Remarks on Jet Production & Odderon in the Multi-Regge Limit

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15^{th} Workshop on Non-Perturbative Quantum Chromodynamics

1. Multi-Regge limit
2. Monte Carlo event generator BFKLex
3. Collinear double logs
4. Solution of BKP equation & open spin chains

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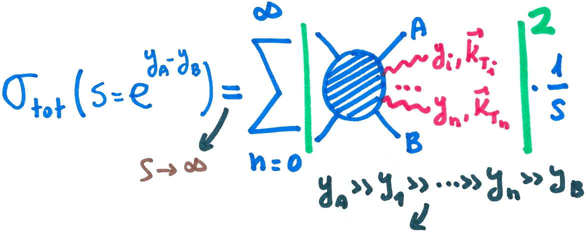
Regge theory preludes QCD. Pomeron in terms of quarks & gluons?

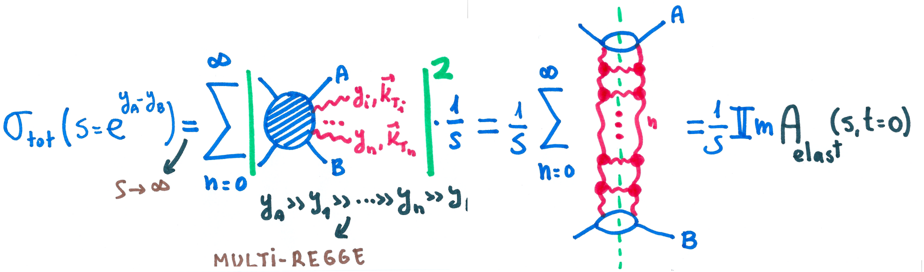
Perturbation theory with large scale $Q > \Lambda_{\rm QCD} \to \alpha_s(Q) \ll 1$.

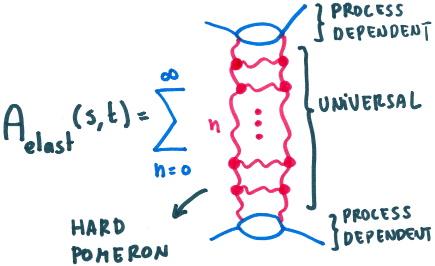
$s \gg t, Q^2 \to \alpha_s(Q) \log\left(\frac{s}{t}\right) \sim \mathcal{O}(1)$. Resummation needed.



$$\sigma_{\rm tot}^{\rm LL} = \sum_{n=0}^{\infty} \mathcal{C}_n^{\rm LL} \alpha_s^n \int_{y_B}^{y_A} dy_1 \int_{y_B}^{y_1} dy_2 \ldots \int_{y_B}^{y_{n-1}} dy_n = \sum_{n=0}^{\infty} \frac{\mathcal{C}_n^{\rm LL}}{n!} \underbrace{\alpha_s^n (y_A - y_B)^n}_{\rm LL}$$

Multi-Regge linked to elastic amplitudes via optical theorem:





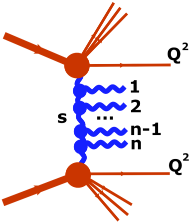
New degree of freedom
= Reggeized gluon

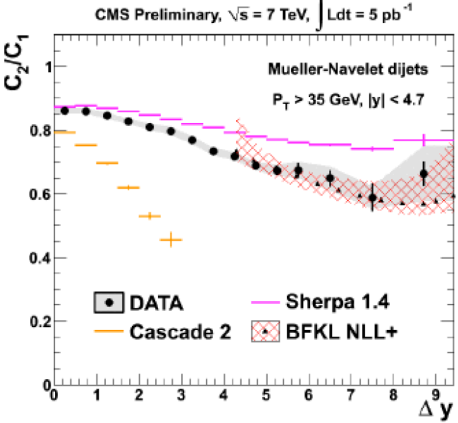
Pomeron = Composite state

2-dim interaction Hamiltonian

Observable proposed (ASV)[2006] (Schwennsen,ASV)[2007]
as ideal to pin down BFKL

$$\mathcal{R}_{2,1} = \frac{<\cos(2\theta)>}{<\cos(\theta)>}$$





Confirmed in 2013

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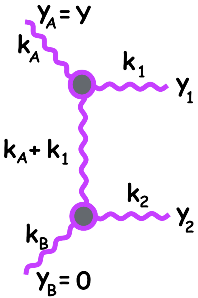
With Grigorios Chachamis

Effective Feynman rules:

Gluon Regge trajectory: $\omega\left(\vec{q}\right) = -\frac{\alpha_s N_c}{\pi} \log \frac{q^2}{\lambda^2}$

Modified propagators in the t-channel:

$$\left(\frac{s_i}{s_0}\right)^{\omega(t_i)} = e^{\omega(t_i)(y_i - y_{i+1})}$$

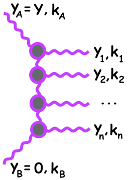


$$\left(\frac{\alpha_s N_c}{\pi}\right)^2 \int d^2\vec{k}_1 \frac{\theta\left(k_1^2 - \lambda^2\right)}{\pi k_1^2} \int d^2\vec{k}_2 \frac{\theta\left(k_2^2 - \lambda^2\right)}{\pi k_2^2} \delta^{(2)}\left(\vec{k}_A + \vec{k}_1 + \vec{k}_2 - \vec{k}_B\right)$$

$$\times \int_0^Y dy_1 \int_0^{y_1} dy_2 e^{\omega\left(\vec{k}_A\right)(Y - y_1)} e^{\omega\left(\vec{k}_A + \vec{k}_1\right)(y_1 - y_2)} e^{\omega\left(\vec{k}_A + \vec{k}_1 + \vec{k}_2\right) y_2}$$

$$\sigma(Q_1, Q_2, Y) = \int d^2\vec{k}_A d^2\vec{k}_B \underbrace{\phi_A(Q_1, \vec{k}_A)\,\phi_B(Q_2, \vec{k}_B)}_{\text{PROCESS-DEPENDENT}} \underbrace{f(\vec{k}_A, \vec{k}_B, Y)}_{\text{UNIVERSAL}}$$

$$f\left(\vec{k}_A, \vec{k}_B, Y\right) = \sum_n \left| \quad \right|^2$$



$$= e^{\omega(\vec{k}_A)Y}\left\{\delta^{(2)}\left(\vec{k}_A - \vec{k}_B\right) + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\alpha_s N_c}{\pi}\int d^2\vec{k}_i \frac{\theta\left(k_i^2 - \lambda^2\right)}{\pi k_i^2}\right.$$

$$\left.\times \int_0^{y_{i-1}} dy_i e^{\left(\omega\left(\vec{k}_A + \sum_{l=1}^{i}\vec{k}_l\right) - \omega\left(\vec{k}_A + \sum_{l=1}^{i-1}\vec{k}_l\right)\right)y_i}\delta^{(2)}\left(\vec{k}_A + \sum_{l=1}^{n}\vec{k}_l - \vec{k}_B\right)\right\}$$

BFKLex: Monte Carlo implementation of full NLO BFKL

NLO is complicated. Example, non-forward $N = 4$ SUSY in adjoint rep.:

$$\mathcal{F}(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}; \mathrm{Y}) = \left(\frac{\mathbf{q}^2\lambda^2}{\mathbf{q}_1^2\mathbf{q}_1'^2}\right)^{\frac{\bar{\alpha}}{2}\left(1-\frac{\zeta_2}{2}\bar{\alpha}\right)\mathrm{Y}} e^{\frac{3}{4}\zeta_3\bar{\alpha}^2\mathrm{Y}} \Bigg\{\delta^{(2)}(\mathbf{q}_1 - \mathbf{q}_2)$$

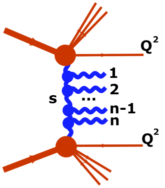
$$+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\left[\int d^2\mathbf{k}_i \frac{\bar{\alpha}}{4}\left(1-\frac{\zeta_2}{2}\bar{\alpha}\right)\frac{\theta\left(\mathbf{k}_i^2-\lambda^2\right)}{\pi\mathbf{k}_i^2}\left(1+\frac{\left(\mathbf{q}_1'+\sum_{l=1}^{i-1}\mathbf{k}_l\right)^2\left(\mathbf{q}_1+\sum_{l=1}^{i}\mathbf{k}_l\right)^2-\mathbf{q}^2\mathbf{k}_i^2}{\left(\mathbf{q}_1'+\sum_{l=1}^{i}\mathbf{k}_l\right)^2\left(\mathbf{q}_1+\sum_{l=1}^{i-1}\mathbf{k}_l\right)^2}\right)\right.$$

$$\left.+\Phi\left(\mathbf{q}_1+\sum_{l=1}^{i-1}\mathbf{k}_l, \mathbf{q}_1+\sum_{l=1}^{i}\mathbf{k}_l\right)\right]\delta^{(2)}\left(\mathbf{q}_1+\sum_{l=1}^{n}\mathbf{k}_l-\mathbf{q}_2\right)$$

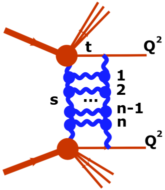
$$\times\int_0^{y_{i-1}} dy_i \left(\frac{\left(\mathbf{q}_1+\sum_{l=1}^{i-1}\mathbf{k}_l\right)^2}{\left(\mathbf{q}_1+\sum_{l=1}^{i}\mathbf{k}_l\right)^2}\right)^{1+\frac{\bar{\alpha}y_i}{2}\left(1-\frac{\zeta_2}{2}\bar{\alpha}\right)} \left(\frac{\left(\mathbf{q}_1'+\sum_{l=1}^{i-1}\mathbf{k}_l\right)^2}{\left(\mathbf{q}_1'+\sum_{l=1}^{i}\mathbf{k}_l\right)^2}\right)^{\frac{\bar{\alpha}y_i}{2}\left(1-\frac{\zeta_2}{2}\bar{\alpha}\right)}\Bigg\}$$

$$
\begin{aligned}
\Phi(\mathbf{q}_1, \mathbf{q}_1+\mathbf{k}) &= \\
&\frac{\bar{\alpha}^2}{32\pi}\frac{1}{\mathbf{q}_1^2(\mathbf{k}+\mathbf{q}_1')^2}\Bigg\{\mathbf{q}^2\left[\ln\left(\frac{\mathbf{q}_1^2}{\mathbf{q}^2}\right)\ln\left(\frac{\mathbf{q}_1'^2}{\mathbf{q}^2}\right)+\ln\left(\frac{(\mathbf{q}_1+\mathbf{k})^2}{\mathbf{q}^2}\right)\ln\left(\frac{(\mathbf{q}_1'+\mathbf{k})^2}{\mathbf{q}^2}\right)\right.\\
&\left.+\frac{1}{2}\ln^2\left(\frac{\mathbf{q}_1^2}{(\mathbf{q}_1+\mathbf{k})^2}\right)+\frac{1}{2}\ln^2\left(\frac{\mathbf{q}_1'^2}{(\mathbf{q}_1'+\mathbf{k})^2}\right)\right]+\frac{1}{2}\frac{\left(\mathbf{q}_1^2(\mathbf{q}_1'+\mathbf{k})^2-\mathbf{q}_1'^2(\mathbf{q}_1+\mathbf{k})^2\right)}{\mathbf{k}^2}\\
&\times\left[\ln\left(\frac{\mathbf{q}_1'^2}{(\mathbf{q}_1'+\mathbf{k})^2}\right)\ln\left(\frac{\mathbf{q}_1'^2(\mathbf{q}_1'+\mathbf{k})^2}{\mathbf{k}^4}\right)-\ln\left(\frac{\mathbf{q}_1^2}{(\mathbf{q}_1+\mathbf{k})^2}\right)\ln\left(\frac{\mathbf{q}_1^2(\mathbf{q}_1+\mathbf{k})^2}{\mathbf{k}^4}\right)\right]\\
&-\frac{\left(\mathbf{q}_1^2(\mathbf{q}_1'+\mathbf{k})^2+\mathbf{q}_1'^2(\mathbf{q}_1+\mathbf{k})^2\right)}{\mathbf{k}^2}\left[\ln^2\left(\frac{\mathbf{q}_1^2}{(\mathbf{q}_1+\mathbf{k})^2}\right)+\ln^2\left(\frac{\mathbf{q}_1'^2}{(\mathbf{q}_1'+\mathbf{k})^2}\right)\right]\\
&+\left[\mathbf{q}^2\left(\mathbf{k}^2-\mathbf{q}_1^2-(\mathbf{q}_1+\mathbf{k})^2\right)+2\mathbf{q}_1^2(\mathbf{q}_1+\mathbf{k})^2-\mathbf{q}_1^2(\mathbf{q}_1'+\mathbf{k})^2-\mathbf{q}_1'^2(\mathbf{q}_1+\mathbf{k})^2\right.\\
&\left.+\frac{\left(\mathbf{q}_1^2(\mathbf{q}_1'+\mathbf{k})^2-\mathbf{q}_1'^2(\mathbf{q}_1+\mathbf{k})^2\right)}{\mathbf{k}^2}\left(\mathbf{q}_1^2-(\mathbf{q}_1+\mathbf{k})^2\right)\right]\mathcal{I}\left(\mathbf{q}_1^2,(\mathbf{q}_1+\mathbf{k})^2,\mathbf{k}^2\right)\\
&+\left[\mathbf{q}^2\left(\mathbf{k}^2-\mathbf{q}_1'^2-(\mathbf{q}_1'+\mathbf{k})^2\right)+2\mathbf{q}_1'^2(\mathbf{q}_1'+\mathbf{k})^2-\mathbf{q}_1'^2(\mathbf{q}_1+\mathbf{k})^2-\mathbf{q}_1^2(\mathbf{q}_1'+\mathbf{k})^2\right.\\
&\left.+\frac{\left(\mathbf{q}_1'^2(\mathbf{q}_1+\mathbf{k})^2-\mathbf{q}_1^2(\mathbf{q}_1'+\mathbf{k})^2\right)}{\mathbf{k}^2}\left(\mathbf{q}_1'^2-(\mathbf{q}_1'+\mathbf{k})^2\right)\right]\mathcal{I}\left(\mathbf{q}_1'^2,(\mathbf{q}_1'+\mathbf{k})^2,\mathbf{k}^2\right)\Bigg\}
\end{aligned}
$$

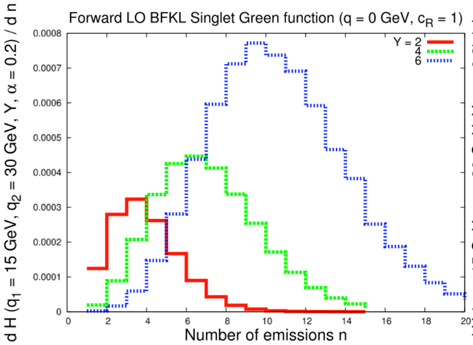
$$
\mathcal{I}\left(\mathbf{p}^2,\mathbf{q}^2,\mathbf{r}^2\right) = \int_0^1\frac{dx}{\mathbf{p}^2(1-x)+\mathbf{q}^2x-\mathbf{r}^2x(1-x)}\ln\left(\frac{\mathbf{p}^2(1-x)+\mathbf{q}^2x}{\mathbf{r}^2x(1-x)}\right).
$$

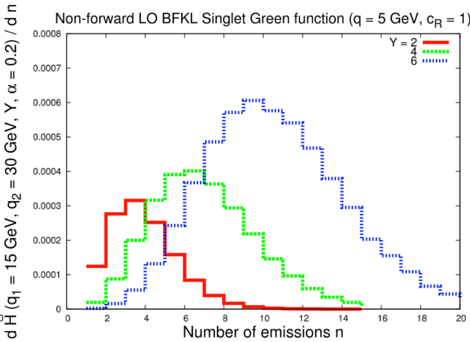


Cut Pomeron: Number of emissions?

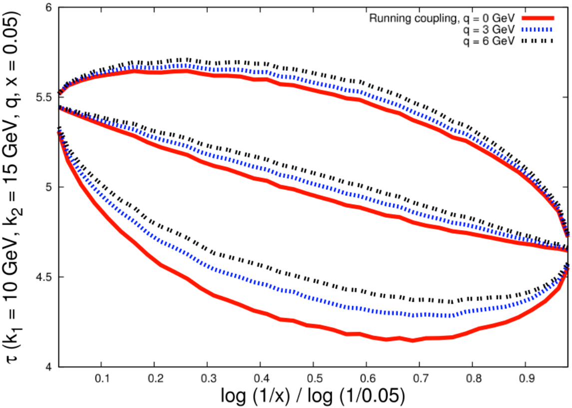


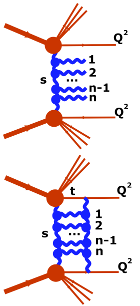
Pomeron: Number of rungs?





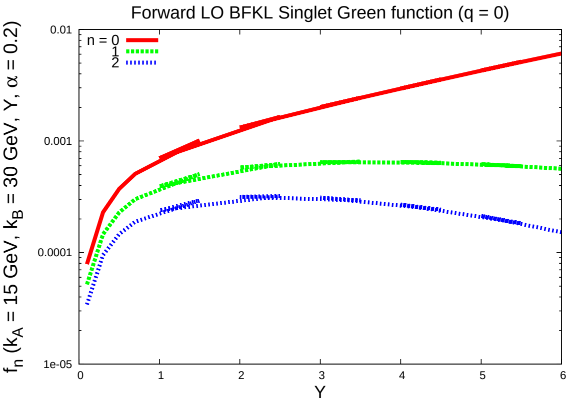
Reggeized (virtual) gluon $|p_T|$ at a given rapidity?





Growth with energy? Depends on the azimuthal angle Fourier component:

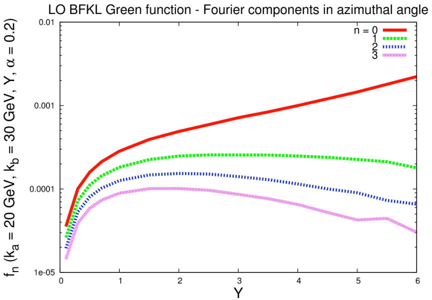
$$f_n\left(|\vec{k}_A|, |\vec{k}_B|, Y\right) = \int_0^{2\pi} \frac{d\theta}{2\pi} f\left(\vec{k}_A, \vec{k}_B, Y\right) \cos\left(n\,\theta\right)$$

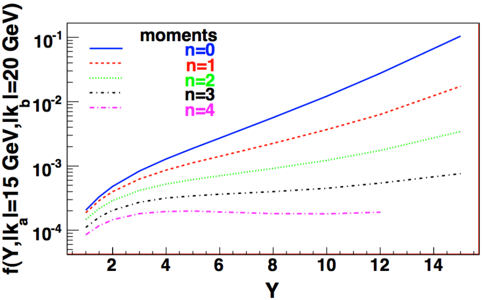


This is a distinct feature of BFKL

BFKL

CCFM





All CCFM projections grow with energy, not in BFKL - 1102.1890

Observables only sensitive to $n > 0$ single out BFKL

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With Grigorios Chachamis
1512.03603 (JHEP)
1511.03548 (PRD)

We can extend the formalism to include collinear regions

$$f = e^{\omega\left(\vec{k}_A\right)Y}\left\{\delta^{(2)}\left(\vec{k}_A-\vec{k}_B\right)+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\alpha_s N_c}{\pi}\int d^2\vec{k}_i\frac{\theta\left(k_i^2-\lambda^2\right)}{\pi k_i^2}\right.$$
$$\left.\times\int_0^{y_{i-1}} dy_i e^{\left(\omega\left(\vec{k}_A+\sum_{l=1}^{i}\vec{k}_l\right)-\omega\left(\vec{k}_A+\sum_{l=1}^{i-1}\vec{k}_l\right)\right)y_i}\delta^{(2)}\left(\vec{k}_A+\sum_{l=1}^{n}\vec{k}_l-\vec{k}_B\right)\right\}$$

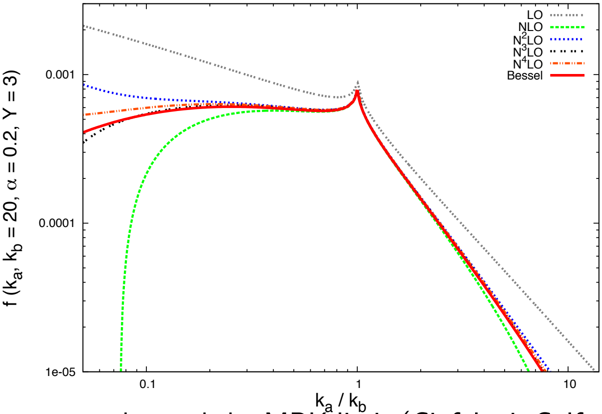
Key at NLL: $\theta\left(k_i^2-\lambda^2\right)\to\theta\left(k_i^2-\lambda^2\right)\underbrace{-\frac{\bar{\alpha}_s}{4}\ln^2\left(\frac{\vec{k}_A^2}{\left(\vec{k}_A+\vec{k}_i\right)^2}\right)}_{\text{NLL}}$

Resum it to all orders (SV-0507317):

$$\theta\left(k_i^2-\lambda^2\right)\to\theta\left(k_i^2-\lambda^2\right)+\sum_{n=1}^{\infty}\frac{(-\bar{\alpha}_s)^n}{2^n n!(n+1)!}\ln^{2n}\left(\frac{\vec{k}_A^2}{\left(\vec{k}_A+\vec{k}_i\right)^2}\right)$$

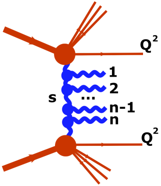
It corresponds to a Bessel function $J_1\left(\sqrt{2\bar{\alpha}_s\ln^2\left(\frac{\vec{k}_A^2}{\left(\vec{k}_A+\vec{k}_i\right)^2}\right)}\right)$

$$\sigma(Q_1, Q_2, Y) = \int d^2\mathbf{k}_a d^2\mathbf{k}_b \, \phi_A(Q_1, \mathbf{k}_a) \, \phi_B(Q_2, \mathbf{k}_b) \, f(\mathbf{k}_a, \mathbf{k}_b, Y)$$



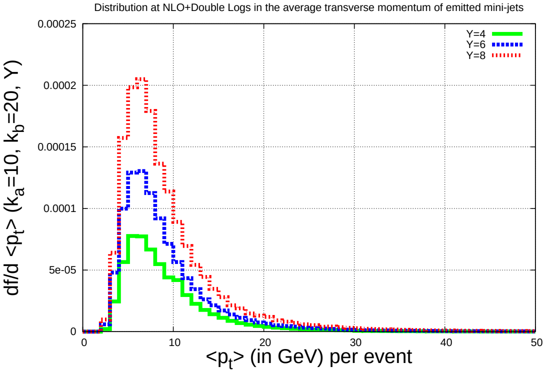
Important to go beyond the MRK limit (Ciafaloni, Colferai, Salam, Stasto).
For BFKL domain we need "δ-like" impact factors $\phi_{A,B}$ & $Q_1 \simeq Q_2$.

Remarks on Jet production & Odderon in the Multi-Regge Limit



Average transverse momentum of emitted mini-jets?

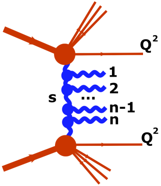
$$\langle p_t \rangle \quad = \quad \frac{1}{N} \sum_{i=1}^{N} |k_i|$$

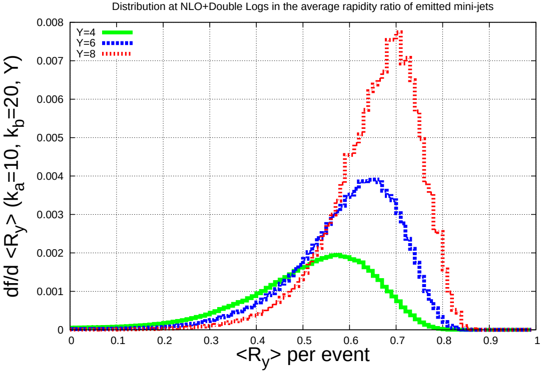


Average rapidity separation among emitted mini-jets?

$$\langle \mathcal{R}_y \rangle = \frac{1}{N+1} \sum_{i=1}^{N+1} \frac{y_i}{y_{i-1}} \quad \simeq 1 + \frac{\Delta}{Y} \ln \frac{\Delta}{Y}$$

if $Y \simeq N\Delta$ in MRK and $Y \gg \Delta$





Higher $\langle \mathcal{R}_y \rangle_{\max}$ for higher energies: $\Delta_{\rm LO} \simeq 0.62$, $\Delta_{\rm LO+DLs} \simeq 0.81$

Lower mini-jet multiplicity when including higher order corrections

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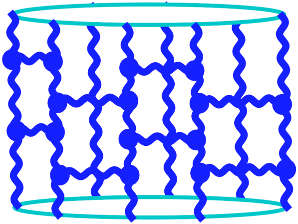
1801.04872 (EPJC)
1606.07349 (PRD)

Amplitudes in the Generalized Leading Logarithmic Approximation (GLLA)



This is an old standing problem in High Energy QCD
Mapped onto a Closed Spin Chain
(Lipatov, Faddeev, Korchemsky,
Janik, Wosiek, Kotanski, Derkachov, Manashov ...)

Monte Carlo integration can be applied in this case (Chachamis-ASV)[2016]
Let us consider singlet exchange in t-channel with 3 Reggeized gluons:

ODDERON

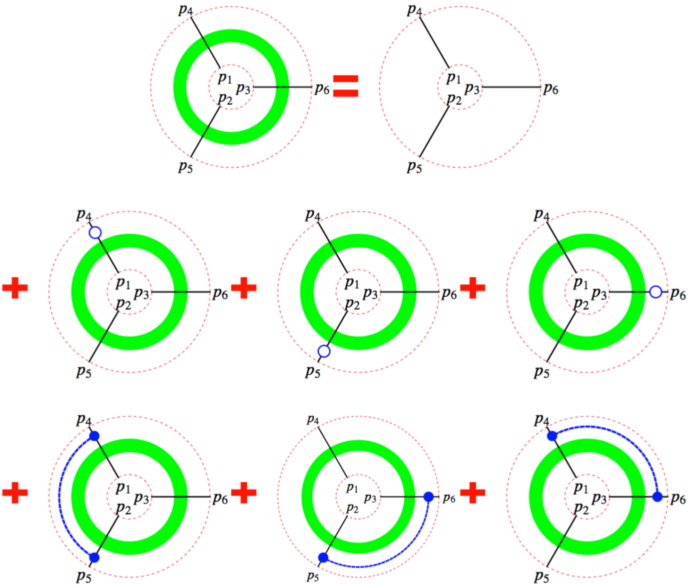
Solution of the IR-finite Bartels-Kwiecinski-Praszalowicz equation:

$$\begin{aligned}(\omega - \omega(\mathbf{p}_1) - \omega(\mathbf{p}_2) - \omega(\mathbf{p}_3))\, f_\omega(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \quad &= \\ \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_4)\, \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_5)\, \delta^{(2)}(\mathbf{p}_3 - \mathbf{p}_6) \\ + \int d^2\mathbf{k}\, \xi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{k})\, f_\omega(\mathbf{p}_1 + \mathbf{k}, \mathbf{p}_2 - \mathbf{k}, \mathbf{p}_3) \\ + \int d^2\mathbf{k}\, \xi(\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_1, \mathbf{k})\, f_\omega(\mathbf{p}_1, \mathbf{p}_2 + \mathbf{k}, \mathbf{p}_3 - \mathbf{k}) \\ + \int d^2\mathbf{k}\, \xi(\mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_2, \mathbf{k})\, f_\omega(\mathbf{p}_1 + \mathbf{k}, \mathbf{p}_2, \mathbf{p}_3 - \mathbf{k})\end{aligned}$$

Square of Lipatov's emission vertex:

$$\xi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{k}) \; = \frac{\alpha_s N_c}{4} \frac{\theta(\mathbf{k}^2 - \lambda^2)}{\pi^2 \mathbf{k}^2} \left(1 + \frac{(\mathbf{p}_1 + \mathbf{k})^2 \mathbf{p}_2^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 \mathbf{k}^2}{\mathbf{p}_1^2 (\mathbf{k} - \mathbf{p}_2)^2}\right)$$

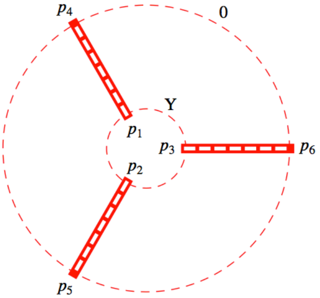
Gluon Regge trajectory: $\omega(\mathbf{p}) = -\frac{\bar{\alpha}_s}{2} \ln \frac{\mathbf{p}^2}{\lambda^2}$


p_4 p_1 p_3 p_6 p_2 p_5 = p_4 p_1 p_3 p_6 p_2 p_5
+ p_4 p_1 p_3 p_6 p_2 p_5 + p_4 p_1 p_3 p_6 p_2 p_5 + p_4 p_1 p_3 p_6 p_2 p_5
+ p_4 p_1 p_3 p_6 p_2 p_5 + p_4 p_1 p_3 p_6 p_2 p_5 + p_4 p_1 p_3 p_6 p_2 p_5

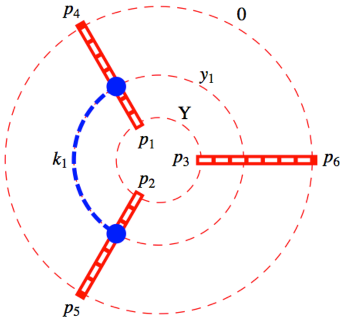
$$f\left(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,Y\right) = \int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}e^{\omega Y}f_\omega\left(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\right)$$

$$\begin{aligned} f\left(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,Y\right) &= \\ & e^{(\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)+\omega(\mathbf{p}_3))Y}\delta^{(2)}\left(\mathbf{p}_1-\mathbf{p}_4\right)\delta^{(2)}\left(\mathbf{p}_2-\mathbf{p}_5\right)\delta^{(2)}\left(\mathbf{p}_3-\mathbf{p}_6\right) \\ & +\sum_{n=1}^{\infty}\left\{\prod_{i=1}^{n}\int_0^{y_{i-1}}dy_i\int d^2\mathbf{k}_i e^{(\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)+\omega(\mathbf{p}_3))(y_{i-1}-y_i)}\mathcal{O}(\mathbf{k}_i)\otimes\right\} \\ & e^{(\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)+\omega(\mathbf{p}_3))y_n}\delta^{(2)}\left(\mathbf{p}_1-\mathbf{p}_4\right)\delta^{(2)}\left(\mathbf{p}_2-\mathbf{p}_5\right)\delta^{(2)}\left(\mathbf{p}_3-\mathbf{p}_6\right) \end{aligned}$$

$$\begin{aligned} \mathcal{O}(\mathbf{k})\otimes f\left(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\right) &\equiv \xi\left(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{k}\right)f\left(\mathbf{p}_1+\mathbf{k},\mathbf{p}_2-\mathbf{k},\mathbf{p}_3\right) \\ &+ \xi\left(\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_1,\mathbf{k}\right)f\left(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k},\mathbf{p}_3-\mathbf{k}\right) \\ &+ \xi\left(\mathbf{p}_1,\mathbf{p}_3,\mathbf{p}_2,\mathbf{k}\right)f\left(\mathbf{p}_1+\mathbf{k},\mathbf{p}_2,\mathbf{p}_3-\mathbf{k}\right) \end{aligned}$$

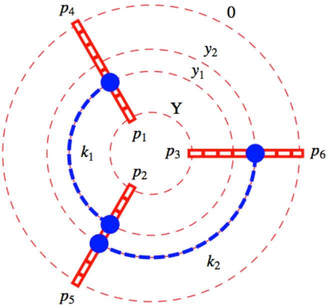


$$= \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_4)\, \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_5)\, \delta^{(2)}(\mathbf{p}_3 - \mathbf{p}_6)\, e^{(\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)+\omega(\mathbf{p}_3))Y}$$



$$= \int d^2\mathbf{k}_1 \int_0^Y dy_1 \delta^{(2)}\left(\mathbf{p}_3 - \mathbf{p}_6\right) \delta^{(2)}\left(\mathbf{k}_1 + \mathbf{p}_1 - \mathbf{p}_4\right) \delta^{(2)}\left(-\mathbf{k}_1 + \mathbf{p}_2 - \mathbf{p}_5\right)$$

$$\times \quad \xi\left(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{k}_1\right) e^{\omega(\mathbf{p}_3)Y} e^{(\omega(\mathbf{k}_1+\mathbf{p}_1)+\omega(\mathbf{p}_2-\mathbf{k}_1))y_1} e^{(\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2))(Y-y_1)}$$



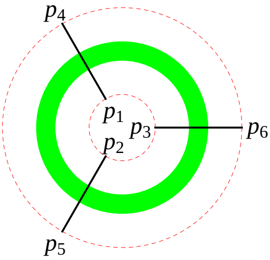
$$= \int d^2\mathbf{k}_1 \int d^2\mathbf{k}_2 \int_0^Y dy_1 \int_0^{y_1} dy_2 \delta^{(2)}\left(k_1 + p_1 - p_4\right) \delta^{(2)}\left(-k_1 + k_2 + p_2 - p_5\right)$$

$$\times \quad \delta^{(2)}\left(-k_2 + p_3 - p_6\right) \xi\left(p_1, p_2, p_3, k_1\right) \xi\left(p_2 - k_1, p_3, k_1 + p_1, k_2\right)$$

$$\times \quad e^{(\omega(p_1)+\omega(p_2))(Y-y_1)} e^{\omega(p_3)(Y-y_2)} e^{\omega(k_1+p_1)y_1} e^{\omega(p_2-k_1)(y_1-y_2)} e^{(\omega(-k_1+k_2+p_2)+\omega(p_3-k_2))y_2}$$

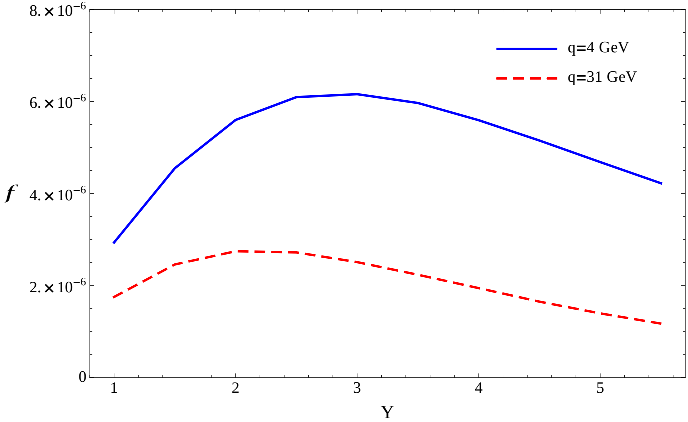
We can pick up two configurations with different transferred momentum:

$$\begin{aligned}
\mathbf{q} &= (4,0) \\
\mathbf{p}_1 &= (10,0) \\
\mathbf{p}_2 &= (20,\pi) \\
\mathbf{p}_3 &= (\mathbf{q}-\mathbf{p}_1)-\mathbf{p}_2 = (14,0) \\
\mathbf{p}_4 &= (20,0) \\
\mathbf{p}_5 &= (25,\pi) \\
\mathbf{p}_6 &= (\mathbf{q}-\mathbf{p}_4)-\mathbf{p}_5 = (9,0)
\end{aligned}
\qquad
\begin{aligned}
\mathbf{q} &= (31,0) \\
\mathbf{p}_1 &= (10,0) \\
\mathbf{p}_2 &= (20,\pi) \\
\mathbf{p}_3 &= (\mathbf{q}-\mathbf{p}_1)-\mathbf{p}_2 = (41,0) \\
\mathbf{p}_4 &= (20,0) \\
\mathbf{p}_5 &= (25,\pi) \\
\mathbf{p}_6 &= (\mathbf{q}-\mathbf{p}_4)-\mathbf{p}_5 = (36,0).
\end{aligned}$$



and study its growth with energy ...

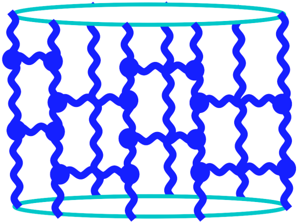
Our solution must contain previous solutions in the literature.
They are singled out by particular impact factors.

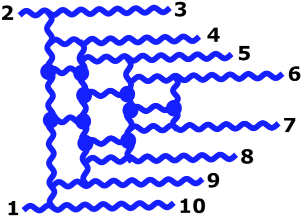


On-going work on phenomenological applications ...

Very similar Reggeon diagrams appear in $N = 4$ SUSY amplitudes (Bartels, Lipatov, SV)[2009,2010]

Most complicated contributions at higher order & arbitrary number of legs:





Mapped onto an Open Spin Chain (Lipatov)
MC integration can be applied also here (Chachamis-ASV)[2018]
Let us consider octet exchange in t-channel with 3 Reggeized gluons:

EIGHT-POINT MHV AMPLITUDE

Solution of the IR-divergent BKP-like equation:

$$\begin{aligned}(\omega - \omega(\mathbf{p}_1) - \omega(\mathbf{p}_2) - \omega(\mathbf{p}_3))\, f_\omega(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \quad = \\ \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_4)\,\delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_5)\,\delta^{(2)}(\mathbf{p}_3 - \mathbf{p}_6) \\ + \int d^2\mathbf{k}\, \xi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{k})\, f_\omega(\mathbf{p}_1 + \mathbf{k}, \mathbf{p}_2 - \mathbf{k}, \mathbf{p}_3) \\ + \int d^2\mathbf{k}\, \xi(\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_1, \mathbf{k})\, f_\omega(\mathbf{p}_1, \mathbf{p}_2 + \mathbf{k}, \mathbf{p}_3 - \mathbf{k})\end{aligned}$$

is

$$\begin{aligned}f_{\mathrm{BKP}}^{\mathrm{adjoint}}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, Y) \quad = \\ e^{(\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)+\omega(\mathbf{p}_3))Y}\,\delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_4)\,\delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_5)\,\delta^{(2)}(\mathbf{p}_3 - \mathbf{p}_6) \\ + \sum_{n=1}^{\infty} \left\{ \prod_{i=1}^{n} \int_0^{y_{i-1}} dy_i \int d^2\mathbf{k}_i\, e^{(\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)+\omega(\mathbf{p}_3))(y_{i-1}-y_i)} \mathcal{O}(\mathbf{k}_i) \otimes \right\} \\ e^{(\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)+\omega(\mathbf{p}_3))y_n}\,\delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_4)\,\delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_5)\,\delta^{(2)}(\mathbf{p}_3 - \mathbf{p}_6)\end{aligned}$$

with

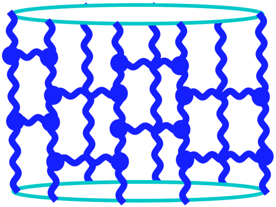
$$\begin{aligned}\mathcal{O}(\mathbf{k}) \otimes f(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \quad &\equiv \quad \xi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{k})\, f(\mathbf{p}_1 + \mathbf{k}, \mathbf{p}_2 - \mathbf{k}, \mathbf{p}_3) \\ &+ \quad \xi(\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_1, \mathbf{k})\, f(\mathbf{p}_1, \mathbf{p}_2 + \mathbf{k}, \mathbf{p}_3 - \mathbf{k})\end{aligned}$$

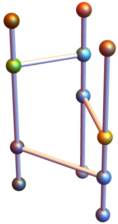
The IR divergencies factor out in a simple form:

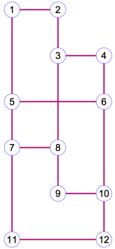
$$f_{\rm BKP}^{\rm adjoint}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,{\rm Y}) \quad = \quad \left(\frac{\lambda^2}{\sqrt{\mathbf{p}_1^2\mathbf{p}_3^2}}\right)^{\frac{\bar{\alpha}_s {\rm Y}}{2}} \underbrace{\widehat{f}_{\rm BKP}^{\rm adjoint}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,{\rm Y})}_{\rm IR-FINITE}$$

where $\widehat{f}_{\rm BKP}^{\rm adjoint}$ is finite when $\lambda \to 0$.

Associated Laplacian matrix: Closed Chain with 3 Reggeons (Odderon)

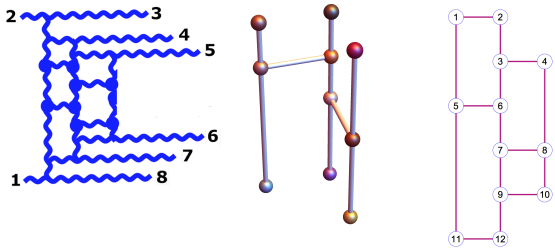






$$\boxed{L} = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 3 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 3 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

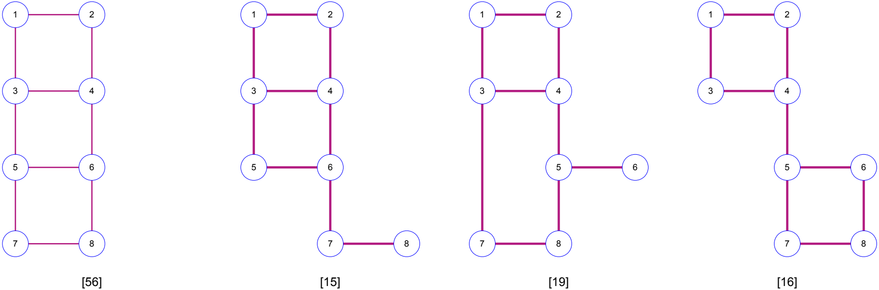
Associated Laplacian matrix: Open Chain with 3 Reggeons

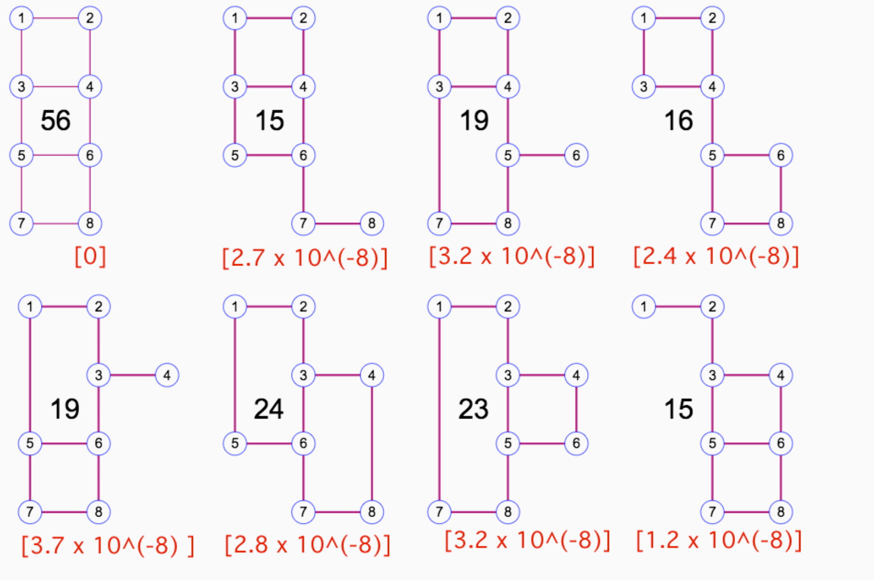


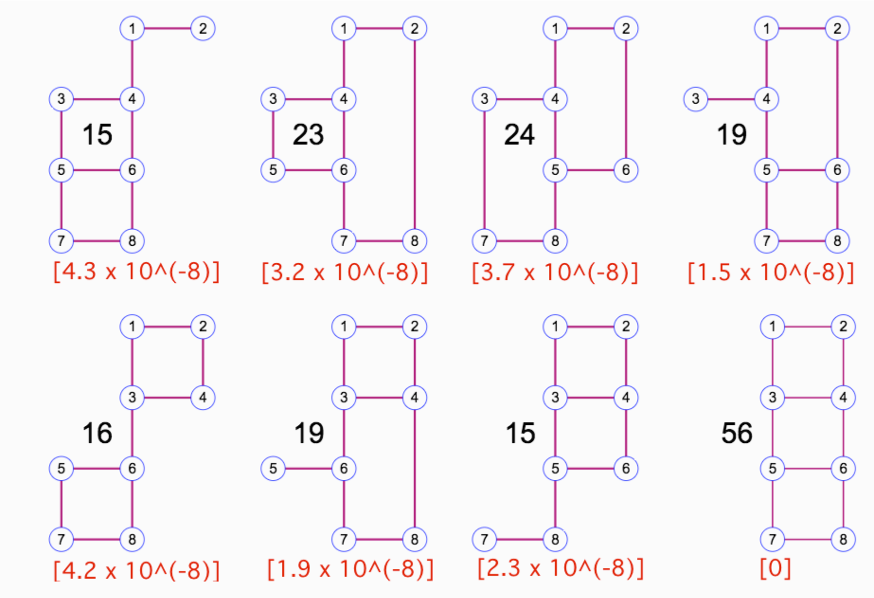
$$
\boxed{L} = \begin{pmatrix}
2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 3 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 3 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2
\end{pmatrix}
$$

Graph Complexity: Number of possible spanning trees
(Connected diagrams crossing all nodes with no loops)

Matrix Tree theorem by Kirchoff: determinant of any principal minor of L




56
[0]
15
[2.7 x 10^(-8)]
19
[3.2 x 10^(-8)]
16
[2.4 x 10^(-8)]
19
[3.7 x 10^(-8)]
24
[2.8 x 10^(-8)]
23
[3.2 x 10^(-8)]
15
[1.2 x 10^(-8)]

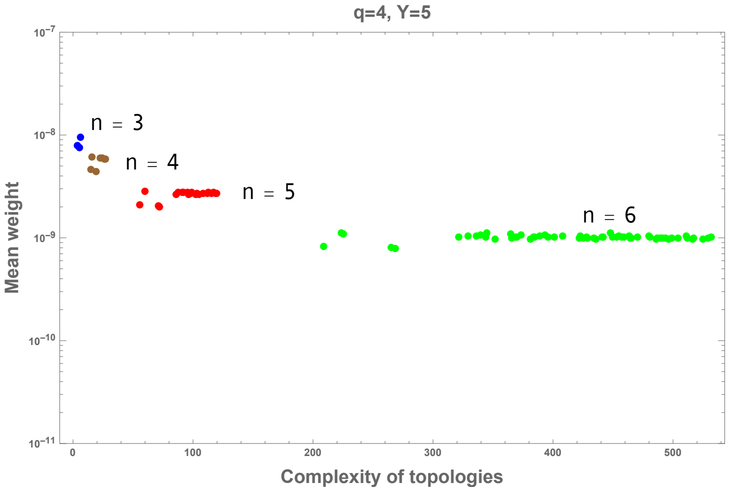

15
[4.3 x 10^(-8)]
23
[3.2 x 10^(-8)]
24
[3.7 x 10^(-8)]
19
[1.5 x 10^(-8)]
16
[4.2 x 10^(-8)]
19
[1.9 x 10^(-8)]
15
[2.3 x 10^(-8)]
56
[0]

Number of Rungs = 4

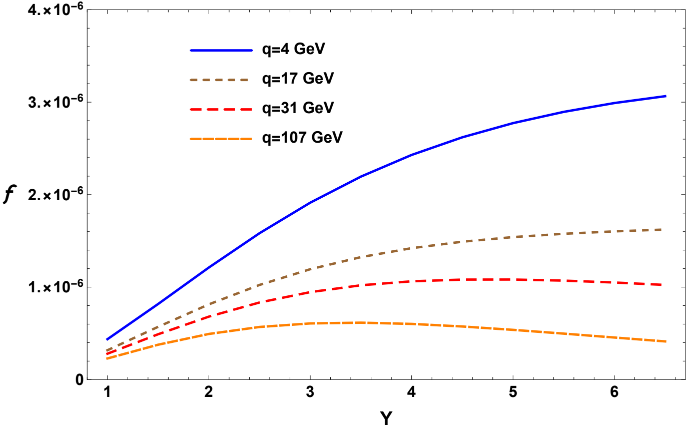
Complexity	# diagrams	Average weight in GGF
15	4	2.6×10^{-8}
16	2	3.3×10^{-8}
19	4	2.6×10^{-8}
23	2	3.2×10^{-8}
24	2	3.3×10^{-8}
56	2	0

A "Complexity Democracy" emerges ...

Average weight per complexity class for Reggeon webs.
Emerging scaling.



Due to integrability?



This is the solution to the "adjoint Odderon"
(open spin chain) relevant in $N = 4$ SUSY amplitudes.

1. Multi-Regge limit
2. Monte Carlo event generator BFKLex
3. Collinear double logs
4. Solution of BKP equation & open spin chains