Elliptic dilogarithms and two-loop Feynman integrals

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based on work to appear with Spencer Bloch

Explicit amplitude computations display rather unexpectedly simple structures allowing to compute many more processes than expected

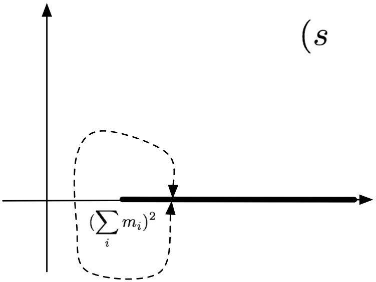
- On-shell recursion methods
- twistor geometry, Graßmanian, Symbol,…
- Dual conformal invariance
- amplitude relations …

All these simplifications hints on simpler structures than the diagrammatic from Feynman rules suggest, and lead to the use of a basis of integrals for expressing the amplitudes

- What are the basic functions entering the expressions for the amplitudes?

Monodromies, periods

- Amplitudes are multivalued quantities : they satisfy monodromy properties when going around the branch cuts for particle production



- As well the amplitude satisfy differential equation with respect to its parameters : kinematic invariants s_{ij}, internal masses m_i, ...
- monodromies and differential equation are typical of periods

Periods

[Kontsevich, Zagier] define periods are follows. $\mathcal{P} \in \mathbb{C}$ is the ring of periods, is $z \in \mathcal{P}$ if $\Re\mathrm{e}(z)$ and $\Im\mathrm{m}(z)$ are of the forms

$$\int_{\Delta \in \mathbb{R}^n} \frac{f(x_i)}{g(x_i)} \prod_{i=1}^{n} dx_i < \infty$$

with $f, g \in \mathbb{Z}[x_1, \cdots, x_n]$ and Δ is algebraically defined by inequalities and equalities.

- Typically form of the Feynman parametrization of a graph
- A Feynman graph with L loops and n edges

$$I_{graph} \propto \int_0^\infty \delta(1 - \sum_{i=1}^{n} x_i) \frac{\mathcal{U}^{n-(L+1)\frac{D}{2}}}{\mathcal{F}^{n-L\frac{D}{2}}} \prod_{i=1}^{n} dx_i$$

- $\mathcal{U}$ is homogenous of degree L in the x_i and $\mathcal{F}$ is homogenous of degree $L+1$ in x_i

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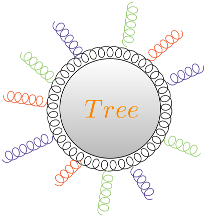
- Typically form of the Feynman parametrization of a graph
- Problem for Feynman graphs $\Delta \cap \{g(x_i) = 0\} \neq \emptyset$: needs to blow-up the intersection points
- As well generaly the domain of integration is not closed $\partial\Delta \neq \emptyset$
- This leads to the notion of “generalized” periods

Part I

Tree-level amplitudes

Tree-level amplitudes in QCD

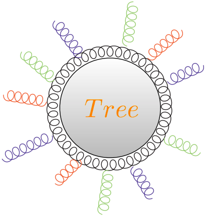


- Tree-level amplitudes in gauge theory are decomposed into color $(n-1)!/2$ color-ordered sub-amplitudes

$$\mathfrak{A}(1,\dots,n) \sim \sum_{\sigma\in\mathfrak{S}_{n-1}/\mathbb{Z}_2} \mathrm{Tr}\left(\lambda^{a_1}\lambda^{a_{\sigma(2)}}\cdots\lambda^{a_{\sigma(n)}}\right)\,\mathcal{A}(1,\sigma(2,\dots,n))$$

- λ^a are generators in the fundamental representation
- $\mathcal{A}(1,\sigma(2,\dots,n))$ are the color ordered amplitudes

Tree-level amplitudes in QCD

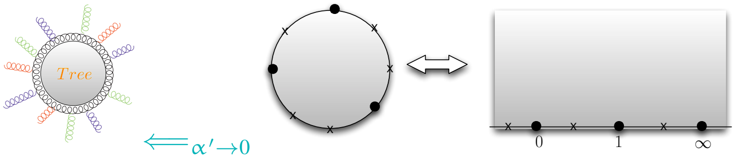


- The color ordered amplitudes are not all independent
- For instance they satisfy the photon decoupling identity

$$\sum_{\sigma\in\mathfrak{S}_{n-1}} \mathcal{A}_n(1,\sigma(2,\dots,n)) = 0$$

- There was the question of the independent amplitudes and their relations

Tree-level amplitudes in QCD



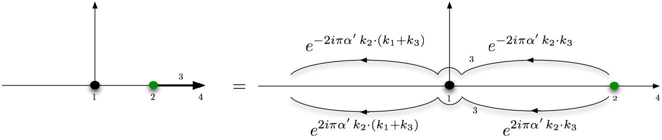
- We evaluate them by considering the $\alpha' \to 0$ limit of open string amplitudes on the disc.
- $PSL(2,\mathbb{R})$ invariance $z_1 = 0$, $z_{n-1} = 1$ and $z_n = +\infty$. (3 marked points)

$$\mathcal{A}(\sigma(1,\ldots,n)) = \int_{x_{\sigma(1)}<\cdots<x_{\sigma(n)}} f(x_i - x_j) \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)^{2\alpha' k_i\cdot k_j} \, d^{n-3}x$$

- The function $f(x_j)$ does not have branch cut but has poles. Depends on the polarisation of the external states.

Monodromies from contour deformation

Contour deformation gives monodromy relations between the ordered amplitudes



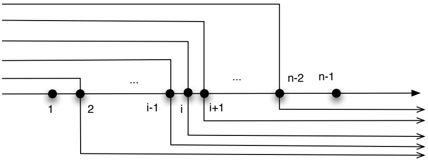
- The monodromy relations lead to a set of linear system of equations relating different ordering of the external states

$$\sum_{\sigma\in\mathfrak{S}_{n-2}} \mathcal{S}[\sigma(2,\ldots,n-1)|\beta(2,\ldots,n-1)]_{k_1}\mathcal{A}_n(n,\sigma(2,\ldots,n-1),1)=0$$

[Bern, Carrasco, Johansson; Bjerrum-bohr, Damgaard, Vanhove; Stieberger; Bjerrum-bohr, Damgaard, Søndergaard, Vanhove]

Momentum kernel

Contour deformation gives monodromy relations between the ordered amplitudes



- This leads to an object named momentum kernel $\mathcal{S}$

$$\mathcal{S}[i_1,\ldots,i_k|j_1,\ldots,j_k]_p := \prod_{t=1}^{k}\left(p\cdot k_{i_t} + \sum_{q>t}^{k} \theta(t,q)\, k_{i_t}\cdot k_{i_q}\right)$$

- $\theta(t,q)=1$ if $(i_t-i_q)(j_t-j_q)<0$ and 0 otherwise
- Exists as well in string theory with $\alpha'\neq 0$ as $\prod \sin\alpha'(\ldots)/\alpha'$

[Bjerrum-bohr, Damgaard, Vanhove; Bjerrum-Bohr, Damgaard, Feng, Sondergaard; Bjerrum-bohr, Damgaard, Søndergaard, Vanhove]

Tree-level Gravity and Gauge amplitudes

- From closed string heterotic string setup we have points on the sphere
- holomorphic factorization $|z|^{\alpha' k_i \cdot k_j} \to x^{\frac{\alpha'}{2} k_i \cdot k_j} y^{\frac{\alpha'}{2} k_i \cdot k_j}$ gives product of ordered disc integrations with relative ordered contour of integrations C_x and C_y

$$\mathfrak{M}(1,\ldots,n) = \int_{C_x} d^{n-3}x \int_{C_y} d^{n-3}y \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)^{\frac{\alpha' k_i \cdot k_j}{2}} (y_i - y_j)^{\frac{\alpha' k_i \cdot k_j}{2}} f(x_{ij}) g(y_{ij})$$

[Kawai,Lewellen, Tye; Tye, Zhang; Bjerrum-Bohr, Damgaard, Søndergaard, Vanhove]

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- The gauge theory and gravity amplitudes take a similar forms

$$\begin{aligned} \mathcal{A}_n^{\text{YM}} &= A^{\text{vector}} \otimes \mathcal{S} \otimes A^{\text{scalar}} \\ \mathcal{M}_n^{\text{Grav}} &= A^{\text{vector}} \otimes \mathcal{S} \otimes A^{\text{vector}} \end{aligned}$$

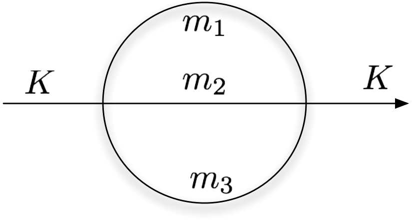
- These relations are generic and independent of any precise parametrisation of the tree-level amplitudes

[Kawai,Lewellen, Tye; Tye, Zhang; Bjerrum-Bohr, Damgaard, Søndergaard, Vanhove]

Part II

Two-loop amplitudes

two-loop integrals



We consider the two-loop sunset integral in two dimensions given by

$$\mathcal{I}^2_\ominus \propto \int_{\mathbb{R}^4} \frac{d^2\ell_1 d^2\ell_2}{(\ell_1^2 - m_1^2)(\ell_2^2 - m_2^2)((\ell_1 + \ell_2 - K^2) - m_3^2)}$$

two-loop integrals

The Feynman parametrisation is given by

$$\mathcal{I}^2_{\ominus} \propto \int_{\substack{x\geqslant 0\\ y\geqslant 0}} \frac{dxdy}{(m_1^2x+m_2^2y+m_3^2)(x+y+xy)-K^2xy}\,.$$

- We are again the setup of [Kontsevich, Zagier]
- The sunset integral is $\int_{\mathcal{D}}\omega$ with the 2-form

$$\omega = \frac{zdx\wedge dy + xdy\wedge dz + ydz\wedge dx}{A_{\ominus}(x,y,z)} \in H^2(\mathbb{P}^2-\mathcal{E})$$

- The graph is based on the elliptic curve $\mathcal{E} : A_{\ominus}(x,y,z)=0$

$$A_{\ominus}(x,y,z) := (m_1^2x+m_2^2y+m_3^2z)(xz+xy+yz)-K^2xyz\,.$$

two-loop integrals

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- The domain of integration is

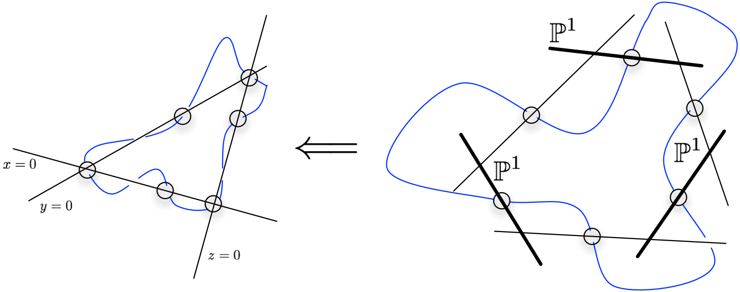
$$\mathcal{D} := \{[x:y:z]\in\mathbb{P}^2 | x\geqslant 0, y\geqslant 0, z\geqslant 0\}$$

the sunset graph motive I

- The elliptic curve intersects the domain of integration $\mathcal{D}$

$$\mathcal{D} \cap \{A_\ominus(x,y,z) = 0\} = \{[1:0:0],[0:1:0],[0:0:1]\}$$

- We need to blow-up work in $\mathbb{P}^2 - \mathcal{E}$



the sunset graph motive II

- The domain of integration $\mathcal{D} \notin H_2(\mathbb{P}^2 - \mathcal{E})$ because $\partial\mathcal{D} \neq \emptyset$
- Need to pass to the relative cohomology
- If $P \to \mathbb{P}^2$ is the blow-up and $\hat{\mathcal{E}}$ is the strict transform of $\mathcal{E}$ (here $\hat{\mathcal{E}} \cong \mathcal{E}$)
- Hexagon $\mathfrak{h}$ union of strict transform of $\partial\mathcal{D} = \{xyz = 0\}$ and the 3 divisors .
- Then in P we have resolved the two problems

$$\mathcal{D} \cap \hat{\mathcal{E}} = \emptyset; \qquad \mathcal{D} \in H_2(P - \hat{\mathcal{E}}, \mathfrak{h} - (\mathfrak{h} \cap \hat{\mathcal{E}}))$$

- The sunset integral is a period of $H^2(P - \hat{\mathcal{E}}, \mathfrak{h} - (\mathfrak{h} \cap \hat{\mathcal{E}}))$

[Bloch, Esnault, Kreimer; Müller-Stach, Weinzeirl, Zayadeh]

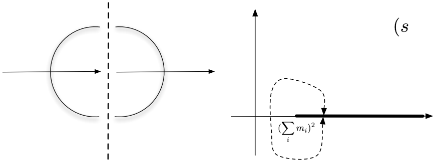
the elliptic curve of the sunset integral

- With all mass equal $m_i = m$ the integral is reduced to with $t = K^2/m^2$

$$\mathcal{I}_\ominus(t) = \int_0^\infty \int_0^\infty \frac{dxdy}{(x+y+1)(x+y+xy) - txy}\,.$$

$$\mathcal{E}_t : \ (x+y+1)(x+y+xy) - txy = 0\,.$$

- Special values
  - At $t = 0$, $t = 1$ and $t = +\infty$ the elliptic curve factorizes.
  - At $t = 9$ we have the 3-particle the threshold $t = K^2/m^2 \in \mathbb{C}\backslash[9, +\infty[$.



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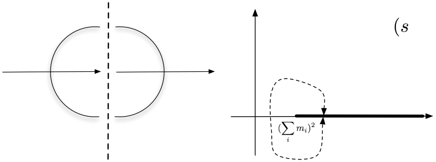

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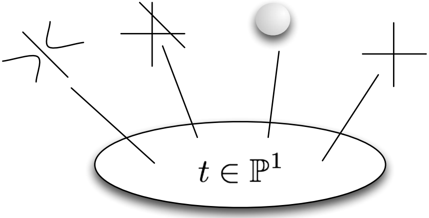

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- We have a family of elliptic curve defining a K_3 surface with 4 singular fibers in [Beauville] classification.
- This is a universal family of $X_1(6)$ modular curves with a point of order 6

the picard-fuchs equation of the sunset integral

$$\mathcal{E}_t : (x+y+1)(x+y+xy) - txy = 0$$

- Since $H^1(\mathcal{E}_t)$ is generated by dx/y and $d(dx/y)/dt$, then exist $p_0(t), p_1(t), p_2(t) \in \mathbb{Z}[t]$ such that

$$p_0(t)\frac{d^2}{dt^2}\left(\frac{dx}{y}\right) + p_1(t)\frac{d}{dt}\left(\frac{dx}{y}\right) + p_2(t)\left(\frac{dx}{y}\right) = d\beta$$

- The homogeneous picard-fuchs operator is

$$L_t = \frac{d}{dt}\left(t(t-1)(t-9)\frac{d}{dt}\right) + (t-3)$$

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- Acting on the integral we have

$$L_t \mathcal{I}^2_{\ominus}(t) = \int_{\mathcal{D}} d\beta = -\int_{\partial\mathcal{D}} \beta \neq 0$$

- We find that (recovering the result of [Laporta, Remiddi])

$$\frac{d}{dt}\left(t(t-1)(t-9)\frac{d\mathcal{I}_{\ominus}(t)}{dt}\right) + (t-3)\mathcal{I}_{\ominus}(t) = -6$$

the sunset integral as an elliptic dilogarithm

- Evaluating the integrals leads to (for details see my talk at String Math 2013)

$$\mathcal{I}_{\ominus}^{2}(t) \sim \alpha_1 \varpi_1(t) + \alpha_2 \varpi_2(t) + \varpi_2(t)\, E_{-1}(q_t)\,,$$

where $\varpi_1(t)$, $\varpi_2(t)$ are the complex and real periods and $E_{-1}(q)$ is defined by

$$E_{-1}(q) \sim \mathcal{I}_{\ominus}^{2}(0) - \frac{1}{2i} \sum_{n \geqslant 0} \left(\mathrm{Li}_2(q^n \zeta_6) + \mathrm{Li}_2(q^n \zeta_6^2) - \mathrm{Li}_2(q^n \zeta_6^4) - \mathrm{Li}_2(q^n \zeta_6^5)\right)$$

- $(\zeta_6)^6 = 1$ and $q = \exp(2i\pi\tau(t))$ with $\tau(t) = \varpi_1(t)/\varpi_2(t)$
- Notice that we have Li_2 and not the Bloch-Wigner D function
- The function is multivalued and convergent