Casimir operator dependences of QCD amplitudes

12th Workshop on QCD, Paris, 10-13 June 2013

Thierry GRANDOU
Institut Non Linéaire de Nice - UMR-CNRS 7335

June 8, 2013

The property of Effective Locality

A functional approach to Lagrangian QCD using exact *Fradkin's representations* for $G_F(x,y|A)$ and $L(A)$, functional *differential* identities, and *linearization* of non-abelian $\mathbf{F}^2$:

- Manifestly gauge invariant (MGI) and Lorentz covariant (MLC)
- Non-Perturbative: Summing over all relevant Feynman graphs
- Displaying a remarkable property, dubbed "Effective Locality", peculiar to the non-abelian structure of QCD

Reminder

- Covariant gauge-dependent gluon propagator,

$$D_{F,\mu\nu}^{ab(\zeta)}(k) = \frac{i\delta_{ab}}{k^2 + i\varepsilon}\left[g_{\mu\nu} - \zeta\, k_\mu k_\nu / k^2\right],\ \zeta = \lambda/(1-\lambda)$$

- Fermionic (quark) propagator in an external gluon field A_μ^a,

$$G_F(x,y|A) = \langle x|[i\gamma^\mu(\partial_\mu - i g A_\mu^a \lambda_a) - m]^{-1}|y\rangle$$

Reminder

- Closed-fermion loop functional,

$$L[A] = Tr \ln[1 - i\, g\, (\gamma A \lambda)\, S_F]\ , \ S_F = G_F[gA = 0]$$

- Example of a functional differential identity

$$\mathcal{F}[\frac{1}{i}\frac{\delta}{\delta j}]\ e^{\frac{i}{2}\int j \cdot D_F^{(\zeta)} \cdot j} = e^{\frac{i}{2}\int j \cdot D_F^{(\zeta)} \cdot j} e^{\mathbf{D}_A^{(\zeta)}}\ \mathcal{F}[A]\Big|_{A = \int D_F^{(\zeta)} \cdot j}$$

where $\mathbf{D}_A^{(\zeta)}$ is the *linkage operator*

$$\mathbf{D}_A^{(\zeta)} = -\frac{i}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, \frac{\delta}{\delta A_\mu^a(x)}\ \mathbf{D}_F^{(\zeta)}\Big|_{\mu\nu}^{ab}(x-y)\, \frac{\delta}{\delta A_\nu^b(y)}$$

Reminder (Fradkin's)

$$\langle p|\mathbf{G}_F[A]|y\rangle = -\frac{1}{(2\pi)^2} e^{-ip\cdot y}\; i\int_0^\infty ds\; e^{-ism^2}\; e^{-\frac{1}{2}\,\mathrm{Tr}\ln(2h)}$$

$$\times \int d[u]\{m - i\gamma\cdot[p - gA(y - u(s))]\}\; e^{\frac{i}{4}\int_0^s ds'\,[u'(s')]^2} e^{ip\cdot u(s)}$$

$$\times \boxed{\left(e^{g\int_0^s ds'\sigma\cdot F(y-u(s'))}\; e^{-ig\int_0^s ds'\,u'(s')\cdot A(y-u(s'))}\right)_+}$$

$h(s_1,s_2) = \int_0^s \mathrm{d}s'\Theta(s_1 - s')\Theta(s_2 - s')$. Auxiliary functional variables, $\Omega^a(s_1)$, $\bar{\Omega}^b(s_2)$, required to circumvent Schwinger proper-time s'-ordering and take both $\mathbf{G}_F[A]$ and $\mathbf{L}[A]$ to *gaussian forms*.

EL not readable on $\mathbf{Z}_{QCD}[j,\eta,\bar{\eta}]$, but on its (even) fermionic momenta.

Reminder (Halpern'77)

The $\chi^a_{\mu\nu}$-field is a (real-valued) *Halpern field* introduced so as to linearize the non-abelian $F^{\mu\nu}F_{\mu\nu}$ dependence of the original QCD Lagrangian density

$$e^{-\frac{i}{4}\int \mathbf{F}^a_{\mu\nu}\mathbf{F}^{\mu\nu}_a} = \mathcal{N}_\chi \int \mathrm{d}[\chi]\, e^{\frac{i}{4}\int \chi^a_{\mu\nu}\chi^{\mu\nu}_a + \frac{i}{2}\int \chi^a_{\mu\nu}\mathbf{F}^{\mu\nu}_a}$$

Collaboration

H.M. Fried, Brown University (RI), USA

Y. Gabellini, Y-M Sheu, INLN.

EL is a definite functional statement

With

$$\mathcal{F}_{\mathrm{I}}[A] = \exp\left[\frac{i}{2}\int A\bar{\mathcal{K}}(2n)A + i\int \bar{\mathcal{Q}}(n)A\right], \quad \mathcal{F}_{\mathrm{II}}[A] = \exp\left(\mathbf{L}[A]\right)$$

The *functional* statement of EL for 2n-points fermionic Green's functions can be read off

$$\begin{aligned} e^{\mathbf{D}_A}\mathcal{F}_{\mathrm{I}}[A]\,\mathcal{F}_{\mathrm{II}}[A] = \quad & \mathcal{N} \quad \exp\left[-\frac{i}{2}\int \bar{\mathcal{Q}}(n)\hat{\mathcal{K}}^{-1}\bar{\mathcal{Q}}(n) + \frac{1}{2}Tr\ln\hat{\mathcal{K}}\right] \\ & \times \quad \exp\left[\frac{i}{2}\int \frac{\delta}{\delta A}\hat{\mathcal{K}}^{-1}\frac{\delta}{\delta A} - \int \bar{\mathcal{Q}}(n)\hat{\mathcal{K}}^{-1}\frac{\delta}{\delta A}\right] \\ & \times \quad \exp\left(\mathbf{L}[A]\right) \end{aligned} \tag{1}$$

at

$$\bar{\mathcal{K}}(2n) = (D_F^{(\zeta)})^{-1} + \hat{\mathcal{K}}(2n), \;\; \hat{\mathcal{K}}(2n)^{ab}_{\mu\nu} = (\, \mathcal{K}_{\mathcal{S}}(2n) + gf\chi\,)^{ab}_{\mu\nu}$$

EL is a functional statement

1. *Because* $\hat{\mathcal{K}} = \mathcal{K}_{\mathcal{S}} + g(f\cdot\chi)$ *is local*

$$\langle x|\mathcal{O}|y\rangle = \mathcal{O}(x)\,\delta^{(4)}(x-y)$$

as well as the extra contributions of $\mathbf{L}[A]$ *to* $\hat{\mathcal{K}}$ *and* $\bar{\mathcal{Q}}$*, the contributions of (1) depend only on the Fradkin variables* $u_i(s_i')$ *and the space-time coordinates* y_i *in a specific but local way*

2. *Nothing in (1) ever refers to* $D_F^{(\zeta)}$ *: Gauge-Invariance is rigorously achieved as a matter of Gauge-Independence*! This is MGI in the most radical sense .. hoped as such by R.P. Feynman in QED (cf. ‘Quantum Field Theory In A Nutshell’, A. Zee)

Any Antecedents ?..

Yes!

- In the pure YM case, early 90's, H. Reinhardt, K. Langfeld, L.v. Smekal discover a surprising effective local interaction

$$\int d^4z\, \partial^\lambda \chi^a_{\lambda\mu}(z)\ \left([(gf\chi)]^{-1}\right)^{\mu\nu}_{ab}(z)\, \partial^\rho \chi^b_{\rho\nu}(z)$$

- H.M. Fried himself in 'Functional Methods and Eikonal Models' (Eds. Frontières, 1990)

- EL 'made easy' to discover within *functional differentiation identities*; very difficult within functional integrations.

QCD amplitudes by EL and *Random Matrices*

.. allow one to calculate $2n$-point fermionic amplitudes in a generic form (At least, at quenched and eikonal approximations, without any further approximation).

These forms comply with a general conjecture (D.D. Ferrante, G.S. and Z. Guralnik, C. Pehlevan. S. Gukov and E. Witten, *etc.*, 2008-2011) so far illustrated on scalar field models, that QFT's GF are expandable in terms of G^{mn}_{pq}-Meijer's special functions.

QCD fermionic amplitudes ..

For $2n = 4$, i.e. a 2-body scattering processus, one quark specie of mass m, impact parameter b in $\varphi(b)$, $\mathcal{T} = (T, T, T, T)$, $D = 4$ copies of the full set of $SU_c(3)$ generators in the fundamental representation,

$$\sim \sum_{\text{monomials}} (\pm 1) \prod_{1 \leq i \leq N}^{\sum q_i = N(N-1)/2} [1 - i(-1)^{q_i}]$$

$$\times\, C \int \mathrm{d}p_1 \,..\, \mathrm{d}p_{N(N-1)/2}\, f(\mathbf{p}) \prod_{J=1}^{2} \int_0^{+\infty} \mathrm{d}\alpha_J^i \, \frac{\sin[\alpha_J^i (\mathcal{O}\mathcal{T})_i]}{\alpha_J^i}$$

$$\times\, G_{34}^{23} \left(iN_c \left(\frac{\alpha_1^i \alpha_2^i}{g\varphi(b)} \right)^2 \frac{\hat{s}(\hat{s} - 4m^2)}{2m^4} \middle| \begin{array}{cccc} \frac{3-2q_i}{4}, & \frac{1}{2}, & 1, & \\ \frac{1}{2}, & \frac{1}{2}, & 1, & 1 \end{array} \right)$$

QCD fermionic amplitudes ..

where partonic $\hat{s}$ and non-perturbative physics $g\varphi(b)$ show up in one and the same expression ...a requirement of an old Dirac's program (cf. S.J. Brodsky and G.F. de Teramond, LF- Quantization in AdS_5/QCD).

$\mathcal{O}(\mathbf{p}) \in O(N)$, $N = D \times (N_c^2 - 1)$. An average over orthogonal matrices is in order, with $f(p_1, \ldots, p_{N(N-1)/2})$ a *Haar measure* on $O(N)$.
A trivial result would come out otherwise under the form,

$$\prod_{i=1}^{N} \mathcal{T}_j^2 = \left(\prod_{a=1}^{N_c^2-1} \frac{1}{4}\lambda_a^2 \right)^4 = \left(\prod_{a=1}^{7} \frac{1}{4}\lambda_a^2 \right) \frac{1}{4}\lambda_8^2 = \mathbf{0}_{3\times 3} \left(\frac{1}{4}\lambda_8^2\right) = 0.$$

QCD fermionic amplitudes ..

Restoring the relevant dependences on $\mathcal{O}(\mathbf{p})$, the amplitude is a finite sum of monomials sharing the same overall color algebraic structure,

$$\prod_{1\leq i\leq N}[1-i(-1)^{q_i}]\int_0^\infty \frac{d\alpha_1^i}{\alpha_1^i}\int_0^\infty \frac{d\alpha_2^i}{\alpha_2^i}\; G_{34}^{23}\left(C^{st}(\alpha_1^i\alpha_2^i)^2|\dots\right)$$

$$<(\sum_{k_i=0}^{\infty}\{k_i,\alpha_1^i\}(\mathcal{O}^{ij}(\mathbf{p})\mathcal{T}_j)^{2k_i+1})(\sum_{k_i'=0}^{\infty}\{k_i',\alpha_2^i\}(\mathcal{O}^{ij}(\mathbf{p})\mathcal{T}_j)^{2k_i'+1})>$$

where, with $R_{ij}=R_{ij}(\Theta_{ij})$ an $\{i-j\}$-plane *rotator*,

$$\mathcal{O}(\mathbf{p})=(R_{12}R_{13}\dots R_{1N})(R_{23}R_{24}\dots R_{2N})\dots\dots(R_{N-1,N})D_\varepsilon$$

$$D_\varepsilon=diag(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_N),\varepsilon_i=\pm1,P(\varepsilon_i=\pm1)=\frac{1}{2}$$

QCD fermionic amplitudes ..

With $\{k_i, \alpha_1^i\} = (-1)^{K_i} (\alpha_J^i)^{2k_i+1}/(2k_i+1)!$, $J = 1, 2$ in the 2 *sine*-function expansions, one gets at $k_i = k_i' = 0$,

$$\prod_{i=1}^{N} [1 - i(-1)^{q_i}] \prod_{J=1}^{2} \int_0^{\infty} \frac{d\alpha_J^i}{\alpha_J^i} \{0, \alpha_J^i\} G_{34}^{23} \left(C^{st} (\alpha_1^i \alpha_2^i)^2 | \ldots \right) K_{00} D C_{2f} \mathbf{1}_{3\times 3}$$

where $< O_{ij}^2 >_{\varepsilon,\Theta} = N^{-1} K_{00}$, and in the fundamental representation (f),

$$C_{2f} = \sum_{a=1}^{N_c^2-1} (\frac{\lambda_a}{2})^2 = \frac{N_c^2 - 1}{2N_c}$$

As usual in Perturbation Theory and a number of non-perturbative models, such as the MIT Bag-Model, the SVM Model, Lattice calculations, *etc.*.

QCD fermionic amplitudes ..

In the same way, but at $k_i = 1, k_i' = 0$ and $k_i = 0, k_i' = 1$, one obtains now,

$$\prod_{i=1}^{N} [1 - i(-1)^{q_i}] \int_0^\infty \frac{\mathrm{d}\alpha_1^i}{\alpha_1^i} \{1, \alpha_1^i\} \int_0^\infty \frac{\mathrm{d}\alpha_2^i}{\alpha_2^i} \{0, \alpha_2^i\} G_{34}^{23} \left(C^{st} (\alpha_1^i \alpha_2^i)^2 | ..\right)$$
$$\times K_{10} \left((DC_{2f})^2 + (DC_{3f}) \right) \mathbf{1}_{N_c \times N_c}$$

where,

$$\sum_{a,b,c=1}^{N_c^1 - 1} d_{abc}\, t^a t^b t^c \equiv C_{3f} \mathbf{1}_{N_c \times N_c}$$

is the second, cubic Casimir operator of the $SU_c(3)$-Lie algebra which is *rank*-2.

QCD fermionic amplitudes ..

At next order, corresponding to terms of order $<(\mathcal{O}^{ij}(\mathbf{p})\mathcal{T}_j)^6>_{\varepsilon,\Theta}$, generated by $\{1,\alpha_2^i\}\times\{1,\alpha_2^i\}$, $\{2,\alpha_2^i\}\times\{0,\alpha_2^i\}$ and $\{0,\alpha_2^i\}\times\{2,\alpha_2^i\}$, one gets for the second line,

$$K_{11}\left\{\left((DC_{2f})^3+\frac{4}{9}(DC_{2f})^2+3(DC_{2f})(DC_{3f})+\frac{1}{9}(DC_{2f})+2(DC_{3f})\right)\mathbf{1}_{3\times 3}\right.$$
$$\left.+\sum_{k,j,l,h,m} 2d_{kjm}d_{klh}\left(T_jT_mT_lT_h\right)+d_{lkh}d_{ljm}\left(T_jT_kT_hT_m\right)\right\}$$

where $K_{11}=K_{20}=K_{02}$.

To proceed, one must choose a basis: Resorting to the Gell-Mann basis of generators, the second contribution is proven to cancel out.

Casimir operator dependences of QCD amplitudes as seen from EL

- At quenched and eikonal approximations at least, QCD amplitudes as seen from *EL*, exhibit a dependence not on the first (quadratic) Casimir solely, C_{2f}, but on the first and second(cubic) Casimirs, C_{2f} and C_{3f}.
- Given the rank-2 character of the $SU_c(3)$ color algebra, it had to be so in a way or other.
- This may be looked upon as a(nother) good sign in favor of the *EL* property.
- Resumming the 2 *sine*-series seems hopeless, though..
- The only efficient way appears to be the famous device ..

'Matrix Reloaded'

The End