z_{13_\perp}|^{\Delta_1+\Delta_3-\Delta_2}} \frac{(s_{23})^{\frac{\omega_2+\omega_3-\omega_1}{2}}}{|z_{23_\perp}|^{\Delta_2+\Delta_3-\Delta_1}} \mu^{-\gamma(\omega_1)-\gamma(\omega_2)-\gamma(\omega_3)}$$

where $\Delta_i = 1 + \omega_i + \gamma_i$ and $\gamma_i = -\frac{1}{2} + i\nu_i$ is a solution of $\omega_i = \aleph(\nu_i)$

To compare to previous results (for tree approximation and for $\omega_1 = \omega_2 + \omega_3$) we need $\Phi(a_1, a_2, a_3)$ at $a_1, a_2, a_3 \ll 1$.

Conclusions

- QCD structure constants in the "triple BFKL limit" $\omega_i \to 0$ at $\frac{g^2}{\omega} \sim 1$ are expressed in terms of the "transverse integral" $\Phi(\gamma_1, \gamma_2, \gamma_3)$.

Outlook

- Calculate this transverse integral and compare to previous result in the limit $n_2 \to n_3$