Energy flow in $\mathcal{N} = 4$ SYM

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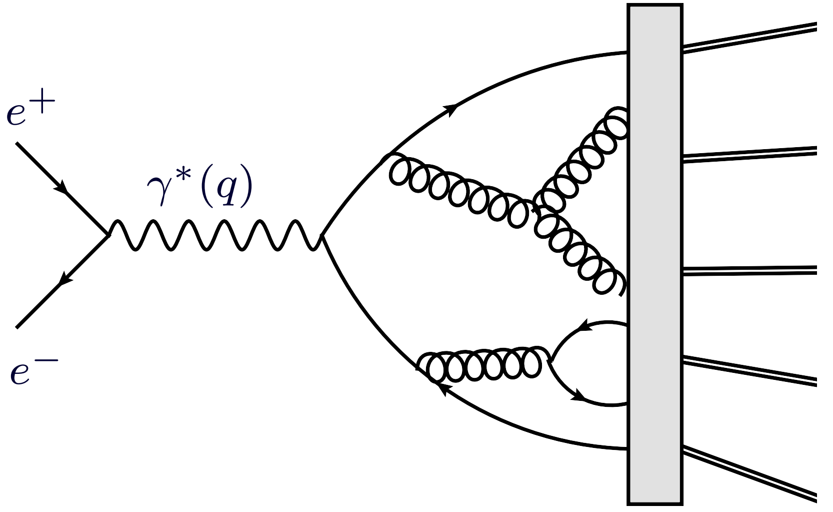
e^+e^- annihilation in $\mathcal{N}=4$ SYM

- ✔ Define IR finite observables in $\mathcal{N}=4$ SYM and evaluate them both at weak/strong coupling
- ✔ Are closely related to the QCD weighted cross-sections for the final states in e^+e^-–annihilation



- ✔ From QCD to $\mathcal{N}=4$ SYM: introduce an analog of the electromagnetic current:

  (protected) half-BPS operator built from the six real scalars

$$O^{IJ}_{\mathbf{20}'}(x) = \mathrm{tr}\left[\Phi^I\Phi^J - \tfrac{1}{6}\delta^{IJ}\Phi^K\Phi^K\right], \qquad (I,J=1,\dots,6)$$

$$O(x,Y) = Y^I Y^J O^{IJ}_{\mathbf{20}'}(x) = Y^I Y^J\, \mathrm{tr}[\Phi^I(x)\Phi^J(x)]$$

The null vector Y^I defines the orientation of the projected operator in the isotopic $SO(6)$ space

What are the properties of the final states created from the vacuum by the operator $O(x,Y)$?

Final states in $\mathcal{N}=4$ SYM

- ✔ To lowest order in the coupling, $O(x,Y)$ produces a pair of scalars out of the vacuum
- ✔ For arbitrary coupling, the state $O(x,Y)|0\rangle$ can be decomposed into an infinite sum over on-shell states with an arbitrary number of scalars (s), gauginos (λ) and gauge fields (g)

$$\int d^4x\, \mathrm{e}^{iqx}\, O(x,Y)|0\rangle = |ss\rangle + |ssg\rangle + |s\lambda\lambda\rangle + \dots$$

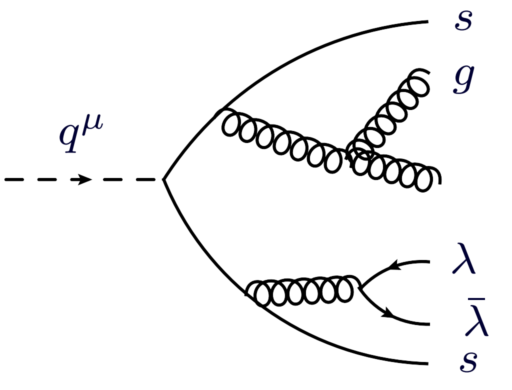
- ✔ The amplitude of creation of a particular final state $|X\rangle$ out of the vacuum

$$\langle X|\int d^4x\, \mathrm{e}^{iqx}\, O(x,Y)|0\rangle = (2\pi)^4\delta^{(4)}(q-p_X)\mathcal{M}_{O_{\mathbf{20'}}\to X}$$

p_X is the total momentum of the state $|X\rangle$

- ✔ The amplitude $\mathcal{M}_{O\to X}$ has the meaning of a (IR divergent) form-factor

$$\mathcal{M}_{O_{\mathbf{20'}}\to X} = \langle X|O(0,Y)|0\rangle$$



Total cross-section of $O_{\mathbf{20'}} \to \text{everything}$

- Analog of the QCD process $\mathrm{e}^+\,\mathrm{e}^- \to$ everything

$$\sigma_{\rm tot}(q) = \sum_X (2\pi)^4 \delta^{(4)}(q-p_X) |\mathcal{M}_{O_{\mathbf{20'}}\to X}|^2$$

- To lowest order in the coupling, the production of a pair of scalars

$$\sigma_{\rm tot}(q;Y) = \frac{1}{2}(N^2-1)(Y\bar{Y})^2 \int \frac{d^4k}{(2\pi)^4}(2\pi)^2\delta_+(k^2)\delta_+((q-k)^2) + \ldots$$

- To higher order in the coupling, each term in the sum $\sum_X$ has IR / collinear divergences
- How to avoid divergences? Use the completeness condition $\sum_X |X\rangle\langle X| = 1$

$$\sigma_{\rm tot}(q) = \int d^4x\ \mathrm{e}^{iqx} \sum_X \langle 0|O(0,\bar{Y})|X\rangle\, \mathrm{e}^{-ixp_X}\, \langle X|O(0,Y)|0\rangle$$

$$= \int d^4x\ \mathrm{e}^{iqx}\, \langle 0|O(x,\bar{Y})O(0,Y)|0\rangle$$ *The operators are not time ordered!*

Wightman correlation function (protected for half-BPS operators)

- All-loop result in $\mathcal{N}=4$ SYM [van Neerven]

$$\sigma_{\rm tot}(q) = \frac{1}{16\pi}(N^2-1)(Y\bar{Y})^2\theta(q^0)\theta(q^2)$$

Perturbative corrections cancel order by order IR finite to any order in the coupling

Weighted cross-section

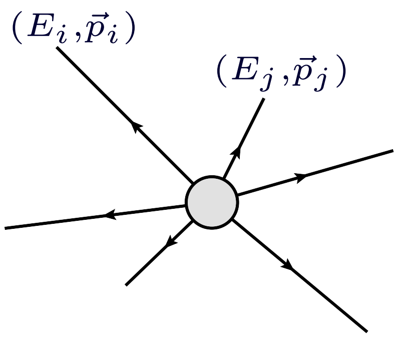
- ✔ More refined information about the final states in $O_{\mathbf{20'}} \to$ everything
- ✔ Assign a weight factor $w(X)$ to the contribution of each state $|X\rangle$

$$\sigma_W(q) = \sigma_{\rm tot}^{-1} \sum_X (2\pi)^4 \delta^{(4)}(q - p_X) w(X) |\mathcal{M}_{O_{\mathbf{20'}} \to X}|^2$$

$$= \sigma_{\rm tot}^{-1} \int d^4x\, \mathrm{e}^{iqx} \sum_X \langle 0|O(x, \bar{Y})|X\rangle w(X) \langle X|O(0,Y)|0\rangle$$

- ✔ Less inclusive quantity as compared with the total cross section, no optical theorem
- ✔ Choose of the weight factors $w(X)$ gives an access to the flow of various quantum numbers of particles (energy, charge, etc) in the final state
- ✔ Popular choice – energy-energy correlations [Basham,Brown,Ellis,Love]

$$w(X) = \sum_{i,j} E_i E_j \delta(\cos\theta_{ij} - \cos\chi)$$



Are known in QCD up to 2 loops numerically

Energy flow

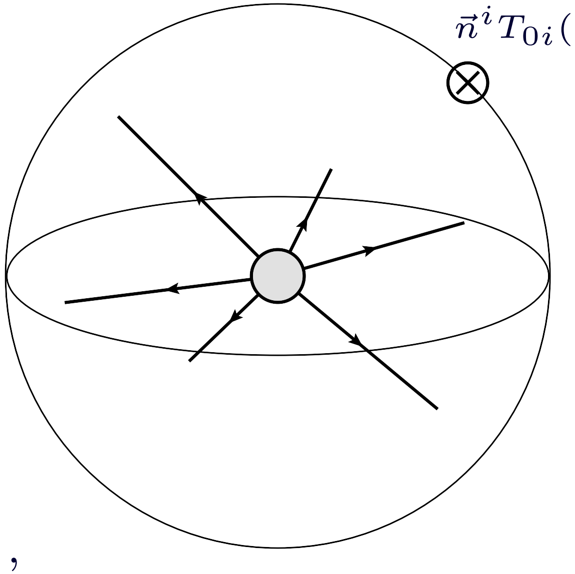
- The total energy in the final state $|X\rangle = |k_1, \ldots, k_\ell\rangle$ that flows into the detector located at spatial infinity in the direction of the vector $\vec{n}$.

$$w_{\mathcal{E}}(k_1, \ldots, k_\ell) = \sum_{i=1}^{\ell} k_i^0\, \delta^{(2)}(\Omega_{\vec{k}_i} - \Omega_{\vec{n}})\,,$$

- Energy flow operator

$$\mathcal{E}(\vec{n})|X\rangle = w_{\mathcal{E}}(X)|X\rangle\,.$$

- Is expressed in terms of the energy-momentum tensor in $\mathcal{N}=4$ SYM [Sveshnikov,Tkachov],[GK,Oderda,Sterman]



$$\mathcal{E}(\vec{n}) = \int_0^\infty dt \lim_{r\to\infty} r^2\, \vec{n}^i T_{0i}(t, r\vec{n})$$

- Representation for $\mathcal{E}(\vec{n})$ in terms of creation and annihilation operators of on-shell states

$$\mathcal{E}(\vec{n}) = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta_+(k^2)\, k^0\, \delta^{(2)}(\Omega_{\vec{n}} - \Omega_{\vec{k}}) \sum_{i=s,\lambda,\bar{\lambda},g} a_i^\dagger(k) a_i(k)\,,$$

Energy correlations

- Single correlator

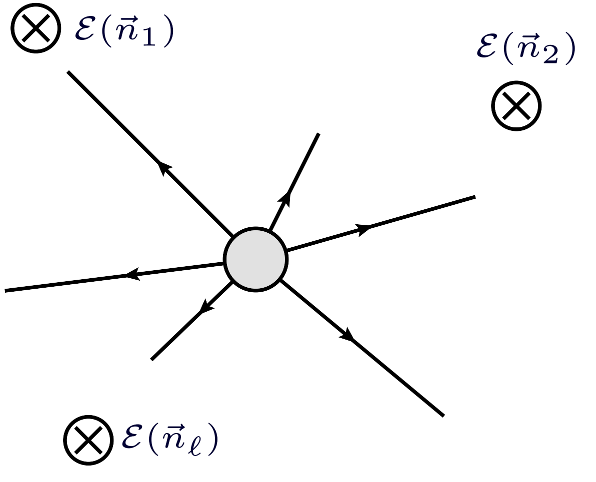
$$\sum_X \langle 0|O(x,\bar{Y})|X\rangle w_{\mathcal{E}}(X)\langle X|O(0,Y)|0\rangle = \sum_X \langle 0|O(x,\bar{Y})\mathcal{E}(\vec{n})|X\rangle\langle X|O(0,Y)|0\rangle$$
$$= \langle 0|O(x,\bar{Y})\mathcal{E}(\vec{n})O(0,Y)|0\rangle$$

Wightman correlation function (no time ordering!) due to real time evolution

- Single energy flow

$$\langle \mathcal{E}(\vec{n}_1)\rangle = \sigma_{\rm tot}^{-1}\int d^4x \; {\rm e}^{iqx}\langle 0|O(x,\bar{Y})\,\mathcal{E}(\vec{n}_1)\,O(0,Y)|0\rangle$$

- Multi-energy correlations [GK,Sterman],[Belitsky,GK,Sterman],[Hofman,Maldacena]



$$\langle \mathcal{E}(\vec{n}_1)\ldots\mathcal{E}(\vec{n}_\ell)\rangle$$
$$= \sigma_{\rm tot}^{-1}\int d^4x \; {\rm e}^{iqx}\langle 0|O(x,\bar{Y})\,\mathcal{E}(\vec{n}_1)\ldots\mathcal{E}(\vec{n}_\ell)\,O(0,Y)|0\rangle$$

Energy flow in the direction of $\vec{n}_1,\ldots,\vec{n}_\ell$

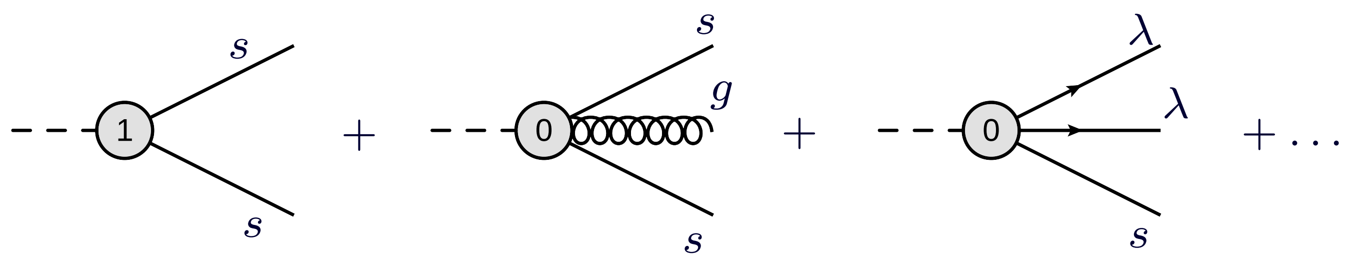
Depends on the relative angles $\cos\theta_{ij} = (\vec{n}_i\cdot\vec{n}_j)$

- The goal is to find $\langle \mathcal{E}(\vec{n}_1)\ldots\mathcal{E}(\vec{n}_\ell)\rangle$ for arbitrary coupling in $\mathcal{N}=4$ SYM

Weighted cross-sections from amplitudes I

- ✔ Transition amplitude

$$\mathcal{M}_{O_{\mathbf{20'}}\to X} =$$



- ✔ One-loop matrix elements (Mandelstam invariants $s_{ij} = (p_i + p_j)^2$ with $p_i^2 = 0$)

$$|\mathcal{M}_{O_{\mathbf{20'}}\to ss}|^2 = \left|\langle s(p_1)s(p_2)|O(0,Y)|0\rangle\right|^2 = \frac{2}{s_{12}}\left[1 + aF_{\rm virt}(q^2)\right]$$

$$|\mathcal{M}_{O_{\mathbf{20'}}\to ssg}|^2 = \left|\langle s(p_1)s(p_2)g(p_3)|O(0,Y)|0\rangle\right|^2 = a\,\frac{s_{12}}{s_{13}s_{23}}$$

$$|\mathcal{M}_{O_{\mathbf{20'}}\to s\lambda\lambda}|^2 = \left|\langle \lambda(p_1)\lambda(p_2)s(p_3)|O(0,Y)|0\rangle\right|^2 = a\,\frac{2}{s_{12}}$$

- ✔ The total transition amplitude to order $O(a)$

$$\sigma_{\rm tot}(q) = \int \text{dPS}_2\,|\mathcal{M}_{O_{\mathbf{20'}}\to ss}|^2 + \int \text{dPS}_3\left(|\mathcal{M}_{O_{\mathbf{20'}}\to ssg}|^2 + |\mathcal{M}_{O_{\mathbf{20'}}\to s\lambda\lambda}|^2\right) + O(a^2)$$

$$= \frac{1}{8\pi}\left[1 + aF_{\rm virt}(q^2)\right] + a\int \text{dPS}_3\,\frac{s_{12}^2 + 2s_{13}s_{23}}{s_{12}s_{13}s_{23}} + O(a^2) = \frac{1}{8\pi} + 0\cdot a + O(a^2)$$

Protected from perturbative corrections

Weighted cross-sections from amplitudes II

- Energy correlations

$$\sigma_{\mathcal{E}}(q) = \sigma_{\rm tot}^{-1}\Big[\int \text{dPS}_2\, w_{\mathcal{E}}(p_1,p_2)\,|\mathcal{M}_{O_{\mathbf{20'}}\to ss}|^2$$
$$+\int \text{dPS}_3\, w_{\mathcal{E}}(p_1,p_2,p_3)\Big(|\mathcal{M}_{O_{\mathbf{20'}}\to ssg}|^2 + |\mathcal{M}_{O_{\mathbf{20'}}\to s\lambda\lambda}|^2\Big) + O(a^2)\Big]$$

- Single detector (space-time orientation is specified by the light-like vector $n^\mu = (1,\vec{n})$)

$$\langle \mathcal{E}(\vec{n})\rangle = \frac{1}{4\pi}\frac{(q^2)^2}{(qn)^3}$$

Protected from loop corrections

- Two detectors (unprotected quantity) [Zhiboedov],[Engelund,Roiban]

$$\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\rangle = -\frac{a}{4(2\pi)^4}\,\frac{q^2}{(n_1n_2)^3}\,\frac{z\ln(1-z)}{(1-z)} + O(a^2)$$

The scaling variable in the rest frame of the source $q^\mu = (q^0,\vec{0})$

$$z = \frac{q^2(n_1n_2)}{2(qn_1)(qn_2)} = (1-\cos\theta_{12})/2\,, \qquad 0<z<1$$

Two-loop corrections to $\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\rangle$ are hard to compute ($\sim 10^2$ diagrams)

Weighted cross-sections from correlation functions I

- ✔ Energy flow operator

$$\langle \mathcal{E}(\vec{n}_1)\rangle \sim \int d^4x\, \mathrm{e}^{iqx} \langle 0|O(x,\bar{Y})\,\mathcal{E}(\vec{n}_1)\,O(0,Y)|0\rangle$$

$$= \underbrace{\int d^4x\, \mathrm{e}^{iqx}}_{\text{Fourier}} \underbrace{\int_0^\infty dt \lim_{r\to\infty} r^2}_{\text{Detector limit}} \underbrace{\langle 0|O(x,\bar{Y})\,T_{0\vec{n}_1}(x_1)\,O(0,Y)|0\rangle}_{\text{Wightman corr. function}}\Big|_{x_1} = (t, r\vec{n}_1)$$

- ✔ Generalization for ℓ detectors

$$\langle \mathcal{E}(\vec{n}_1)\ldots\mathcal{E}(\vec{n}_\ell)\rangle = \text{Fourier} \times \text{Limit}\left[\langle 0|O(x,\bar{Y})\,T_{0\vec{n}_1}(x_1)\ldots T_{0\vec{n}_\ell}(x_\ell)\,O(0,Y)|0\rangle\Big|_{x_i=(t_i,r_i\vec{n}_i)}\right]$$

- ✔ How to compute energy flow correlators:
  - ✗ Compute corr.function $\langle O(x)T(x_1)\ldots T(x_\ell)O(0)\rangle$ in Euclid
  - ✗ Continue to Minkowski with Wightman prescription
  - ✗ Take detector limit + perform Fourier
- ✔ Correlation functions in $\mathcal{N}=4$ SYM have a lot of symmetry :
  - ✗ $\langle O(x)T(x_1)O(0)\rangle$ is fixed by conformal symmetry $\to$ exact result for $\langle\mathcal{E}(\vec{n}_1)\rangle$ [Hofman,Maldacena]
  - ✗ $\langle O(x)T(x_1)T(x_2)O(0)\rangle$ is not fixed by conformal symmetry

Weighted cross-sections from correlation functions II

Hidden beauty of $\mathcal{N}=4$ SYM:

- ✔ Quantum corrections to various correlation functions are determined by the same scalar function

$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle_E = \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2}\Phi(u,v;a)$$

$$\langle O(x_1)T(x_2)T(x_3)O(x_4)\rangle_E = \frac{1}{(x_{12}^2 x_{23}^2 x_{34}^2)^2} P(\partial_u,\partial_v)\Phi(u,v;a)$$

Conformal ratios

$$u = x_{12}^2 x_{34}^2/(x_{13}^2 x_{24}^2)\,, \qquad v = x_{23}^2 x_{41}^2/(x_{13}^2 x_{24}^2)$$

- ✔ Universal function in $\mathcal{N}=4$ SYM at weak coupling [Eden,Schubert,Sokatchev],[Bianchi et al]

$$\Phi(u,v) = a\,\Phi^{(1)}(u,v) + a^2\left(\frac{1}{2}(1+u+v)\left[\Phi^{(1)}(u,v)\right]^2 + 2\left[\Phi^{(2)}(u,v) + \frac{1}{u}\Phi^{(2)}(v/u,1/u) + \frac{1}{v}\Phi^{(2)}(1/v,u/v)\right]\right) + O(a^3)$$

$\Phi^{(1)}(u,v)$ 'box' integral, $\Phi^{(2)}(u,v)$ 'double' box integral

- ✔ 'Permutation symmetry' allows us to determine $\Phi_{\rm weak}(u,v)$ to six loops [Eden,Heslop,GK,Sokatchev]
- ✔ AdS/CFT correspondences predicts $\Phi(u,v)$ at strong coupling [Arutyunov, Frolov]

From Euclid to Minkowski

- ✔ Brute force method: compute anew using Schwinger-Keldysh technique (too hard)
- ✔ Better method: analytically continue correlation functions from Euclid to Minkowski+Wightman
- ✔ Warm-up example: free scalar propagator $D_{\text{Euclid}}(x) = \langle \phi(x)\phi(0) \rangle \sim 1/x^2$

$$\langle 0|\phi(x)\phi(0)|0\rangle = \sum_n \langle 0|\phi(x)|n\rangle\langle n|\phi(0)|0\rangle$$

$$= \sum_{E_n>0} \mathrm{e}^{-iE_n(x^0-i0)+i\vec{p}\vec{x}}\langle 0|\phi(0)|n\rangle\langle n|\phi(0)|0\rangle \sim \frac{1}{(x^0-i0)^2-\vec{x}^2}$$

- ✔ How to get Wightman correlation functions ('magic' recipe) [Mack]
  - ✗ Go to Mellin space:

$$\Phi_{\text{Euclid}} = \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} M(j_1, j_2; a)\, u^{j_1} v^{j_2}\,, \qquad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2}$$

  - ✗ Substitute $x_{ij}^2 \;\to\; x_{ij,+}^2 = x_{ij}^2 - i0\cdot x_{ij}^0$

$$\Phi_{\text{Wightman}} = \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} M(j_1, j_2; a) \left(\frac{x_{12,+}^2 x_{34,+}^2}{x_{13,+}^2 x_{24,+}^2}\right)^{j_1} \left(\frac{x_{23,+}^2 x_{41,+}^2}{x_{13,+}^2 x_{24,+}^2}\right)^{j_2}$$

- ✔ $M(j_1, j_2; a)$ is known both at weak and strong coupling in planar $\mathcal{N}=4$ SYM

Energy correlations

✔ Energy correlations for arbitrary coupling

$$\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\rangle = \frac{1}{(4\pi^2)^2}\frac{q^2}{(n_1 n_2)^3}\mathcal{F}_{\mathcal{E}}(z;a)\,, \qquad z = (1-\cos\theta_{12})/2$$

✗ Weak coupling:

$$\mathcal{F}_{\mathcal{E}}(z;a<1) = -\frac{a}{4}\,\frac{z\ln(1-z)}{(1-z)} + O(a^2)$$

[Zhiboedov],[Engelund,Roiban]

✗ Strong coupling:

$$\mathcal{F}_{\mathcal{E}}(z;a\to\infty) = 8\,z^3 + O(1/\sqrt{a})$$

[Hofman,Maldacena]

✔ All-loop representation

$$\mathcal{F}_{\mathcal{E}}(z;a) = \int_{-\delta-i\infty}^{-\delta+i\infty}\frac{dj_1 dj_2}{(2\pi i)^2}\underbrace{M(j_1,j_2;a)}_{\text{corr.function}}\underbrace{K_{\mathcal{E}}(j_1,j_2)}_{\text{detector}}\left(\frac{1-z}{z}\right)^{j_1+j_2}$$

Detector function is independent on the coupling

$$K_{\mathcal{E}}(j_1,j_2) \sim \frac{\Gamma(1-j_1-j_2)}{\Gamma(j_1+j_2)[\Gamma(1-j_1)\Gamma(1-j_2)]^2}$$

$$M(j_1,j_2;a) = \underbrace{aM^{(1)}(j_1,j_2) + a^2 M^{(2)}(j_1,j_2)}_{\text{are known}} + \dots$$

✔ Analytical expression for $\mathcal{F}_{\mathcal{E}}(z;a)$ at two loops, extension to higher loops is feasible

Conclusions and open questions

- Energy correlations are good/nontrivial physical observables in $\mathcal{N} = 4$ SYM
- Relation to energy flow correlations in QCD (most complicated part)?
- All symmetries of $\mathcal{N} = 4$ SYM are preserved, what is the manifestation of integrability?
- Interpolation between weak and strong coupling?
- Other proposals for ‘good’ observables?