New Skins for an Old Ceremony

The Conformal Bootstrap and the Ising Model

Sheer El-Showk
École Polytechnique & CEA Saclay

Based on:

arXiv:1203.6064 with M. Paulos, D. Poland, S. Rychkov,
D. Simmons-Duffin, A. Vichi

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June 12, 2013

Twelfth Workshop on Non-Perturbative QCD, IAP

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Motivation & Approach

Why return to the bootstrap?

1. Conformal symmetry very powerful tool *not fully exploited* in $D > 2$.
2. Completely non-perturbative tool to study field theories
   - Does not require SUSY, large N, or weak coupling.
3. In $D = 2$ conformal symmetry enhanced to *Virasoro* symmetry
   - Allows us to *completely solve* some CFTs ($c < 1$).
4. Long term hope: generalize this to $D > 2$?

Approach

- Use only "global" conformal group, valid in all D.
- Our previous result:
  - Constrained "landscape of CFTs" in $D = 2, 3$ using conformal bootstrap.
  - Certain CFTs (e.g. Ising model) sit at *boundary* of solution space.
- New result: "solve" spectrum & OPE of CFTs (in any D) on boundary.
  - Check against the $D = 2$ Ising model.
- The Future: Apply this to $D = 3$ Ising model?

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Outline

1. Motivation
2. The Ising Model
3. CFT Refresher
4. The Bootstrap & the *Extremal Functional Method*
5. Results: the 2d Ising model
6. The (Near) Future
7. Conclusions/Comments

The Ising model

Basic Definition

- Lattice theory with nearest neighbor interactions

$$H = -J \sum_{<i,j>} s_i s_j$$

with $s_i = \pm 1$ (this is $O(N)$ model with $N = 1$).

Relevance

- Historical: 2d Ising model solved exactly. [Onsager, 1944].
- Relation to $\mathcal{E}$-expansion.
- "Simplest" CFT (universality class)
- Describes:
  1. Ferromagnetism
  2. Liquid-vapour transition
  3. ...

Continuum Limit

- To study fixed point can take continuum limit (and $\sigma(x) \in \mathbb{R}$)

$$H = \int d^D x \left[(\nabla \sigma(x))^2 + t\,\sigma(x)^2 + a\,\sigma(x)^4 \right]$$

- In $D < 4$ coefficient a is relevant and theory *flows* to a fixed point.

$\mathcal{E}$-expansion

Wilson-Fisher set $D = 4 - \mathcal{E}$ and study critical point perturbatively.

Setting $\mathcal{E} = 1$ can compute anomolous dimensions in $D = 3$:

$$[\sigma] = 0.5 \to 0.518\ldots$$
$$[\epsilon] := [\sigma^2] = 1 \to 1.41\ldots$$
$$[\epsilon'] := [\sigma^4] = 2 \to 3.8\ldots$$

New Perspective

At fixed point conformal symmetry emerges:

- Strongly constrains data of theory.
- Can we use symmetry to fix e.g. $[\sigma], [\epsilon], [\epsilon'], \ldots$?
- Can we also fix interactions this way?

CFT Refresher

Spectrum and OPE

CFT Background

CFT defined by specifying:

- Spectrum $\mathcal{S} = \{\mathcal{O}_i\}$ of primary operators dimensions, spins: (Δ_i, l_i)
- Operator Product Expansion (OPE)

$$\mathcal{O}_i(x) \cdot \mathcal{O}_j(0) \sim \sum_k C_{ij}^k D(x, \partial_x) \mathcal{O}_k(0)$$

$\mathcal{O}_i$ are primaries. Diff operator $D(x, \partial_x)$ encodes *descendent* contributions.

This data fixes all correlators in the CFT:

- 2-pt & 3-pt fixed:

$$\langle \mathcal{O}_i \mathcal{O}_j \rangle = \frac{\delta_{ij}}{x^{2\Delta_i}}, \qquad \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle \sim C_{ijk}$$

- Higher pt functions contain no new dynamical info:

$$\underbrace{\langle \underbrace{\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)}_{\sum_k C_{12}^k D(x_{12}, \partial_{x_2}) \mathcal{O}_k(x_2)} \underbrace{\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)}_{\sum_l C_{34}^l D(x_{34}, \partial_{x_4})(x_3) \mathcal{O}_l(x_4)} \rangle}_{\sum_{k,l} C_{12}^k C_{34}^l D(x_{12}, x_{34}, \partial_{x_2}, \partial_{x_4}) \langle \mathcal{O}_k(x_2) \mathcal{O}_l(x_4) \rangle}$$

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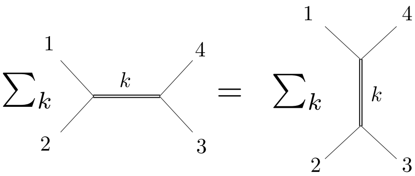
Crossing Symmetry

CFT Background

This procedure is not unique: $\langle\phi_1\phi_2\phi_3\phi_4\rangle$



Consistency requires equivalence of two different contractions

$$\sum_k C^k_{12} C^k_{34}\, G^{12;34}_{\Delta_k, l_k}(u,v) = \sum_k C^k_{14} C^k_{23}\, G^{14;23}_{\Delta_k, l_k}(u,v)$$

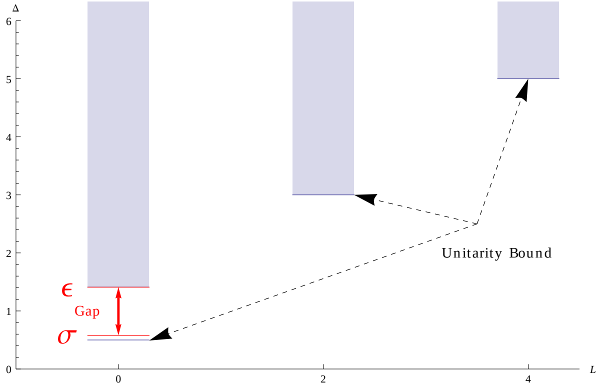
Functions $G^{ab;cd}_{\Delta_k, l_k}$ are *conformal blocks* (of "small" conformal group):

- Each G_{Δ_k, l_k} corresponds to one operator $\mathcal{O}_k$ in OPE.
- Entirely *kinematical*: all dynamical information is in C^k_{ij}.
- u, v are independent conformal cross-ratios: $u = \frac{x_{12}x_{34}}{x_{13}x_{24}}, v = \frac{x_{14}x_{23}}{x_{13}x_{24}}$
- Crossing symmetry give non-perturbative constraints on (Δ_k, C^k_{ij}).

How Strong is Crossing Symmetry?

Constraining the spectrum

Figure: A Putative Spectrum in $D = 3$



- Unitarity implies:

$$\Delta \geq \frac{D-2}{2} \quad (l=0),$$
$$\Delta \geq l + D - 2 \quad (l \geq 0)$$

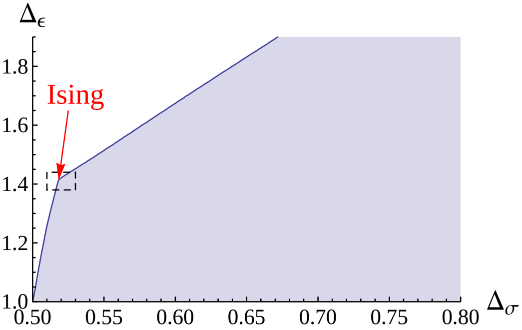
- “Carve” landscape of CFTs by imposing gap in scalar sector.
- Fix lightest scalar: σ.
- Vary next scalar: ϵ.
- Spectrum otherwise *unconstrained*: allow any other operators.

Constraining Spectrum using Crossing Symmetry

Is crossing symmetry consistent with a gap?

σ four-point function: $\langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle$

Crossing symmetric values of σ-ϵ



Blue = solution may exists.

White = No solution exists.

- Certain values of σ, ϵ inconsistent with crossing symmetry.
- Solutions to crossing:
  1. white region $\Rightarrow$ 0 solutions.
  2. blue region $\Rightarrow$ ∞ solutions.
  3. boundary $\Rightarrow$ 1 solution (unique)!
- Ising model special in two ways:
  1. On boundary of allowed region.
  2. At kink in boundary curve.

The Extremal Functional Method

Extremal Functional Method

Solving CFTs on the boundary via Crossing

1. Consider four *identical* scalars:

$$\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle \qquad \dim(\sigma) = \Delta_\sigma$$

2. Crossing equations simplify: sum of functions with positive coefficients.

$$\sum_{\mathcal{O}_k} \underbrace{(C^k_{\sigma\sigma})^2}_{p_k} \underbrace{\left[G^{12;34}_{\Delta_k,l_k}(x) - G^{14;23}_{\Delta_k,l_l}(x)\right]}_{F_k(x)} = 0$$

3. Convert to geometric cone problem by expanding in derivatives:

$$p_1 \underbrace{(F_1, F_1', F_1'', \ldots)}_{\vec{v}_1} + p_2 \underbrace{(F_2, F_2', F_2'', \ldots)}_{\vec{v}_2} + p_3 \underbrace{(F_3, F_3', F_3'', \ldots)}_{\vec{v}_3} + \cdots = \vec{0}$$

4. Each vector $\vec{v}_k$ represents the contribution of an operator $\mathcal{O}_k$.
5. If $\{\vec{v}_1, \vec{v}_2, \ldots\}$ span a positive cone there is no solution.
6. Efficient numerical methods to check if vectors $\vec{v}_k$ span a cone.
7. When cone "unfolds" solution is unique!

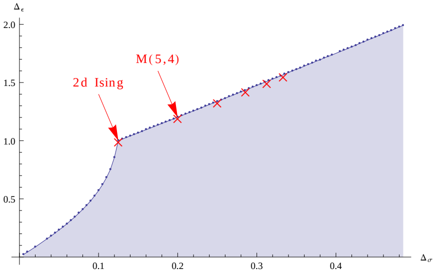
Spectrum and OPE from EFM?

Checking the extremal functional method

How Powerful is Crossing Symmetry?

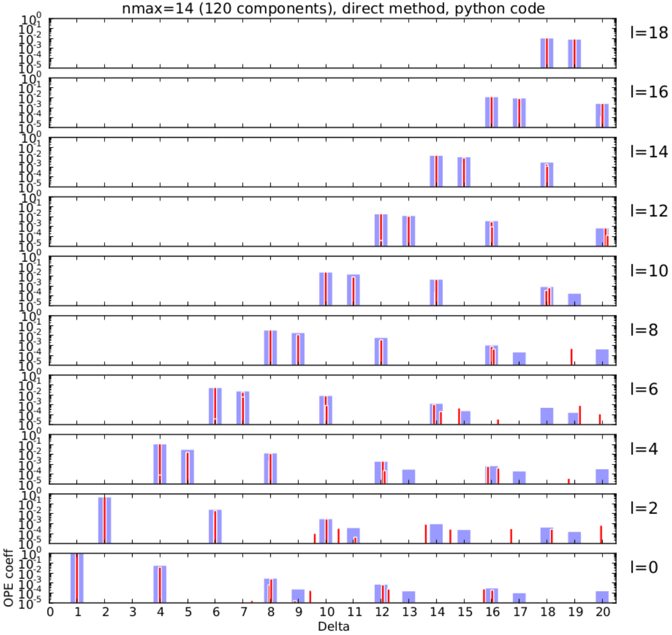
To check our technique lets apply to 2d Ising model.

- Same plot in 2d.
- Completely solvable theory.
  - Using Virasoro symmetry can compute full spectrum & OPE.



Can we reproduce using crossing symmetry & only “global” conformal group?

Spectrum from Extremal Functional Method



Crossing Symmetry vs. Exact Results

Exact (Virasoro) results compared to unique solution at “kink” on boundary:

Spin 0

L	Bootstrap Δ	Virasoro Δ	Δ Error (in %)	Bootstrap OPE	Virasoro OPE	OPE Error (in %)
0	1.	1	0.0000106812	0.500001	0.5	0.000140121
0	4.00145	4	0.03625	0.0156159	0.015625	0.0582036
0	8.035	8	0.4375	0.00019183	0.000219727	12.6962
0	12.175	12	1.45833	3.99524×10^{-6}	6.81196×10^{-6}	41.3496

Mileage from Crossing Symmetry?

- 12 OPE coefficients to $< 1\%$ error.
- Spectrum better:
  1. In 2d Ising expect operators at $L, L+1, L+4$.
  2. We find this structure up to $L = 20$
     ~ 38 operator dimensions $< 1\%$ error!

What about 3d Ising Model?

Current “State-of-the-Art”

3d Ising model

Using $\mathcal{E}$-expansion, Monte Carlo and other techniques find partial spectrum:

Field:	σ	ϵ	ϵ'	$T_{\mu\nu}$	$C_{\mu\nu\rho\lambda}$
Dim (Δ):	0.5182(3)	1.413(1)	3.84(4)	3	5.0208(12)
Spin (l):	0	0	0	2	4

Only 5 operators and no OPE coefficients known for 3d Ising!

Lots of room for improvement!

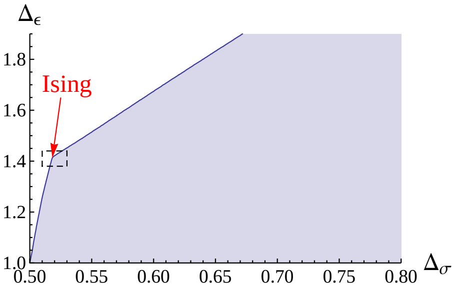
Our Goal

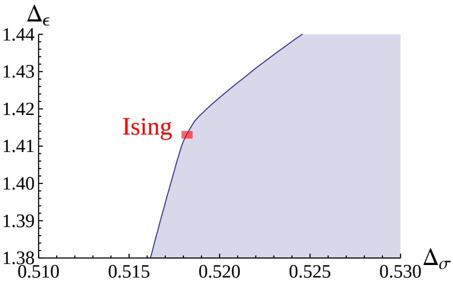
Compute these anomolous dimensions (and many more) and OPE coefficients using the bootstrap applied along the boundary curve.

Spectrum of the 3d Ising Model

Computing 3d Spectrum from Boundary Functional?

A first problem: what point on the boundary? what is correct value of σ?





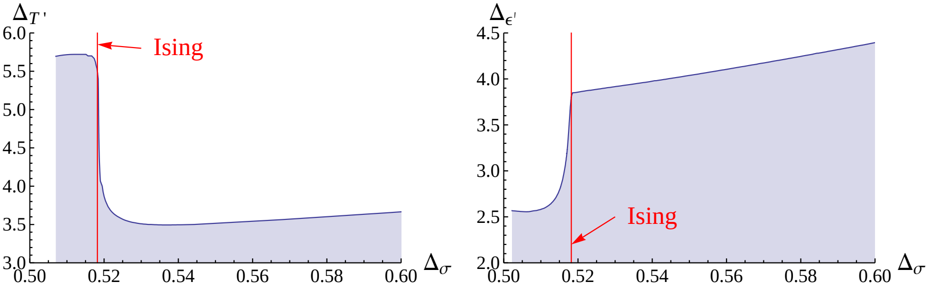
1. In $D = 2$ we know σ by other means.
2. "Kink" is not so sharp when we zoom in.
3. Gets sharper as we increase number of constraints
   $\Rightarrow$ should taylor expand to higher order!

Origin of the Kink

Re-arrangment of spectrum?

Spectrum near the kink undergoes rapid re-arrangement.

Plots for next Scalar and Spin 2 Field



1. "Kink" in (ϵ, σ) plot due to rapid rearrangement of *higher dim spectrum*.
2. Important to determine σ to high precision.
3. Does this hint at some analytic structure we can use?

The Future

What's left to do?

Honing in on the Ising model?

- Fix dimension of σ in 3d Ising using "kink" or other features.
- Use boundary functional to compute spectrum, OPE for 3d Ising.
- Compare with lattice or experiment!
- Additional constraints: add another correlator $\langle \sigma\sigma\epsilon\epsilon \rangle$.
- Study spectrum, OPE as a function of spacetime dimension.

Exploring the technology

- How specific is this structure to Ising model?
- Can we impose more constraints and find new "kinks" for other CFTs?
- Can any CFT be "solved" by imposing a few constraints (gaps) and then solving crossing symmetry?
- What about SCFTs? Need to know structure of supersymmetric conformal blocks.

Thanks

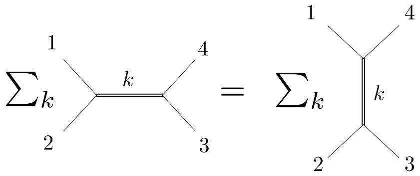
Implementing Crossing Symmetry

So how do we enforce crossing symmetry in practice?

Consider four *identical* scalars: $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \qquad \dim(\phi) = \Delta_\phi$

Recall crossing symmetry constraint:

$$\sum_{\mathcal{O}_k}(C^k_{\phi\phi})^2\, G^{12;34}_{\Delta_k,l_k}(x) = \sum_{\mathcal{O}_k}(C^k_{\phi\phi})^2\, G^{14;23}_{\Delta_k,l_k}(x)$$



Crossing Symmetry Nuts and Bolts

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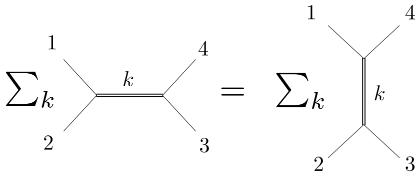
Move everything to LHS:

$$\sum_{\mathcal{O}_k} (C^k_{\phi\phi})^2\, G^{12;34}_{\Delta_k,l_k}(x) - \sum_{\mathcal{O}_k} (C^k_{\phi\phi})^2\, G^{14;23}_{\Delta_k,l_k}(x) = 0$$



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Express as sum of functions with positive coefficients:

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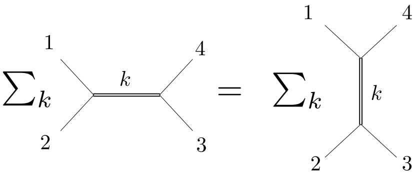

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Functions $F_k(x)$ are formally infinite dimensional vectors.

$$p_1 \underbrace{(F_1, F_1', F_1'', \ldots)}_{\vec{v}_1} + p_2 \underbrace{(F_2, F_2', F_2'', \ldots)}_{\vec{v}_2} + p_3 \underbrace{(F_3, F_3', F_3'', \ldots)}_{\vec{v}_3} + \cdots = \vec{0}$$

1. Each vector $\vec{v}_k$ represents the contribution of an operator $\mathcal{O}_k$.
2. If $\{\vec{v}_1, \vec{v}_2, \ldots\}$ span a positive cone there is no solution.
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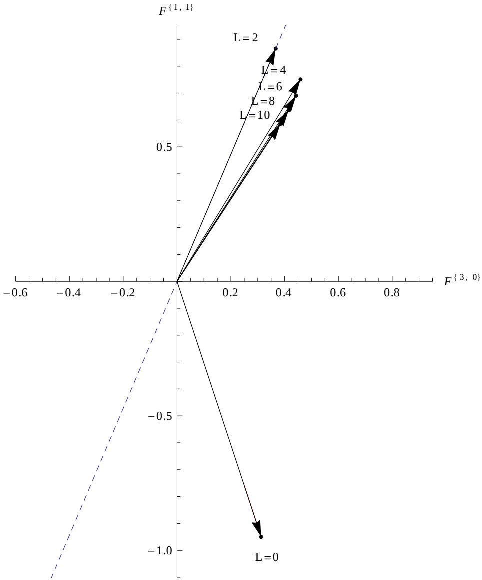
Cones in Derivative Space



Two-derivative truncation

- Consider $\langle\sigma\sigma\sigma\sigma\rangle$ with $\Delta(\sigma) = 0.515$.
- We plot e.g. $(F^{(1,1)}, F^{(3,0)})$.

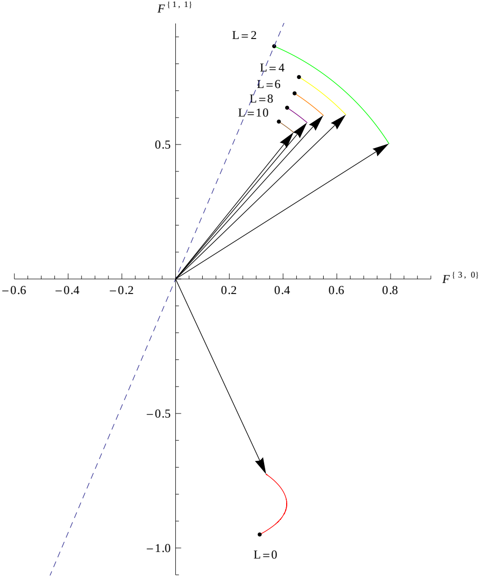
- Consider putative spectrum $\{\Delta_k, l_k\}$

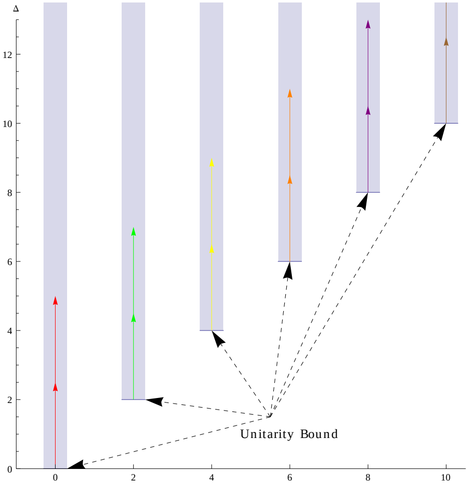
$$\Delta = \Delta_{unitarity}$$
$$l = 0 \text{ to } 10$$

- Vectors *represent operators*.
- All vectors lie inside cone
  $\Rightarrow$ Inconsistent spectrum!

Cones in Derivative Space

Derivatives $\Longleftrightarrow$ Putative Spectrum





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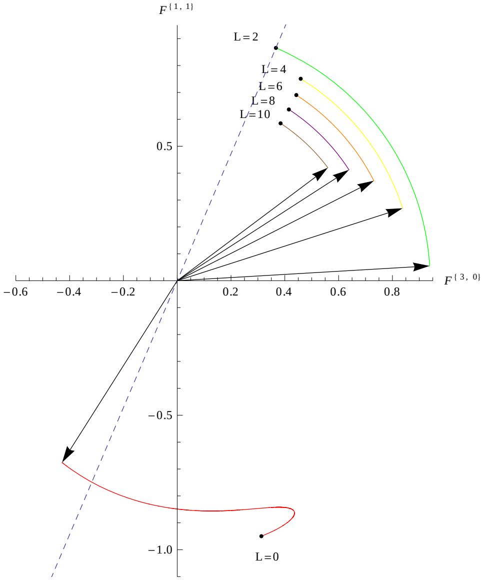


- Allow even more operators in putative spectrum.
- *Scalar channel* plays essential role.
  $\Rightarrow$ vectors *span* plane.
  $\Rightarrow$ In particular can find $p_k \geq 0$

$$\sum_k p_k\, F_{\Delta_k, l_k} = 0$$

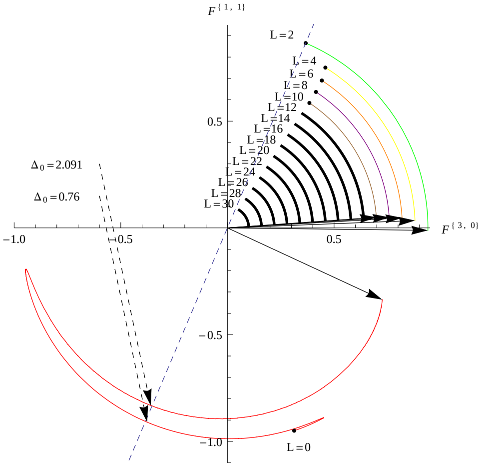
$\Rightarrow$ crossing sym. can be satisfied.

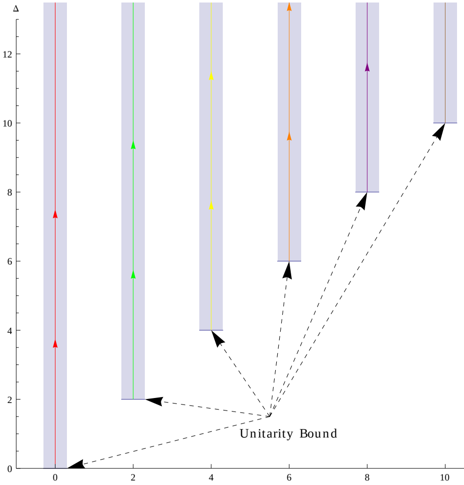
Why does this work?

- Cone boundary defined by low-lying operators.
- Higher Δ, l operators less important.
- Follows from convergence of CB expansion.

Cones in Derivative Space

Derivatives $\Longleftrightarrow$ Putative Spectrum





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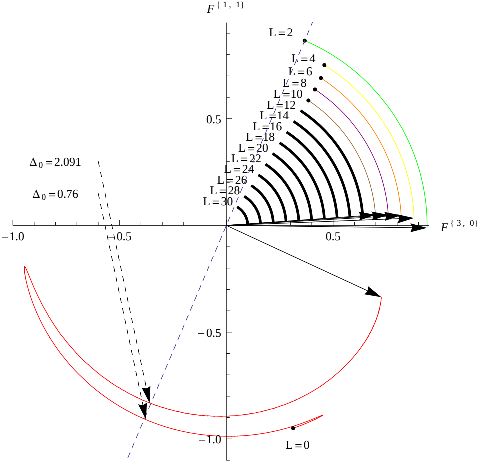
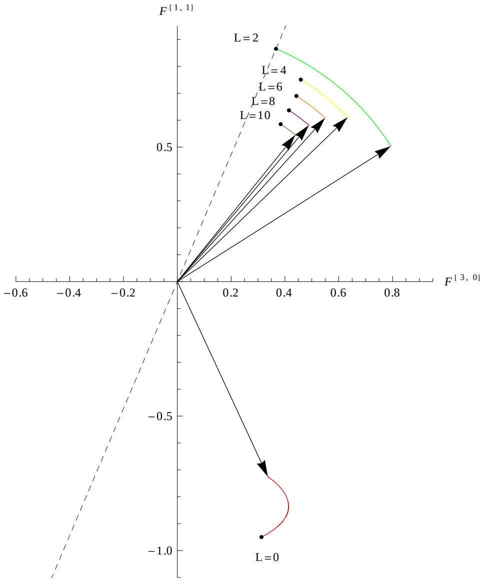


Carving the Landscape of CFTs

1. Plot imposes necessary conditions.
2. Carve "landscape" via exclusion.

Any CFT in $D = 3$ with $\dim(\sigma) = 0.515$ must have another scalar with $0.76 \le \Delta \le 2.091$.

The “Extremal Functional Method”



Uniqueness of “Boundary Functional”

– Consider $\Delta_0 < 0.76$

- No combination of vecs give a zero.

$$\sum_i p_i \vec{F}_i \neq 0 \quad \text{for} \quad p_i > 0$$

– Consider $\Delta_0 > 0.76$

- Many ways to for vects to give zero.
- Families of possible $\{p_i\}$.
- Neither spectrum nor OPE fixed.

– Consider $\Delta_0 = 0.76$

- Only one way to form zero.
- Non-zero p_i fixed $\Rightarrow$ unique spectrum.
- Value of $p_i := (C^k_{ii})^2$ fixed $\Rightarrow$ unique OPE.
- Non-zero p_i: $\begin{matrix} \Delta \sim 0.76, & L = 0 \\ \Delta = 2, & L = 2 \end{matrix}$

– NOTE: Num operators = num components of $\vec{F}_i$

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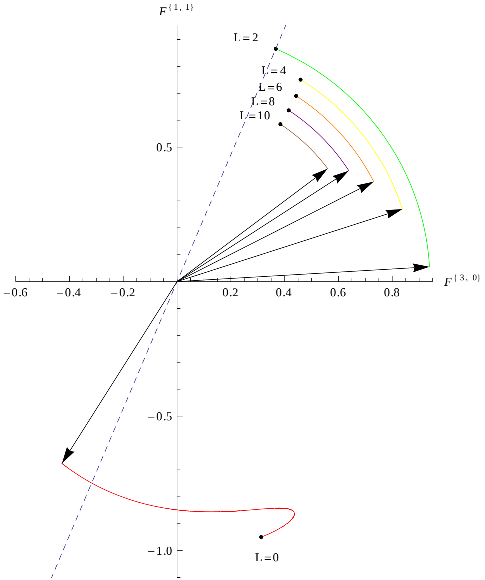

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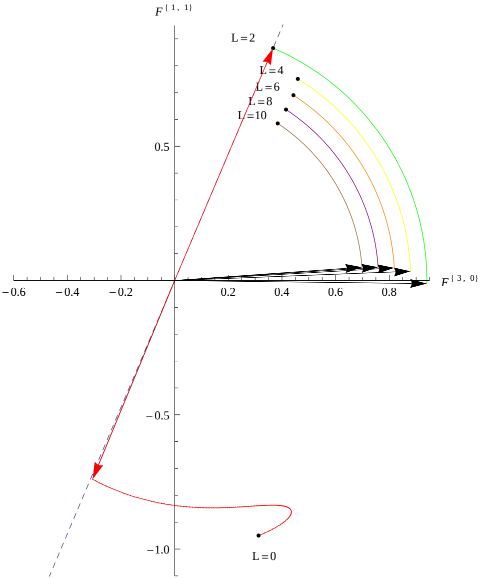

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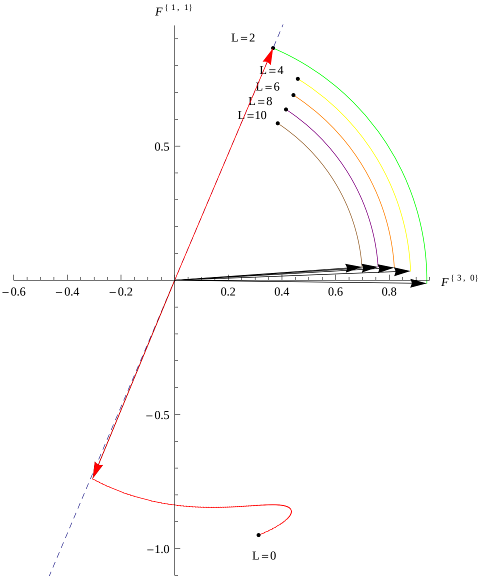

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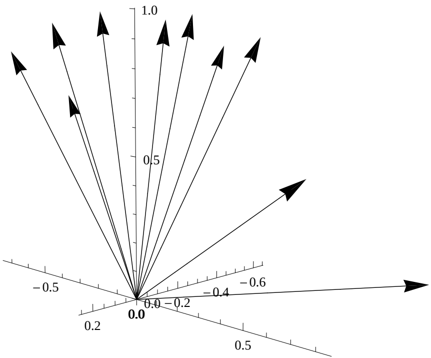
Cones in Derivative Space

Vectors form cone $\Rightarrow$ no solution.

No Solutions to Crossing



1. Axes are derivatives:

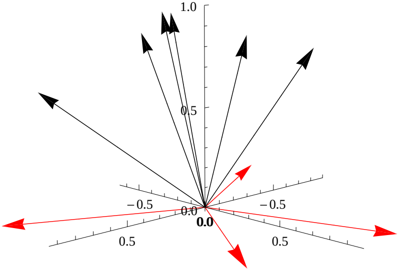
$$F'_{\Delta,l}(x), F''_{\Delta,l}(x), F'''_{\Delta,l}(x)$$

2. Vectors represents operators
3. All operators lie *inside* half-space.
4. $\vec{0}$ not in positive cone.

Cones in Derivative Space

Cone "unfolds" giving unique solution.

Unique Solution to Crossing



1. Axes are derivatives:

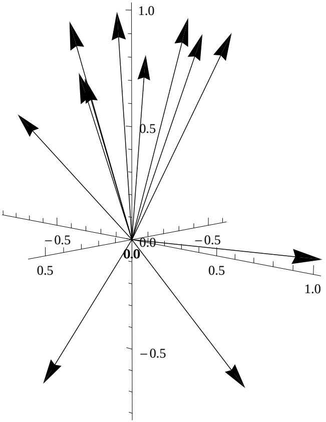
$$F'_{\Delta,l}(x), F''_{\Delta,l}(x), F'''_{\Delta,l}(x)$$

2. Vectors represents operators
3. Boundary of cone (red) spans a plane.
4. $\vec{0}$ in span of red vectors.

Cones in Derivative Space

As more operators added solutions no longer unique.

Many Solutions to Crossing



1. Axes are derivatives:

$$F'_{\Delta,l}(x), F''_{\Delta,l}(x), F'''_{\Delta,l}(x)$$

2. Vectors represents operators
3. Vectors span full space.
4. Many ways to form $\vec{0}$.