PRD71(2005), JPG32(2006),
PRD77(2008), NPBPS234(2013),
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Dispersive approach to QCD and hadronic vacuum polarization function

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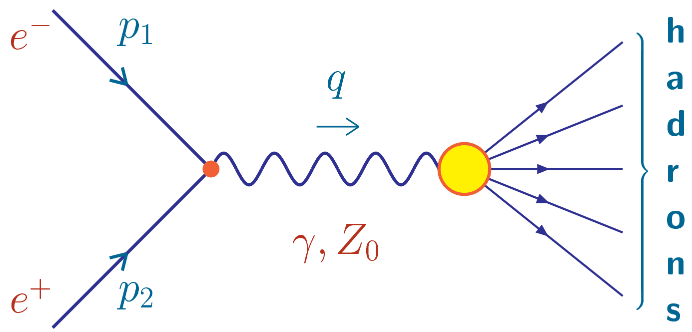
INTRODUCTION

Hadronic vacuum polarization function $\Pi(q^2)$ plays a central role in various issues of QCD and Standard Model. In particular, the theoretical description of some strong interaction processes and hadronic contributions to electroweak observables is inherently based on $\Pi(q^2)$:

- **electron–positron annihilation into hadrons**
- **hadronic τ lepton decay**
- **muon anomalous magnetic moment**
- **running of the electromagnetic coupling**

GENERAL DISPERSION RELATIONS

The cross–section of $e^+e^- \to$ hadrons:



$$\sigma = 4\pi^2 \frac{2\alpha^2}{s^3} L^{\mu\nu} \Delta_{\mu\nu},$$

where $s = q^2 = (p_1 + p_2)^2 > 0$,

$$L_{\mu\nu} = \frac{1}{2}\Big[q_\mu q_\nu - g_{\mu\nu}q^2 - (p_1 - p_2)_\mu (p_1 - p_2)_\nu\Big],$$

$$\Delta_{\mu\nu} = (2\pi)^4 \sum_\Gamma \delta(p_1 + p_2 - p_\Gamma)\, \big\langle 0|J_\mu(-q)|\Gamma\big\rangle\langle\Gamma|J_\nu(q)|0\rangle,$$

and $J_\mu = \sum_f Q_f :\bar{q}\,\gamma_\mu\, q:$ **is the electromagnetic quark current.**

Kinematic restriction: **hadronic tensor** $\Delta_{\mu\nu}(q^2)$ **acquires non–zero values only for** $q^2 \geq m^2$**, since otherwise no hadron state** Γ **could be excited**

■ Feynman (1972); Adler (1974).

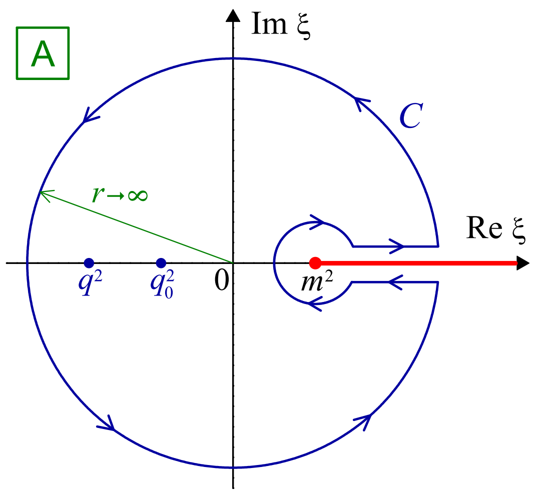
The hadronic tensor can be represented as $\Delta_{\mu\nu} = 2\,\mathrm{Im}\,\Pi_{\mu\nu}$,

$$\Pi_{\mu\nu}(q^2) = i\int e^{iqx}\langle 0|\, T\{J_\mu(x)\, J_\nu(0)\}\,|0\rangle\, d^4x = i(q_\mu q_\nu - g_{\mu\nu}q^2)\frac{\Pi(q^2)}{12\pi^2}.$$

Kinematic restriction: $\Pi(q^2)$ **has the only cut** $q^2 \ge m^2$.

Dispersion relation for $\Pi(q^2)$:

$$\Delta\Pi(q^2,\, q_0^2) = \frac{1}{2\pi i}(q^2 - q_0^2)\oint_C \frac{\Pi(\xi)}{(\xi - q^2)(\xi - q_0^2)}\,d\xi$$
$$= (q^2 - q_0^2)\int_{m^2}^{\infty} \frac{R(s)}{(s - q^2)(s - q_0^2)}\,ds,$$



where $\Delta\Pi(q^2,\, q_0^2) = \Pi(q^2) - \Pi(q_0^2)$ **and** $R(s)$ **denotes the measurable ratio of two cross–sections (** $R(s) \equiv 0$ **for** $s < m^2$ **)**

$$R(s) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\Big[\Pi(s + i\varepsilon) - \Pi(s - i\varepsilon)\Big] = \frac{\sigma\left(e^+e^- \to \text{hadrons};\, s\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-;\, s\right)}.$$

For practical purposes it is proves to be convenient to deal with the Adler function ($Q^2 = -q^2 \geq 0$)

$$D(Q^2) = \frac{d\,\Pi(-Q^2)}{d\ln Q^2}, \qquad D(Q^2) = Q^2 \int_{m^2}^{\infty} \frac{R(s)}{(s+Q^2)^2}\, ds$$

- Adler (1974); De Rujula, Georgi (1976); Bjorken (1989).

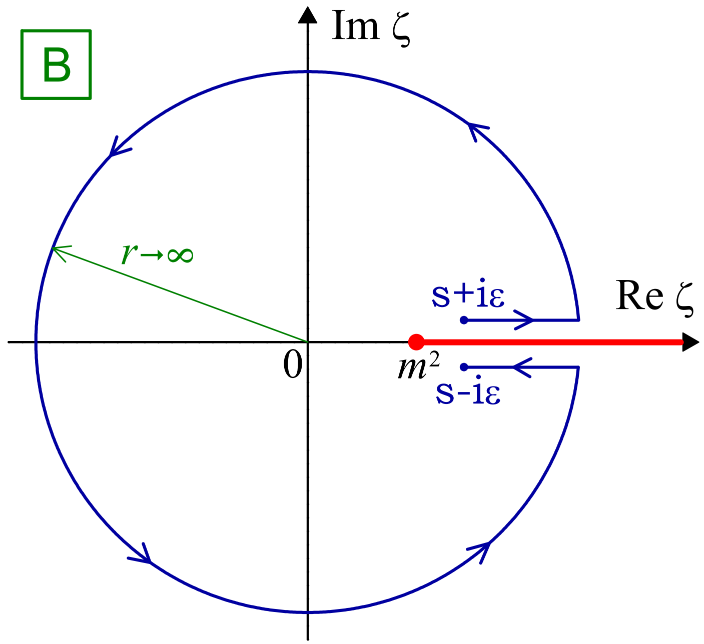
The inverse relations between the functions on hand read

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\,\frac{d\zeta}{\zeta},$$

- Radyushkin (1982); Krasnikov, Pivovarov (1982)

$$\Delta\Pi(-Q^2, -Q_0^2) = -\int_{Q_0^2}^{Q^2} D(\zeta)\frac{d\zeta}{\zeta}$$

- Nesterenko (2013).



The integration contour in complex ζ–plane lies in the region of analyticity of the integrand.

The complete set of relations between $\Pi(q^2)$, $R(s)$, and $D(Q^2)$:

$$\Delta\Pi(q^2,\, q_0^2) = (q^2 - q_0^2) \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma - q^2)(\sigma - q_0^2)}\, d\sigma = - \int_{-q_0^2}^{-q^2} D(\zeta) \frac{d\zeta}{\zeta},$$

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[\Pi(s + i\varepsilon) - \Pi(s - i\varepsilon)\Big] = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta) \frac{d\zeta}{\zeta},$$

$$D(Q^2) = - \frac{d\,\Pi(-Q^2)}{d\,\ln Q^2} = Q^2 \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma + Q^2)^2}\, d\sigma.$$

Their derivation requires only the location of cut of $\Pi(q^2)$ and its UV asymptotic. Neither additional approximations nor phenomenological assumptions are involved.

Nonperturbative constraints:

- $\Pi(q^2)$: **only cut** $q^2 \geq m^2$;
- $R(s)$: **vanishes for** $s < m^2$, **embodies** π^2**–terms;**
- $D(Q^2)$: **only cut** $Q^2 \leq -m^2$, **vanishes at** $Q^2 \to 0$.

DISPERSIVE APPROACH TO QCD

Functions on hand in terms of common spectral density:

$$\Delta\Pi(q^2,\, q_0^2) = \Delta\Pi^{(0)}(q^2,\, q_0^2) + \int_{m^2}^{\infty} \rho(\sigma) \ln\left(\frac{\sigma - q^2}{\sigma - q_0^2}\frac{m^2 - q_0^2}{m^2 - q^2}\right)\frac{d\,\sigma}{\sigma},$$

$$R(s) = R^{(0)}(s) + \theta(s - m^2) \int_{s}^{\infty} \rho(\sigma)\frac{d\,\sigma}{\sigma},$$

$$D(Q^2) = D^{(0)}(Q^2) + \frac{Q^2}{Q^2 + m^2}\int_{m^2}^{\infty} \rho(\sigma)\frac{\sigma - m^2}{\sigma + Q^2}\frac{d\,\sigma}{\sigma},$$

$$\rho(\sigma) = \frac{1}{\pi}\frac{d}{d\,\ln\sigma}\,\mathrm{Im}\lim_{\varepsilon\to 0_+} p(\sigma - i\varepsilon) = -\frac{d\,r(\sigma)}{d\,\ln\sigma} = \frac{1}{\pi}\,\mathrm{Im}\lim_{\varepsilon\to 0_+} d(-\sigma - i\varepsilon),$$

with $\Delta\Pi^{(0)}(q^2,\, q_0^2)$, $R^{(0)}(s)$, and $D^{(0)}(Q^2)$ being leading–order terms, $p(q^2)$, $r(s)$, and $d(Q^2)$ being the strong corrections

- **Nesterenko, Papavassiliou (2005, 2006); Nesterenko (2007–2013).**

The obtained integral representations automatically embody all the aforementioned nonperturbative constraints. Their derivation requires only the general dispersion relations and the asymptotic ultraviolet behavior of $\Pi(q^2)$. Neither additional approximations nor model–dependent assumptions were involved.

The leading–order terms of the functions on hand:

$$\bar{\Pi}^{(0)}(q^2) = \frac{2}{\tan^2\varphi}\left(1 - \frac{\varphi}{\tan\varphi}\right) - \frac{2}{3},$$

$$R^{(0)}(s) = \theta(s - m^2)\left(1 - \frac{m^2}{s}\right)^{3/2},$$

$$D^{(0)}(Q^2) = 1 + \frac{3}{\xi}\left[1 - \sqrt{1+\xi^{-1}}\,\sinh^{-1}\!\left(\xi^{1/2}\right)\right],$$

where $\bar{\Pi}(q^2) = \Delta\Pi(0, q^2)$**,** $\sin^2\varphi = q^2/m^2$**,** $\xi = Q^2/m^2$

■ **Feynman (1972); Akhiezer, Berestetsky (1965).**

Perturbative contribution to the spectral density:

$$\rho_{\text{pert}}(\sigma) = \frac{1}{\pi}\frac{d}{d\,\ln\sigma}\,\text{Im}\lim_{\varepsilon\to 0_+} p_{\text{pert}}(\sigma - i\varepsilon) = -\frac{d\,r_{\text{pert}}(\sigma)}{d\,\ln\sigma} = \frac{1}{\pi}\,\text{Im}\lim_{\varepsilon\to 0_+} d_{\text{pert}}(-\sigma - i\varepsilon).$$

The following model for spectral density will be employed:

$$\rho(\sigma) = \frac{4}{\beta_0}\frac{1}{\ln^2(\sigma/\Lambda^2) + \pi^2} + \frac{\Lambda^2}{\sigma}$$

■ Nesterenko (2011–2013).

In the massless limit ($m = 0$) integral representations read

$$\Delta\Pi(q^2,\, q_0^2) = -\ln\left(\frac{-q^2}{-q_0^2}\right) + \int_0^\infty \rho(\sigma)\ln\left[\frac{1-(\sigma/q^2)}{1-(\sigma/q_0^2)}\right]\frac{d\,\sigma}{\sigma},$$

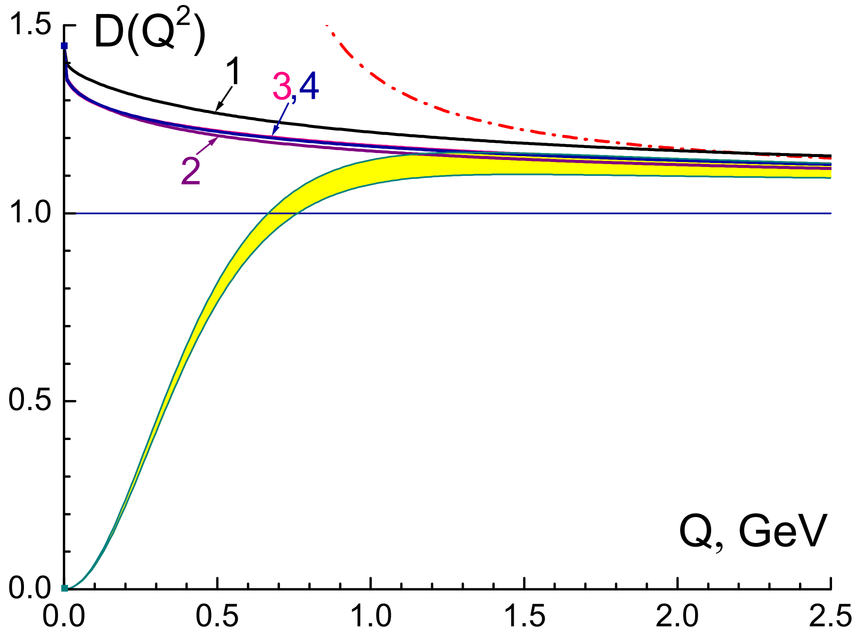
$$R(s) = \theta(s)\left[1 + \int_s^\infty \rho(\sigma)\frac{d\,\sigma}{\sigma}\right], \qquad D(Q^2) = 1 + \int_0^\infty \frac{\rho(\sigma)}{\sigma + Q^2}\,d\,\sigma.$$

For $\rho(\sigma) = \text{Im}\,d_{\text{pert}}(-\sigma - i0_+)/\pi$ two highlighted equations become identical to those of the APT ■ Shirkov, Solovtsov (1997–2007).

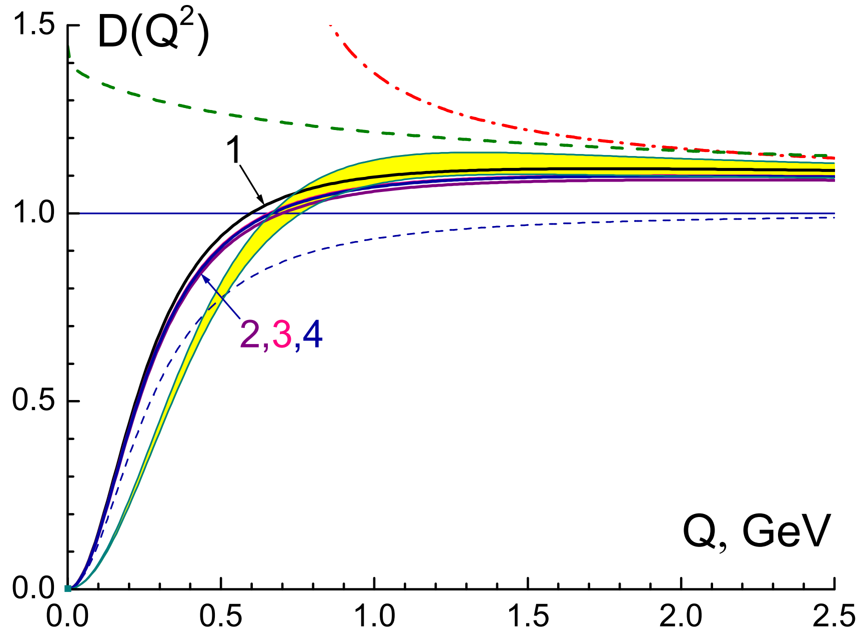
But it is essential to keep the threshold m^2 nonvanishing.

Adler function

massless limit $(m^2 = 0)$



nonvanishing m^2

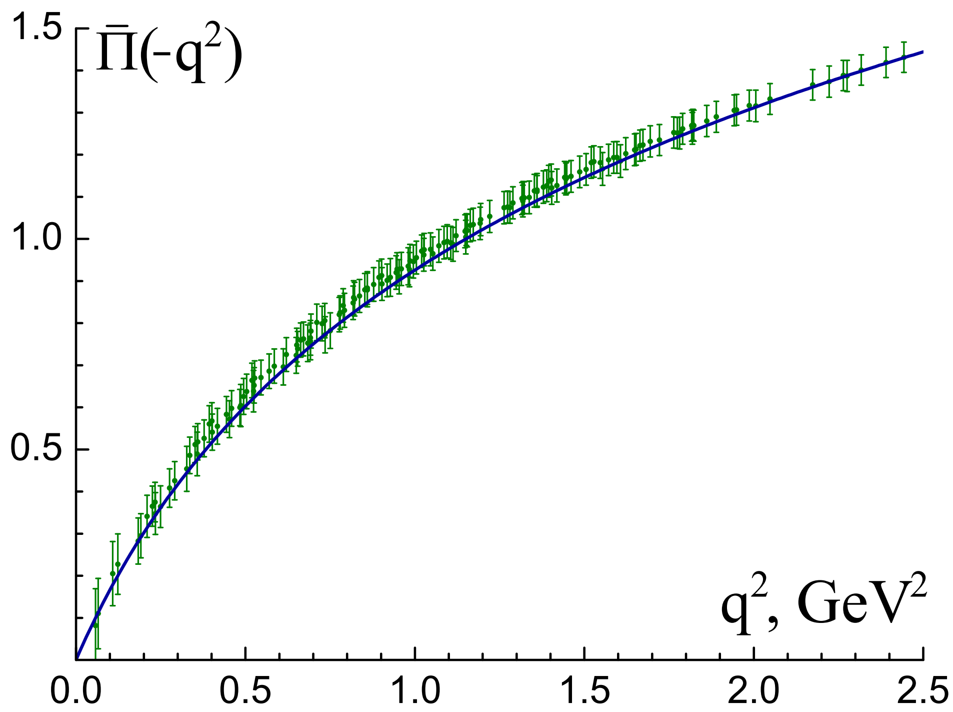


- **Nesterenko, Papavassiliou (2006); Nesterenko (2007–2009).**

Reliability of approaches:

Pert. theory	Massless APT	Dispersive approach
$Q \gtrsim 1.5\,\mathrm{GeV}$	$Q \gtrsim 1.0\,\mathrm{GeV}$	entire energy range

Hadronic vacuum polarization function

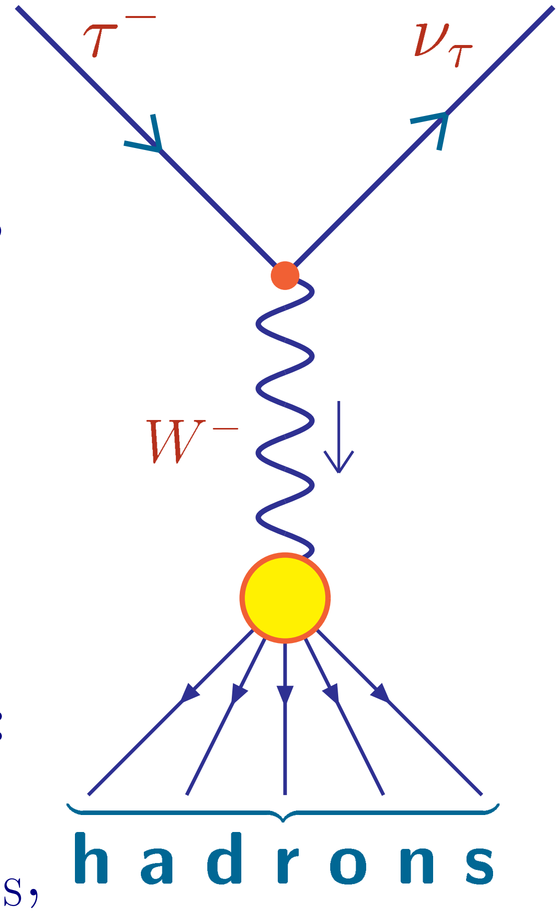


Solid curve presents DispQCD result for $\bar{\Pi}(q^2) = \Delta\Pi(0, q^2)$**, whereas its lattice prediction is shown by data points.**

- Della Morte, Jager, Juttner, Wittig (2011); Nesterenko (2013).

DispQCD result is in a good agreement with lattice data in the entire energy range.

Inclusive τ lepton hadronic decay



The interest to this process is due to

- **The only lepton with hadronic decays**
- **No need in phenomenological models**
- **Precise experimental data**
- **Probes infrared hadron dynamics**

The experimentally measurable quantity:

$$R_\tau = \frac{\Gamma(\tau^- \to \text{hadrons}^- \, \nu_\tau)}{\Gamma(\tau^- \to e^- \, \bar{\nu}_e \, \nu_\tau)} = R_{\tau,\mathrm{V}} + R_{\tau,\mathrm{A}} + R_{\tau,\mathrm{S}},$$

$$R_{\tau,\mathrm{V}} = R_{\tau,\mathrm{V}}^{J=0} + R_{\tau,\mathrm{V}}^{J=1} = 1.783 \pm 0.011 \pm 0.002,$$

$$R_{\tau,\mathrm{A}} = R_{\tau,\mathrm{A}}^{J=0} + R_{\tau,\mathrm{A}}^{J=1} = 1.695 \pm 0.011 \pm 0.002.$$

■ ALEPH Collaboration (1998–2008).

The theoretical prediction for the quantities on hand reads

$$R^{J=1}_{\tau,\mathrm{V/A}} = \frac{N_{\mathrm{c}}}{2}\,|V_{\mathrm{ud}}|^2\,S_{\mathrm{EW}}\Big(\Delta^{\mathrm{V/A}}_{\mathrm{QCD}} + \delta'_{\mathrm{EW}}\Big),$$

$N_{\mathrm{c}}=3$, $|V_{\mathrm{ud}}|=0.9738\pm0.0005$, $S_{\mathrm{EW}}=1.0194\pm0.0050$, $\delta'_{\mathrm{EW}}=0.0010$,

$$\Delta^{\mathrm{V/A}}_{\mathrm{QCD}} = 2\int_{m^2_{\mathrm{V/A}}}^{M^2_\tau} f\Big(\frac{s}{M^2_\tau}\Big) R^{\mathrm{V/A}}(s)\,\frac{ds}{M^2_\tau}\,,$$

where $M_\tau = 1.777\,\mathrm{GeV}$, $f(x) = (1-x)^2\,(1+2x)$,

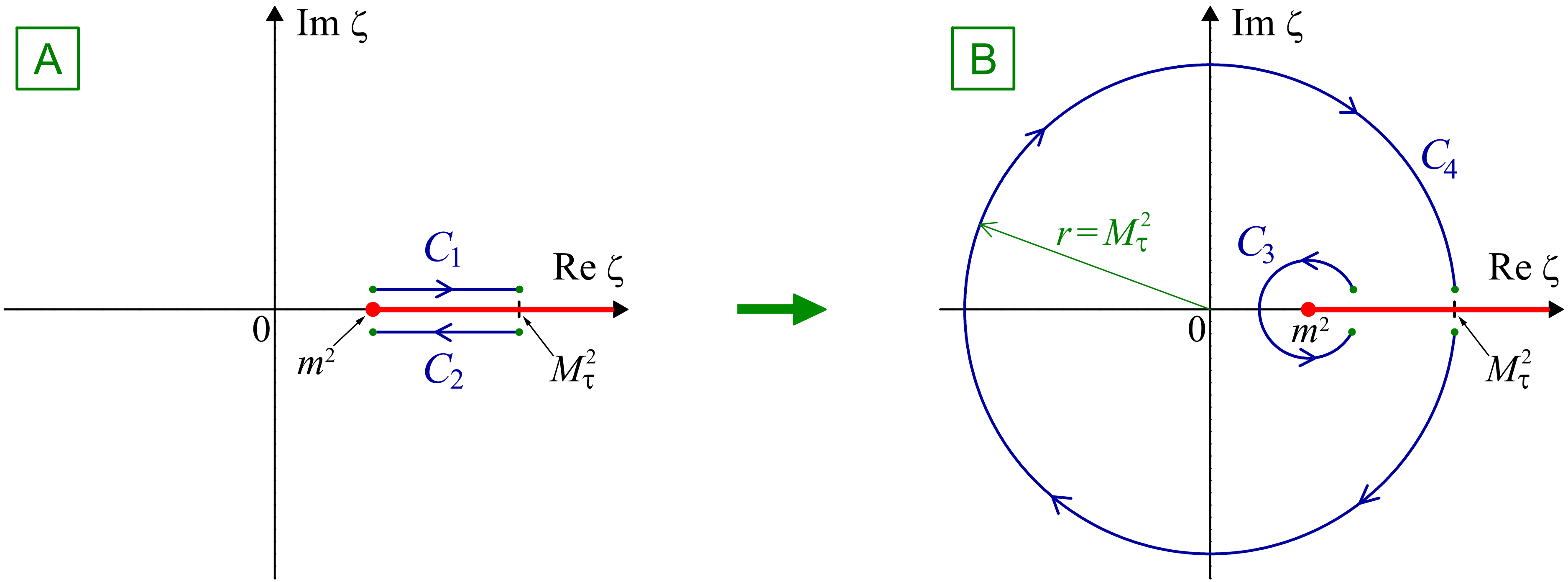
$$R^{\mathrm{V/A}}(s)=\frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\Big[\Pi^{\mathrm{V/A}}(s+i\varepsilon)-\Pi^{\mathrm{V/A}}(s-i\varepsilon)\Big]=\frac{1}{\pi}\,\mathrm{Im}\lim_{\varepsilon\to 0_+}\Pi^{\mathrm{V/A}}(s+i\varepsilon)$$

- Braaten, Narison, Pich (1992); Pivovarov (1992).

Integration by parts leads to

$$\Delta_{\mathrm{QCD}} = g(1)R(M^2_\tau) - g(\chi)R\big(m^2\big) + \frac{1}{2\pi i}\int_{C_1+C_2} g\left(\frac{\zeta}{M^2_\tau}\right) D(-\zeta)\frac{d\zeta}{\zeta},$$

where $\chi = m^2/M^2_\tau$ **and** $g(x) = x(2-2x^2+x^3)$.



$$\Delta_{\rm QCD} = g(1)R(M_\tau^2) - g(\chi)R\left(m^2\right) + \frac{1}{2\pi i}\int\limits_{C_3+C_4} g\left(\frac{\zeta}{M_\tau^2}\right)D(-\zeta)\frac{d\zeta}{\zeta}$$

Despite the aforementioned remarks, in the perturbative analysis the massless limit ($m = 0$) is assumed, that gives

$$\Delta_{\rm QCD} = \frac{1}{2\pi}\lim_{\varepsilon\to 0_+}\int_{-\pi+\varepsilon}^{\pi-\varepsilon}\left[1 - g\left(-e^{i\theta}\right)\right]D\left(M_\tau^2 e^{i\theta}\right)d\theta.$$

Inclusive τ decay within perturbative approach:

Commonly, perturbative $D(Q^2)$ is directly employed here

$$D(Q^2) \simeq D^{(\ell)}_{\mathrm{pert}}(Q^2) = D^{(0)}_{\mathrm{pert}}(Q^2) + \sum_{j=1}^{\ell} d_j \left[\alpha^{(\ell)}_{\mathrm{pert}}(Q^2)\right]^j, \quad Q^2 \to \infty$$

with $\alpha^{(1)}_{\mathrm{pert}}(Q^2) = 4\pi/[\beta_0 \ln(Q^2/\Lambda^2)]$**,** $\beta_0 = 11 - 2n_{\mathrm{f}}/3$**, and** $d_1 = 1/\pi$**.**

In what follows the one–loop level ($\ell = 1$**) with** $n_{\mathrm{f}} = 3$ **active flavors will be assumed.**

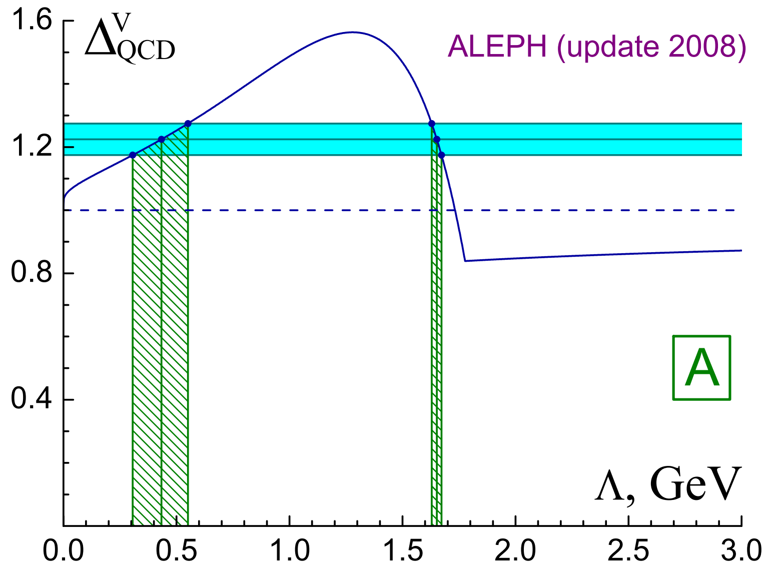
The one–loop perturbative expression for $\Delta^{\mathrm{V/A}}_{\mathrm{QCD}}$ **reads**

$$\Delta^{\mathrm{V/A}}_{\mathrm{pert}} = 1 + \frac{4}{\beta_0} \int_0^{\pi} \frac{\lambda A_1(\theta) + \theta A_2(\theta)}{\pi(\lambda^2 + \theta^2)}\, d\theta,$$

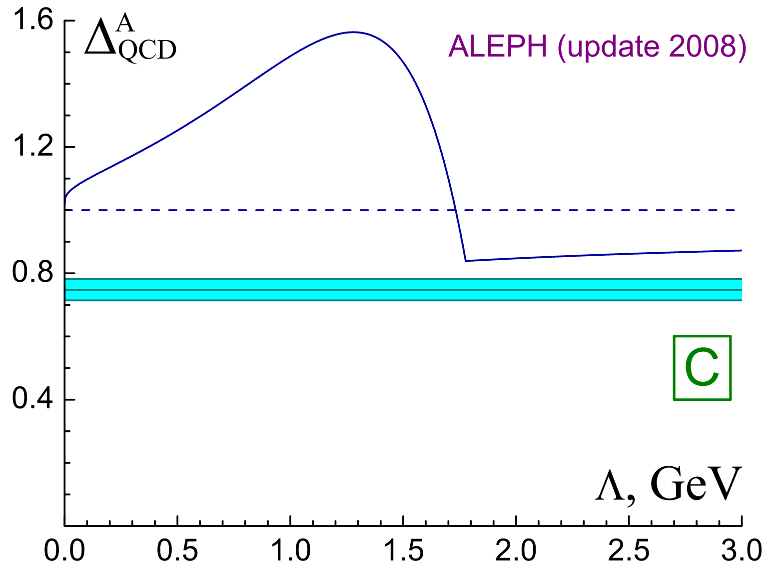
where $\lambda = \ln\left(M_\tau^2/\Lambda^2\right)$**,** $A_1(\theta) = 1 + 2\cos(\theta) - 2\cos(3\theta) - \cos(4\theta)$**,** $A_2(\theta) = 2\sin(\theta) - 2\sin(3\theta) - \sin(4\theta)$**.**

Perturbative approach gives $\Delta^{\rm V}_{\rm pert} \equiv \Delta^{\rm A}_{\rm pert}$, but $\Delta^{\rm V}_{\rm exp} \neq \Delta^{\rm A}_{\rm exp}$:

$\Delta^{\rm V}_{\rm exp} = 1.224 \pm 0.050$, $\Delta^{\rm A}_{\rm exp} = 0.748 \pm 0.034$ **[ALEPH–2008 data]**



$\boxed{\Lambda = \left(434^{+117}_{-127}\right)\,\text{MeV}}$ $\Lambda = \left(1652^{+21}_{-23}\right)\,\text{MeV}$



no solution

V-channel: perturbative approach gives two equally justified solutions, but only highlighted one is usually retained.

A-channel: perturbative approach fails to describe experimental data on inclusive τ lepton hadronic decay.

Inclusive τ decay within dispersive approach:

Description of the inclusive τ lepton hadronic decay within DispQCD enables one to properly account for

- **effects due to hadronization** $(m \neq 0)$
- **nonperturbative constraints on the functions on hand**

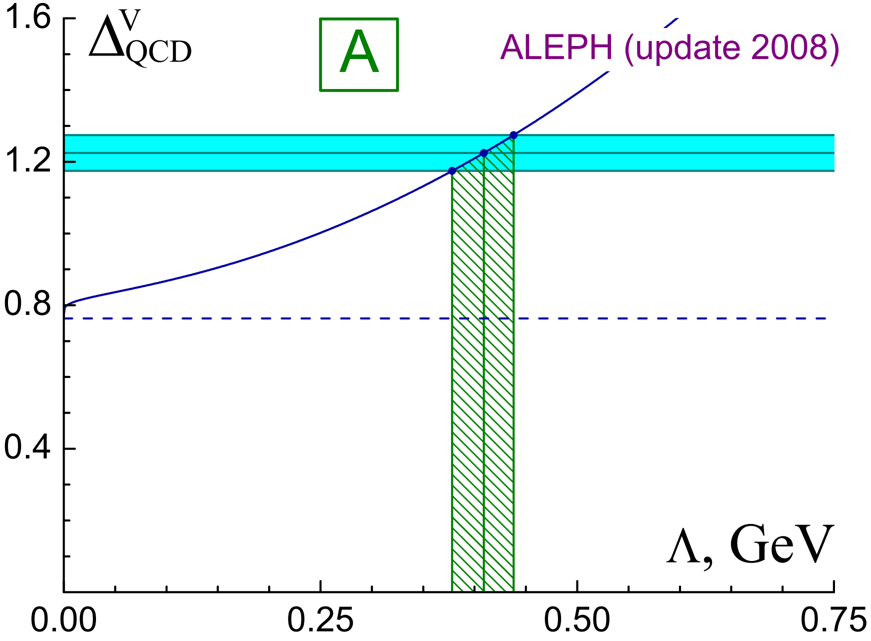
The use of initial expression for $\Delta_{\rm QCD}^{\rm V/A}$ with obtained above integral representations eventually leads to

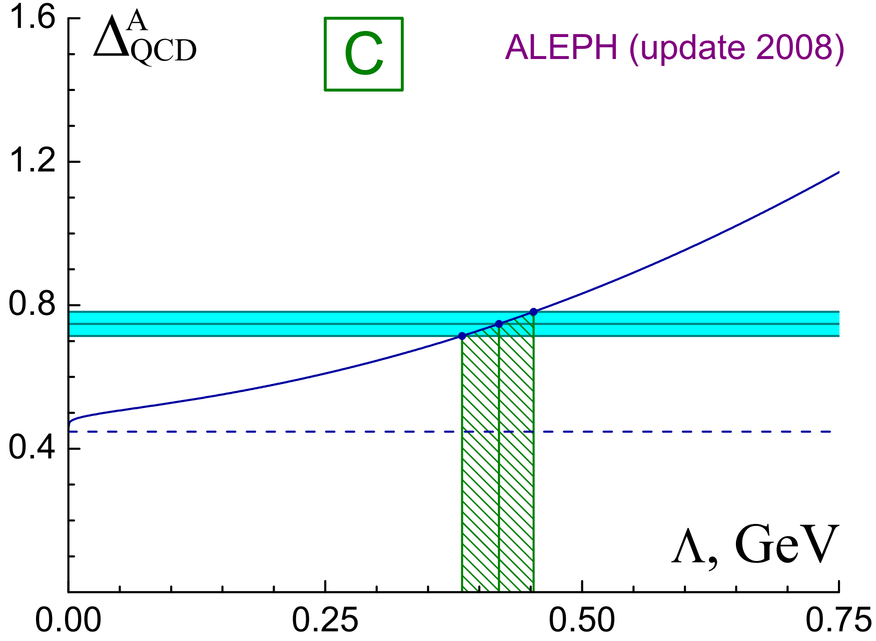
$$\Delta_{\rm QCD}^{\rm V/A} = \sqrt{1-\zeta_{\rm V/A}}\left(1 + 6\zeta_{\rm V/A} - \frac{5}{8}\zeta_{\rm V/A}^2 + \frac{3}{16}\zeta_{\rm V/A}^3\right) + \int_{m_{\rm V/A}^2}^{\infty} H\!\left(\frac{\sigma}{M_\tau^2}\right)\rho(\sigma)\,\frac{d\sigma}{\sigma}$$

$$-3\zeta_{\rm V/A}\left(1 + \frac{1}{8}\zeta_{\rm V/A}^2 - \frac{1}{32}\zeta_{\rm V/A}^3\right)\ln\!\left[\frac{2}{\zeta_{\rm V/A}}\left(1 + \sqrt{1-\zeta_{\rm V/A}}\right) - 1\right],$$

with $\zeta_{\rm V/A} = m_{\rm V/A}^2/M_\tau^2$, $H(x) = g(x)\,\theta(1-x) + g(1)\,\theta(x-1) - g\big(\zeta_{\rm V/A}\big)$

■ **Nesterenko (2011–2013).**





$$\Lambda = \bigl(408 \pm 30\bigr)\ \mathrm{MeV} \qquad\qquad \Lambda = \bigl(418 \pm 35\bigr)\ \mathrm{MeV}$$

The comparison of obtained result with experimental data yields nearly identical values of the QCD scale parameter Λ in vector and axial–vector channels, that testifies to the self–consistency of the developed approach.

SUMMARY

- The integral representations for $\Pi(q^2)$, $R(s)$, and $D(Q^2)$ are derived within dispersive approach to QCD

- These representations embody the nonperturbative constraints and retain the effects due to hadronization

- The obtained results are in a good agreement with lattice data and low–energy experimental predictions

- The developed approach is capable of describing experimental data on inclusive τ lepton hadronic decay in vector and axial–vector channels in a self–consistent way