NLO BFKL and anomalous dimensions of light-ray operators

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Outline

- Light-ray operators.
- Regge limit in the coordinate space.
- "BFKL" representation of 4-point correlation function in $\mathcal{N} = 4$ SYM.
- "DGLAP" representation of 4-point correlation function.
- Anomalous dimensions from DGAP vs BFKL representations.

Problem

Gluon operators of leading twist

$$\mathcal{O}_n^g \equiv F_{-i}^a \nabla_-^{n-2} F_-^{ai}$$

Anomalous dimension (in gluodynamics)

$$\gamma_n = \frac{2}{\pi}\alpha_s N_c \Big[-\frac{1}{n(n-1)} - \frac{1}{(n+1)(n+2)} + \psi(n+1) + \gamma_E - \frac{11}{12} \Big] + O(\alpha_s^2)$$

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BFKL gives $\gamma(n\alpha_s)$ at the non-physical point $n \to 1$

$$\gamma_n = \Big[A_n^{\text{LO BFKL}} + \omega B_n^{\text{NLO BFKL}} + ...\Big]\Big(\frac{\alpha_s N_c}{\pi\omega}\Big)^n \qquad \omega \equiv n - 1$$

LO: Jaroszewicz (1980), NLO: Lipatov, Fadin, Camici, Ciafaloni (1998)

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Q: which one?   A: gluon light-ray (LR) operator

Light-ray operators and parton densities

Gluon light-ray (LR) operator of twist 2

$$F^a_{-i}(x'_+ + x_\perp)[x'_+, x_+]^{ab}F^{b\ i}_-(x_+ + x_\perp)$$

Forward matrix element - gluon parton density

$$z^\mu z^\nu \langle p|F^a_{\mu\xi}(z)[z, 0]^{ab}F^{b\xi}_\nu(0)|p\rangle^\mu \overset{z^2=0}{=} 2(pz)^2 \int_0^1 dx_B \ x_B D_g(x_B, \mu) \cos(pz)x_B$$

Evolution equation (in gluodynamics)

$$\mu^2 \frac{d}{d\mu^2} F^a_{-i}(x'_+ + x_\perp)[x'_+, x_+]^{ab}F^{b\ i}_-(x_+ + x_\perp)$$

$$= \int_{x_+}^{x'_+} dz'_+ \int_{x_+}^{z'_+} dz_+ \ K(x'_+, x_+; z'_+, z_+; \alpha_s)F^a_{-i}(z'_+ + x_\perp)[z'_+, z_+]^{ab}F^{b\ i}_-(z_+ + x_\perp)$$

"Forward" LR operator

$$F(L_+, x_\perp) = \int dx_+ \ F^a_{-i}(L_+ + x_+ + x_\perp)[L_+ + x_+, x_+]^{ab}F^{bi}_-(x_+ + x_\perp)$$

Light-ray operators and parton densities

Expansion in ("forward") local operators

$$F(L_+, x_\perp) = \sum_{n=2}^{\infty} \frac{L_+^{n-2}}{(n-2)!} \mathcal{O}_n^g(x_\perp), \quad \mathcal{O}_n^g \equiv \int dx_+ F_{-i}^a \nabla_-^{n-2} F_-^{ai}(x_+, x_\perp)$$

Evolution equation for $F(L_+, x_\perp)$

$$\mu \frac{d}{d\mu} F(L_+, x_\perp) = \int_0^1 du \, K_{gg}(u, \alpha_s) F(uL_+, x_\perp)$$

$$\Rightarrow \gamma_n(\alpha_s) = -\int_0^1 du \, u^{n-2} K_{gg}(u, \alpha_s) \qquad \mu \frac{d}{d\mu} \mathcal{O}_n^g = -\gamma_n(\alpha_s) \mathcal{O}_n^g$$

$u^{-1} K_{gg}$ - DGLAP kernel

$$u^{-1} K_{gg}(u) = \frac{2\alpha_s N_c}{\pi} \left(\bar{u}u + \left[\frac{1}{\bar{u}u} \right]_+ - 2 + \frac{11}{12} \delta(\bar{u}) \right) + \text{higher orders in } \alpha_s$$

Conformal LR operator ($j = \frac{1}{2} + i\nu$)

$$F^\mu(L_+, x_\perp) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} (L_+)^{-\frac{3}{2} + i\nu} \mathcal{F}^\mu_{\frac{1}{2} + i\nu}(x_\perp)$$

$$\mathcal{F}^\mu_j(x_\perp) = \int_0^\infty dL_+ \, L_+^{1-j} F^\mu(L_+, x_\perp)$$

Evolution equation for "forward" conformal light-ray operators

$$\Rightarrow \mu^2 \frac{d}{d\mu^2} \mathcal{F}_j(z_\perp) = \int_0^1 du \, K_{gg}(u, \alpha_s) u^{j-2} \mathcal{F}_j(z_\perp)$$

$\Rightarrow \gamma_j(\alpha_s)$ is an analytical continuation of $\gamma_n(\alpha_s)$

4-point correlation function (CF)

$$A(x,y,x',y') = [(x-y)^2(x'-y')^2]^{2+\gamma_k} N_c^2 \langle \mathcal{O}(x)\mathcal{O}(y)\mathcal{O}(x')\mathcal{O}(y')\rangle$$

$\mathcal{O} = \phi_I^a \phi_I^a$ - Konishi operator (γ_k - anomalous dimension)

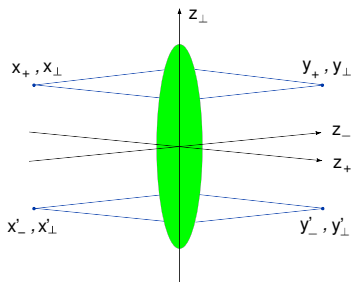In a conformal theory the amplitude is a function of two conformal ratios

$$
\begin{aligned}
A &= F(R,R') \\
R &= \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2}, \qquad R' = \frac{(x-y)^2(x'-y')^2}{(x-y')^2(x'-y)^2}
\end{aligned}
$$

At large N_c

$$A(x,y,x',y') = A(g^2 N_c) \qquad\qquad g^2 N_c = \lambda \; - \; \text{'t Hooft coupling}$$

Regge limit in the coordinate space

Regge limit: $x_+ \to \rho x_+$, $x'_+ \to \rho x'_+$, $y_- \to \rho' y_-$, $y'_- \to \rho' y_{--}$ $\rho, \rho' \to \infty$
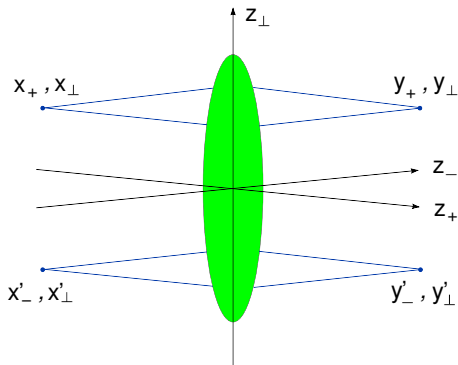

Full 4-dim conformal group: $A = F(R, r)$

$$R = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2} \to \frac{\rho^2\rho'^2 x_+ x'_+ y_- y'_-}{(x-x')^2_\perp (y-y')^2_\perp} \to \infty$$

$$r = \frac{[(x-y)^2(x'-y')^2 - (x'-y)^2(x-y')^2]^2}{(x-x')^2(y-y')^2(x-y)^2(x'-y')^2}$$

$$\to \frac{[(x'-y')^2_\perp x_+ y_- + x'_+ y'_-(x-y)^2_\perp + x_+ y'_-(x'-y)^2_\perp + x'_+ y_-(x-y')^2_\perp]^2}{(x-x')^2_\perp (y-y')^2_\perp x_+ x'_+ y_- y'_-}$$

Regge limit: $x_+ \to \rho x_+$, $x'_+ \to \rho x'_+$, $y_- \to \rho' y_-$, $y'_- \to \rho' y_{--}$
  $\rho, \rho' \to \infty$

Regge limit symmetry: 2-dim conformal group $SL(2, C)$ formed from
P_1, P_2, M^{12}, D, K_1 and K_2 which leave the plane $(0, 0, z_\perp)$ invariant.

Pomeron in a conformal theory

$$A(x, y; x', y') \overset{s \to \infty}{=} \frac{i}{2} \int d\nu \, f_+(\aleph(\lambda, \nu)) F(\lambda, \nu) \Omega(r, \nu) R^{\aleph(\lambda,\nu)/2}$$

L. Cornalba (2007)

$f_+(\omega) = \frac{e^{i\pi\omega} - 1}{\sin \pi \omega}$ - signature factor

$$\Omega(r, \nu) = \frac{\sin \nu \rho}{\sinh \rho}, \qquad \cosh \rho = \frac{\sqrt{r}}{2}$$

- solution of the eqn $(\Box_{H_3} + \nu^2 + 1)\Omega(r, \nu) = 0$

The dynamics is described by:
$\aleph(\lambda, \nu)$ - pomeron intercept,
and
$F(\lambda, \nu)$ - "pomeron residue".

$$A(x, y; x', y') \overset{s \to \infty}{=} \frac{i}{2} \int d\nu \, f_+(\aleph(\lambda, \nu)) F(\lambda, \nu) \Omega(r, \nu) R^{\aleph(\lambda, \nu)/2}$$

Pomeron intercept:

$$\aleph(\nu, \lambda) = \frac{\lambda}{4\pi^2} \left[\chi(\nu) + \frac{\lambda}{16\pi^2} \delta(\nu) \right] + O(\lambda^3)$$

$\chi(\nu) = 2\psi(1) - \psi(\frac{1}{2} + i\nu) - \psi(\frac{1}{2} - i\nu) = -S_1(-\frac{1}{2} + i\nu) + \text{c.c.}$ - BFKL intercept,

$\delta(\nu) = -\frac{3}{2}\zeta(3) + \pi^2 \ln 2 + \frac{\pi^2}{3} S_1(-\frac{1}{2} + i\nu) + 2S_3(-\frac{1}{2} + i\nu) + \pi^2 S_{-1}(-\frac{1}{2} + i\nu) - 4S_{-2,1}(-\frac{1}{2} + i\nu) + \text{c.c.}$

- NLO BFKL intercept $S_i(x)$ - harmonic sums   Lipatov, Kotikov (2000)

"Pomeron residue" $F(\nu, \lambda)$:

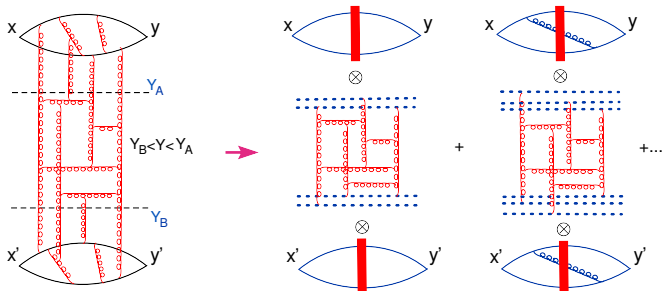$$F(\nu, \lambda) = \lambda^2 F_0(\nu) + \lambda^3 F_1(\nu) + ...$$

$F_0(\nu) = \frac{\pi \sinh \pi\nu}{4\nu \cosh^3 \pi\nu}$ Cornalba, Costa, Penedones (2007)

$F_1(\nu)$ = see below   G. Chirilli and I.B. (2009)

$$[(x-y)^2(x'-y')^2]^{2+\gamma_k}\langle T\{\hat{O}(x)\hat{O}(y)\hat{O}(x')\hat{O}(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

$a_0 = \frac{x_+y_+}{(x-y)^2}$, $b_0 = \frac{x'_-y'_-}{(x'-y')^2}$ $\Leftrightarrow$ impact factors do not scale with energy
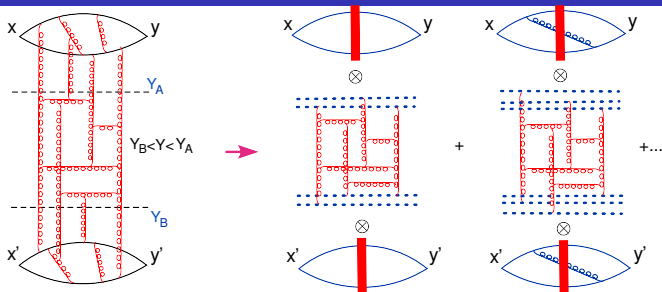$\Rightarrow$ all energy dependence is contained in $[\mathrm{DD}]^{a_0,b_0}$ ($a_0b_0 = R$)

$$[(x-y)^2(x'-y')^2]^{2+\gamma_k}\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}(y')\}\rangle$$

$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

Dipole-dipole scattering $\chi(\gamma) \equiv 2C - \psi(\gamma) - \psi(1-\gamma)$

$$[DD] = \int d\nu \int dz_0 \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^{\frac{1}{2}+i\nu}\left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^{\frac{1}{2}-i\nu} D(\frac{1}{2}+i\nu;\lambda)R^{\omega(\nu)/2}$$

$$D(\gamma;\lambda) = \frac{\Gamma(-\gamma)\Gamma(\gamma-1)}{\Gamma(1+\gamma)\Gamma(2-\gamma)}\left\{1 - \frac{\lambda}{4\pi^2}\left[\frac{\chi(\gamma)}{\gamma(1-\gamma)} - \frac{\pi^2}{3}\right] + O(\lambda^2)\right\}$$

$$[(x-y)^2(x'-y')^2]^{2+\gamma_k}\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\, \text{IF}^{a_0}(x,y;z_1,z_2)[\text{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\text{IF}^{b_0}(x',y';z'_1,z'_2)$$

Result :   (G.A. Chirilli and I.B., 2010)

$$F(\nu) = \frac{N_c^2}{N_c^2-1}\frac{4\pi^4\alpha_s^2}{\cosh^2\pi\nu}\left\{1+\frac{\alpha_s N_c}{\pi}\left[\frac{\pi^2}{2}-\frac{2\pi^2}{\cosh^2\pi\nu}-\frac{8}{1+4\nu^2}\right]+O(\alpha_s^2)\right\}$$

Light-cone OPE in Cornalba's formula

$$[x_{12}^2 x_{34}^2]^{2+\gamma_K} \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{i}{2}\int d\nu \, \frac{\tanh \pi\nu}{\nu} F(\nu)\Omega(r,\nu)R^{\frac{1}{2}\aleph(\nu)}\tilde{f}_+(\aleph(\nu))$$

$\aleph(\nu)$ - pomeron intercept, $\tilde{f}_+(\omega) = (e^{i\pi\omega} - 1)/\sin \pi\omega$ - signature factor

$$\Omega(r,\nu) = \frac{\nu}{2\pi^2}\frac{\sin 2\nu\rho}{\sinh \rho}, \quad \cosh \rho = \frac{\sqrt{r}}{2}$$

In the double limit (Regge) + ($x_{12}^2 \to 0$)

$$R = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} \to \frac{x_{1_+}x_{2_+}x_{3_-}x_{4_-}}{x_{12_\perp}^2 x_{34_\perp}^2}$$

$$r \to \frac{x_{12_+}^2(x_{3_-}x_{14_\perp}^2 - x_{4_-}x_{13_\perp}^2)^2}{x_{1_+}x_{2_+}x_{3_-}x_{4_-}x_{12_\perp}^2 x_{34_\perp}^2}$$

$$\Omega(r,\nu) \to \frac{\nu}{2\pi^2 i}\big(r^{-\frac{1}{2}+i\nu} - r^{-\frac{1}{2}-i\nu}\big)$$

Cornalba's formula as $x_{12\perp}^2 \to 0$

$$[x_{12}^2 x_{34}^2]^{2+\gamma_K} \int dx_{2_+} dx_{3_-} \ \langle \mathcal{O}(L_+ + x_{2_+}, x_{1\perp}) \mathcal{O}(x_{2_+}, x_{2\perp}) \mathcal{O}(L_- + x_{3_-}, x_{2\perp}) \mathcal{O}(x_{3_-}, x_{3\perp}) \rangle$$

$$= \frac{i\alpha_s^2}{8} \pi^2 L_+ L_- \int_0^1 du\, dv \int d\nu \frac{\tanh \pi\nu}{\nu \cosh^2 \pi\nu} \left(\frac{x_{12\perp}^2 x_{34\perp}^2 \bar{u}u\bar{v}v}{[x_{13\perp}^2 v + x_{14}^2 \bar{v}]^2} \right)^{\frac{1}{2}+i\nu} \left(\frac{L_+^2 L_-^2 \bar{u}u\bar{v}v}{x_{12\perp}^2 x_{34\perp}^2} \right)^{\aleph(\nu)/2} f_+$$

$$= \frac{i\alpha_s^2}{8} \pi^2 \int_0^1 dv \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\xi\, f(\aleph(\xi)) \qquad \xi \equiv \frac{1}{2} + i\nu$$

$$\times \frac{(\bar{v}v)^{1-\xi+\frac{\aleph(\xi)}{2}} \cos \pi\xi}{(\xi - \frac{1}{2}) \sin^3 \pi\xi} \frac{B(2 - \xi + \frac{\aleph(\xi)}{2})}{[x_{13\perp}^2 v + x_{14}^2 \bar{v}]^{2+\aleph(\xi)}} \left(\frac{x_{12\perp}^2 x_{34\perp}^2}{[x_{13\perp}^2 v + x_{14}^2 \bar{v}]^2} \right)^{-\xi - \frac{\aleph(\xi)}{2}} (L_+ L_-)^{1+\aleph(\xi)}$$

= "BFKL" representation of the amplitude in the   (Regge) + $(x_{12}^2 \to 0)$ limit

Supermultiplet of twist-2 operators

SU_4 singlet operators.   (Korchemsky et al, 2003)

$$S_{1n}(z) = \mathcal{O}_g^n(z) + \frac{n-1}{8}\mathcal{O}_\lambda^n(z) - \frac{n(n-1)}{8}\mathcal{O}_\phi^n(z)$$

$$S_{2n}(z) = \mathcal{O}_g^n(z) - \frac{1}{8}\mathcal{O}_\lambda^n(z) + \frac{n(n+1)}{24}\mathcal{O}_\phi^n(z)$$

$$S_{3n}(z) = \mathcal{O}_g^n(z) - \frac{n+2}{4}\mathcal{O}_\lambda^n(z) - \frac{(n+1)(n+2)}{8}\mathcal{O}_\phi^n(z)$$

$$\mathcal{O}_\lambda^n(x_\perp) = \int dx_+ \, i\bar\lambda^a \nabla^{n-1}\sigma_-\lambda^a(x_+, x_\perp)$$

$$\mathcal{O}_\phi^n(z) = \int dx_+ \, \bar\phi^{a,I}\nabla^n\phi^{a,I}(x_+, x_\perp)$$

All operators have the same anomalous dimension

$$\gamma_n^{S_1}(\alpha_s) \equiv \gamma_n(\alpha_s) = \frac{2\alpha_s}{\pi}N_c[\psi(n-1) + C] + O(\alpha_s^2), \qquad \gamma_n^{S_2} = \gamma_{n+1}^{S_1}, \qquad \gamma_n^{S_3} = \gamma_{n+2}^{S_1}$$

Supermultiplet of LR operators

Gluino and scalar LR operators

$$\Lambda(L_+, x_\perp) = \frac{i}{2} \int dx'_+ \, [\bar{\lambda}^a(L_+ + x_+ + x_\perp)[x'_+ + x_+, x_+]^{ab}\sigma_- \nabla_- \lambda^b(x_+ + x_\perp) + \text{c.c}]$$

$$\Phi(L_+, x_\perp) = \int dx'_+ \, \phi^{a,I}(L_+ + x_+ + x_\perp)[x'_+ + x_+, x_+]^{ab}\nabla^2_- \phi^{b,I}(x_+ + x_\perp)$$

$$\Lambda_j(x_\perp) = \int_0^\infty dL_+ \, L_+^{-j+1}\Lambda(L_+, x_\perp), \quad \Phi_j(x_\perp) = \int_0^\infty dL_+ \, L_+^{-j+1}\Phi(L_+, x_\perp)$$

SU_4 singlet LR operators.

$$S_{1j}(x_\perp) = F_j(x_\perp) + \frac{j-1}{8}\Lambda_j(x_\perp) - \frac{j(j-1)}{8}\Phi_j(x_\perp)$$

$$S_{2j}(x_\perp) = F_j(x_\perp) - \frac{1}{8}\Lambda_j(x_\perp) + \frac{j(j+1)}{24}\Phi_j(x_\perp)$$

$$S_{3j}(x_\perp) = F_j(x_\perp) - \frac{j+2}{4}\Lambda_j(x_\perp) - \frac{(j+1)(j+2)}{8}\Phi_j(x_\perp)$$

All operators have the same anomalous dimension

$$\gamma_j^{S_1}(\alpha_s) \equiv \gamma_j(\alpha_s) = \frac{2\alpha_s}{\pi}N_c[\psi(j-1) + C] + O(\alpha_s^2), \quad \gamma_j^{S_2} = \gamma_{j+1}^{S_1}, \quad \gamma_j^{S_3} = \gamma_{j+2}^{S_1}$$

Three-point CF of LR operator and two locals

Three-point correlation function of local operators ($\mathcal{O} = \phi_I^a \phi_I^a$ - Konishi operator)

$$(\mu^2 z_{23}^2)^{-\gamma_K} \langle S_{1n}(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3)\rangle = \sum \frac{c_n[1+(-1)^n]}{z_{12}^2 z_{13}^2 z_{23}^2} \left(\frac{z_{12_-}}{z_{12}^2} - \frac{z_{13_-}}{z_{13}^2}\right)^n \left(\frac{\mu^{-2} z_{23}^2}{z_{12}^2 z_{13}^2}\right)^{\frac{1}{2}\gamma(n,\alpha_s)}$$

$$\Rightarrow$$

$$x_{23_\perp}^2 (\mu^2 x_{23_\perp}^2)^{\gamma_K} \int dx_{1_+} \langle S_{1n}(x_{1_+}, x_{1_\perp})\mathcal{O}(x_{2_-}, x_{2_\perp})\mathcal{O}(x_{3_-}, x_{3_\perp})\rangle$$
$$= -2\pi i \frac{c_n[1+(-1)^n]}{x_{23_\perp}^4} \frac{\Gamma(1+2n+\gamma_n)}{\Gamma^2(1+n+\frac{1}{2}\gamma_n)} \left(\frac{x_{23_\perp}^2}{x_{2_-}x_{3_-}}\right)^{1+\frac{\gamma_n}{2}} \left(\frac{x_{12_\perp}^2}{x_{2_-}} + \frac{x_{13_\perp}^2}{x_{3_-}}\right)^{-1-n-\gamma_n}$$

CF of light-ray operator and two local operators

$$x_{23_\perp}^2 (\mu^2 x_{23_\perp}^2)^{\gamma_K} \langle S_{1j}(x_{1_\perp})\mathcal{O}(x_{2_-}, x_{2_\perp})\mathcal{O}(x_{3_-}, x_{3_\perp})\rangle$$
$$= -2\pi i \frac{c_j[1+e^{i\pi j}]}{x_{23_\perp}^4} \frac{\Gamma(1+2j+\gamma_j)}{\Gamma^2(1+j+\frac{1}{2}\gamma_j)} \left(\frac{x_{23_\perp}^2}{x_{2_-}x_{3_-}}\right)^{1+\frac{\gamma_j}{2}} \left(\frac{x_{12_\perp}^2}{x_{2_-}} + \frac{x_{13_\perp}^2}{x_{3_-}}\right)^{-1-j-\gamma_j}$$

Three-point CF of LR operator and two locals

Proof: compare conformal Ward identities for $\int dx_+ S_{1n}(x_+, x_\perp)$ and $S_{1j}(x_\perp)$

$$i[D, \int dx_+ S_{1n}(x_+, x_\perp)] = \int dx_+ \left(x_+ \frac{\partial}{\partial x_+} + n + 2 + (x - x_0)_i^\perp \frac{\partial}{\partial x_i^\perp}\right) S_{1n} = (n+1) \int dx_+ S_{1n}(x_+, x_\perp)$$

vs

$$i[D, \Phi(x_+, x_\perp)\nabla_-^2 \Phi(x'_+, x_\perp)] = \left[4 + x_+ \frac{\partial}{\partial x_+} + x'_+ \frac{\partial}{\partial x'_+} + (x - x_0)_i^\perp \frac{\partial}{\partial x_i^\perp}\right] \Phi(x_+, x_\perp)\nabla_-^2 \Phi(x'_+, x_\perp)$$

$$\Rightarrow \ i[D, \Phi_j(x_\perp)] = \left[j + 1 + (x - x_0)_i^\perp \frac{\partial}{\partial x_i^\perp}\right] \Phi_j(x_\perp)$$

Similarly

$$i[K_+, \int dx_+ \Phi(x_\perp)\nabla_-^n \Phi(x_\perp)] = 4\left[n + (x - x_0)_i \frac{\partial}{\partial x_i^\perp}\right] \int dx_+ \, x_+ \Phi(x_\perp)\nabla_-^n \Phi(x_\perp)$$

vs

$$i[K_+, \int dx_+ \int_0^\infty ds_+ \, (s_+)^{-j+1} \nabla_- \Phi(x_+ + \frac{s_+}{2} + x_\perp)\nabla_- \Phi(x_+ - \frac{s_+}{2} + x_\perp)]$$

$$= 4\left[j + (x - x_0)_i^\perp \frac{\partial}{\partial x_i^\perp}\right] \int dx_+ \, x_+ \int_0^\infty ds_+ \, (s_+)^{-j+1} \Phi(x_+ + \frac{s_+}{2} + x_\perp)\Phi(x_+ - \frac{s_+}{2} + x_\perp)$$

CF of light-ray operator and two local operators "in forward kinematics"

$$x_{23_\perp}^2 (\mu^2 x_{23_\perp}^2)^{\gamma_K} \int dx_{3_-} \langle S_j^\mu(x_{1_\perp}) \mathcal{O}(L_- + x_{3_-}, x_{2_\perp}) \mathcal{O}(x_{3_-}, x_{3_\perp}) \rangle$$
$$= \int_0^1 du \, \frac{c(j, \alpha_s)[1 + e^{i\pi j}] L_-^j (\bar{u}u)^j}{[x_{12_\perp}^2 u + x_{13_\perp}^2 \bar{u}]^{1+j}} \left(\frac{\bar{u}u\mu^{-2} x_{23_\perp}^2}{[x_{12_\perp}^2 u + x_{13_\perp}^2 \bar{u}]^2} \right)^{\frac{1}{2}\gamma(j, \alpha_s)}$$

Light-cone OPE

CF of light-ray operator and two local operators "in forward kinematics"

$$
x_{23_\perp}^2 (\mu^2 x_{23_\perp}^2)^{\gamma_K} \int dx_{3-} \langle S_j^\mu(x_{1_\perp}) \mathcal{O}(L_- + x_{3-}, x_{2_\perp}) \mathcal{O}(x_{3-}, x_{3_\perp}) \rangle
$$
$$
= \int_0^1 du \, \frac{c(j, \alpha_s)[1 + e^{i\pi j}] L_-^j (\bar{u}u)^j}{[x_{12_\perp}^2 u + x_{13_\perp}^2 \bar{u}]^{1+j}} \left(\frac{\bar{u}u\mu^{-2} x_{23_\perp}^2}{[x_{12_\perp}^2 u + x_{13_\perp}^2 \bar{u}]^2} \right)^{\frac{1}{2}\gamma(j, \alpha_s)}
$$

Limit $x_{23_\perp} \equiv \Delta \to 0$

$$
\Delta^2 (\mu^2 \Delta^2)^{\gamma_K} \int dx_{3-} \langle S_j^+(x_{1_\perp}) \mathcal{O}(L_- + x_{3-}, x_{2_\perp}) \mathcal{O}(x_{3-}, x_{3_\perp}) \rangle
$$
$$
= \frac{\Gamma^2\left(1 + j + \frac{\gamma}{2}\right)}{\Gamma(2 + 2j + \gamma)} \, \frac{c(j, \alpha_s)[1 + e^{i\pi j}] L_-^j}{(x_{12_\perp}^2)^{1+j} (\mu^2 x_{12_\perp}^2)^{-\gamma(j,\alpha_s)}} (\mu\Delta)^{\gamma(j,\alpha_s)}
$$

Light-ray operators in (+) and (-) directions:

$$\Phi^+(L_+, x_\perp) = \int dx'_+ \, \nabla_- \phi^{a,I}(L_+ + x_+ + x_\perp)[x'_+ + x_+, x_+]^{ab} \nabla_- \phi^{b,I}(x_+ + x_\perp)$$

$$\Phi_j^+(x_\perp) = \int_0^\infty dL_+ \, L_+^{-j+1} \Phi(L_+, x_\perp)$$

$$\Phi^-(L_-, x_\perp) = \int dx'_- \, \nabla_+ \phi^{a,I}(L_- + x_- + x_\perp)[x'_- + x_-, x_-]^{ab} \nabla_+ \phi^{b,I}(x_- + x_\perp)$$

$$\Phi_j^-(x_\perp) = \int_0^\infty dL_- \, L_+^{-j+1} \Phi(L_-, x_\perp)$$

and similarly for Λ_j's and G_j's

$$\text{CF of two LRs}: \quad \langle S_{j=\frac{3}{2}+i\nu}^+(x_{1_\perp}) S_{j'=\frac{3}{2}+i\nu'}^-(x_{3_\perp}) \rangle = \frac{\delta(\nu - \nu') a(j, \alpha_s)}{(x_{13_\perp}^2)^{j+1}(x_{13_\perp}^2 \mu^2)^{\gamma(j, \alpha_s)}}$$

$\Rightarrow$ Light-cone OPE

$$x_{12_\perp}^2 (\mu^2 x_{12_\perp}^2)^{\gamma_K} \int dx_{1_+} \ \mathcal{O}(L_+ + x_{2_+}, x_{1_\perp}) \mathcal{O}(x_{2_+}, x_{2_\perp})$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj \ c(j, \alpha_s)[1 + e^{i\pi j}] L_+^j (\mu^2 x_{12_\perp}^2)^{\gamma(j,\alpha_s)} S_j^+ (x_{1_\perp})$$

⇒ Light-cone OPE

$$x_{12_\perp}^2 (\mu^2 x_{12_\perp}^2)^{\gamma_\kappa} \int dx_{1_+} \ \mathcal{O}(L_+ + x_{2_+}, x_{1_\perp})\mathcal{O}(x_{2_+}, x_{2_\perp})$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj \ c(j, \alpha_s)[1 + e^{i\pi j}]L_+^j (\mu^2 x_{12_\perp}^2)^{\gamma(j, \alpha_s)} S_j^+(x_{1_\perp})$$

⇒ "DGLAP" result (at $x_{12}^2 \to 0$)

$$\mu^{-4}(\mu^4 x_{12}^2 x_{34}^2)^{2+\gamma_k} \int dx_{2_+} dx_{3_-} \ \langle \mathcal{O}(L_+ + x_{2_+}, x_{1_\perp})\mathcal{O}(x_{2_+}, x_{2_\perp})\mathcal{O}(L_- + x_{4_-}, x_{3_\perp})\mathcal{O}(x_{4_-}, x_{4_\perp})$$

"DGLAP" represenation of 4-point CFs

⇒ Light-cone OPE

$$x_{12_\perp}^2 (\mu^2 x_{12_\perp}^2)^{\gamma_\kappa} \int dx_{1_+} \, \mathcal{O}(L_+ + x_{2_+}, x_{1_\perp}) \mathcal{O}(x_{2_+}, x_{2_\perp})$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj \, c(j, \alpha_s)[1 + e^{i\pi j}] L_+^j (\mu^2 x_{12_\perp}^2)^{\gamma(j,\alpha_s)} S_j^+(x_{1_\perp})$$

⇒ "DGLAP" result (at $x_{12}^2 \to 0$)

$$\mu^{-4}(\mu^4 x_{12}^2 x_{34}^2)^{2+\gamma_k} \int dx_{2_+} dx_{3_-} \, \langle \mathcal{O}(L_+ + x_{2_+}, x_{1_\perp}) \mathcal{O}(x_{2_+}, x_{2_\perp}) \mathcal{O}(L_- + x_{4_-}, x_{3_\perp}) \mathcal{O}(x_{4_-}, x_{4_\perp})$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj \, C(j, \alpha_s)[1 + e^{i\pi j}] L_+^j (\mu^2 x_{12_\perp}^2)^{\frac{1}{2}\gamma(j,\alpha_s)} \langle S_j^\mu(x_{2_\perp}) \mathcal{O}(L_- + x_{4_-}, x_{3_\perp}) \mathcal{O}(x_{4_-}, x_{4_\perp}) \rangle$$

"DGLAP" represenation of 4-point CFs

$\Rightarrow$ Light-cone OPE

$$x_{12_\perp}^2 (\mu^2 x_{12_\perp}^2)^{\gamma_\kappa} \int dx_{1+} \; \mathcal{O}(L_+ + x_{2+}, x_{1_\perp}) \mathcal{O}(x_{2+}, x_{2_\perp})$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj \; c(j, \alpha_s)[1 + e^{i\pi j}] L_+^j (\mu^2 x_{12_\perp}^2)^{\gamma(j,\alpha_s)} S_j^+(x_{1_\perp})$$

$\Rightarrow$ "DGLAP" result (at $x_{12}^2 \to 0$)

$$\mu^{-4}(\mu^4 x_{12}^2 x_{34}^2)^{2+\gamma_k} \int dx_{2+} dx_{3-} \; \langle \mathcal{O}(L_+ + x_{2+}, x_{1_\perp}) \mathcal{O}(x_{2+}, x_{2_\perp}) \mathcal{O}(L_- + x_{4-}, x_{3_\perp}) \mathcal{O}(x_{4-}, x_{4_\perp}) \rangle$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj \; C(j, \alpha_s)[1 + e^{i\pi j}] L_+^j (\mu^2 x_{12_\perp}^2)^{\frac{1}{2}\gamma(j,\alpha_s)} \langle S_j^\mu(x_{2_\perp}) \mathcal{O}(L_- + x_{4-}, x_{3_\perp}) \mathcal{O}(x_{4-}, x_{4_\perp}) \rangle$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj \; C(j, \alpha_s) \int_0^1 dv \; \frac{[1 + e^{i\pi j}](L_+ L_-)^j}{[x_{12_\perp}^2 v + x_{13_\perp}^2 \bar{v}]^{1+j}} \left(\frac{x_{12_\perp}^2 x_{34}^2}{[x_{13_\perp}^2 v + x_{14_\perp}^2 \bar{v}]^2} \right)^{\frac{1}{2}\gamma(j,\alpha_s)} (\bar{v}v)^{j+\frac{1}{2}\gamma(j,\alpha_s)}$$

DGLAP *vs* BFKL for 4-point CFs

"DGLAP" result (leading twist)

$$\mu^{-4}(\mu^4 x_{12}^2 x_{34}^2)^{2+\gamma_k} \int dx_{2_+} dx_{3_-} \ \langle \mathcal{O}(L_+ + x_{2_+}, x_{1_\perp}) \mathcal{O}(x_{2_+}, x_{2_\perp}) \mathcal{O}(L_- + x_{4_-}, x_{3_\perp}) \mathcal{O}(x_{4_-}, x_{4_\perp}) \rangle$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj \ C(j, \alpha_s) \int_0^1 dv \ \frac{[1 + e^{i\pi j}](L_+ L_-)^j}{[x_{12_\perp}^2 v + x_{13_\perp}^2 \bar{v}]^{1+j}} \left(\frac{x_{12}^2 x_{34}^2}{[x_{13_\perp}^2 v + x_{14_\perp}^2 \bar{v}]^2} \right)^{\frac{1}{2}\gamma(j,\alpha_s)} (\bar{v}v)^{j+\frac{1}{2}\gamma(j,\alpha_s)}$$

"DGLAP" result (leading twist)

$$\mu^{-4}(\mu^4 x_{12}^2 x_{34}^2)^{2+\gamma_k} \int dx_{2_+} dx_{3_-} \; \langle \mathcal{O}(L_+ + x_{2_+}, x_{1_\perp}) \mathcal{O}(x_{2_+}, x_{2_\perp}) \mathcal{O}(L_- + x_{4_-}, x_{3_\perp}) \mathcal{O}(x_{4_-}, x_{4_\perp}) \rangle$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj \; C(j, \alpha_s) \int_0^1 dv \; \frac{[1 + e^{i\pi j}](L_+ L_-)^j}{[x_{12_\perp}^2 v + x_{13_\perp}^2 \bar{v}]^{1+j}} \left(\frac{x_{12}^2 x_{34}^2}{[x_{13_\perp}^2 v + x_{14_\perp}^2 \bar{v}]^2} \right)^{\frac{1}{2}\gamma(j, \alpha_s)} (\bar{v} v)^{j + \frac{1}{2}\gamma(j, \alpha_s)}$$

BFKL result $(\aleph(\xi, \alpha_s) = \frac{\alpha_s N_c}{\pi} \chi(\xi) + ...$ - pomeron intercept)

$$[x_{12}^2 x_{34}^2]^{2+\gamma_K} \int dx_{2_+} dx_{3_-} \; \langle \mathcal{O}(L_+ + x_{2_+}, x_{1\perp}) \mathcal{O}(x_{2_+}, x_{2\perp}) \mathcal{O}(L_- + x_{3_-}, x_{2\perp}) \mathcal{O}(x_{3_-}, x_{3\perp}) \rangle$$

$$= \frac{i\alpha_s^2}{8} \pi^2 \int_0^1 dv \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\xi \; f(\aleph(\xi)) \qquad\qquad \xi \equiv \frac{1}{2} + i\nu$$

$$\times \frac{(\bar{v}v)^{1-\xi+\frac{\aleph(\xi)}{2}} \cos \pi\xi}{(\xi - \frac{1}{2}) \sin^3 \pi\xi} \frac{B(2 - \xi + \frac{\aleph(\xi)}{2})}{[x_{13\perp}^2 v + x_{14\perp}^2 \bar{v}]^{2+\aleph(\xi)}} \left(\frac{x_{12\perp}^2 x_{34\perp}^2}{[x_{13\perp}^2 v + x_{14\perp}^2 \bar{v}]^2} \right)^{-\xi - \frac{\aleph(\xi)}{2}} (L_+ L_-)^{1+\aleph(\xi)}$$

"DGLAP" result (leading twist)

$$\mu^{-4}(\mu^4 x_{12}^2 x_{34}^2)^{2+\gamma_k} \int dx_{2_+} dx_{3_-} \langle \mathcal{O}(L_+ + x_{2_+}, x_{1_\perp}) \mathcal{O}(x_{2_+}, x_{2_\perp}) \mathcal{O}(L_- + x_{4_-}, x_{3_\perp}) \mathcal{O}(x_{4_-}, x_{4_\perp}) \rangle$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dj \, C(j,\alpha_s) \int_0^1 dv \, \frac{[1+e^{i\pi j}](L_+ L_-)^j}{[x_{12_\perp}^2 v + x_{13_\perp}^2 \bar{v}]^{1+j}} \left(\frac{x_{12_\perp}^2 x_{34_\perp}^2}{[x_{13_\perp}^2 v + x_{14_\perp}^2 \bar{v}]^2} \right)^{\frac{1}{2}\gamma(j,\alpha_s)} (\bar{v}v)^{j+\frac{1}{2}\gamma(j,\alpha_s)}$$

BFKL result ($\aleph(\xi,\alpha_s) = \frac{\alpha_s N_c}{\pi}\chi(\xi) + ...$ - pomeron intercept)

$$[x_{12}^2 x_{34}^2]^{2+\gamma_K} \int dx_{2_+} dx_{3_-} \langle \mathcal{O}(L_+ + x_{2_+}, x_{1\perp}) \mathcal{O}(x_{2_+}, x_{2\perp}) \mathcal{O}(L_- + x_{3_-}, x_{2\perp}) \mathcal{O}(x_{3_-}, x_{3\perp}) \rangle$$

$$= \frac{i\alpha_s^2}{8}\pi^2 \int_0^1 dv \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\xi \, f\big(\aleph(\xi)\big) \qquad \xi \equiv \frac{1}{2} + i\nu$$

$$\times \frac{(\bar{v}v)^{1-\xi+\frac{\aleph(\xi)}{2}} \cos \pi\xi}{(\xi-\frac{1}{2})\sin^3 \pi\xi} \frac{B\big(2-\xi+\frac{\aleph(\xi)}{2}\big)}{[x_{13\perp}^2 v + x_{14\perp}^2 \bar{v}]^{2+\aleph(\xi)}} \left(\frac{x_{12\perp}^2 x_{34\perp}^2}{[x_{13\perp}^2 v + x_{14\perp}^2 \bar{v}]^2} \right)^{-\xi-\frac{\aleph(\xi)}{2}} (L_+ L_-)^{1+\aleph(\xi)}$$

We compare these results around $\xi \sim j - 1 \sim 0$

$$\Rightarrow \ 1 + \aleph(\xi,\alpha_s) = j \ \text{ and } \ \gamma(j,\alpha_s) = -2\xi - \aleph(\xi)$$

$$\aleph(\xi) - 2\aleph(\xi)\aleph'(\xi) \simeq \frac{\alpha_s N_c}{\pi\xi} + \frac{\zeta(3)\alpha_s^2}{\xi} + ... \ \rightarrow \ \gamma_j = -2\frac{\alpha_s N_c}{\pi(j-1)} + [0 + \zeta(3)(j-1)]\big(\frac{\alpha_s N_c}{\pi(j-1)}\big)^3 + ...$$

$$\gamma_j = -2\frac{\alpha_s N_c}{\pi(j-1)} \ + \ [0 + \zeta(3)(j-1)]\left(\frac{\alpha_s N_c}{\pi(j-1)}\right)^3 + \ldots$$

is an anomalous dimension of light-ray operator $F\nabla^{j-2}F(x_\perp)$

1 Conclusions:

- NLO BFKL gives anomalous dimensions of light-ray operators $F_{+i}\nabla^{\omega-1}F_+{}^i$ as $\omega \to 0$ (in all orders in α_s)

1 Conclusions:
- NLO BFKL gives anomalous dimensions of light-ray operators $F_{+i}\nabla^{\omega-1}F_{+}{}^{i}$ as $\omega \to 0$ (in all orders in α_s)

2 Outlook
- In QCD
- 3-point CF of $\langle F\nabla^{\omega_1-1}F(x_{1_\perp})\ F\nabla^{\omega_2-1}F(x_{2_\perp})\ F\nabla^{\omega_3-1}F(x_{3_\perp})\rangle$ as $\omega_1, \omega_2, \omega_3 \to 0$ (joint project with V. Kazakov and E. Sobko):