Systematics of the Hadron Spectrum from Conformal Quantum Mechanics and Holographic QCD

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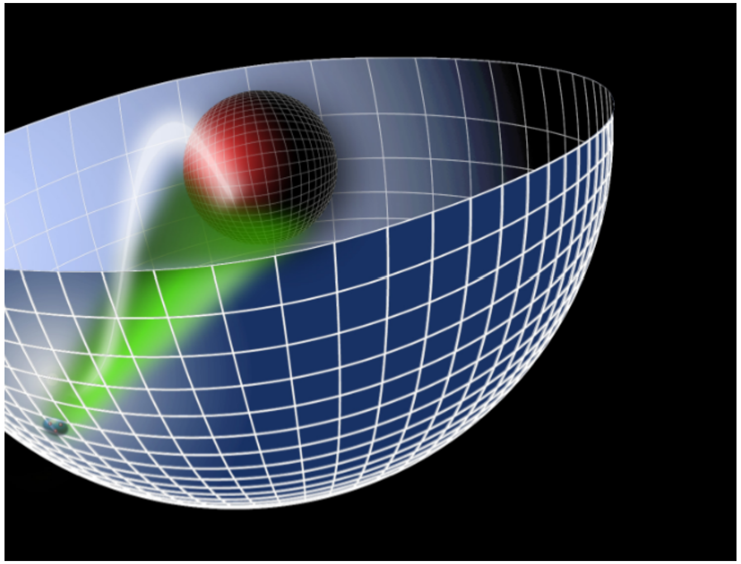

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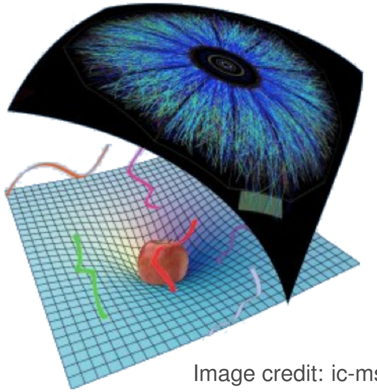

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In collaboration with Stan Brodsky (SLAC) and Hans G. Dosch (Heidelberg)

1 Effective Confinement from Underlying Conformal Invariance

[S. J. Brodsky, GdT and H.G. Dosch, arXiv:1302.4105]

- Incorporate in an effective theory the fundamental dilatation symmetry of the 4-dim QCD Lagrangian in the chiral limit of massless quarks

- Invariance properties of one dimensional field theory under the full conformal group from dAFF action

  [V. de Alfaro, S. Fubini and G. Furlan (dAFF) [Nuovo Cim. A **34**, 569 (1976)]

$$S = \tfrac{1}{2} \int dt \left(\dot{Q}^2 - \frac{g}{Q^2} \right)$$

  where g is a dimensionless number (Casimir operator which depends on the representation)

- The equation of motion

$$\ddot{Q} - \frac{g}{Q^3} = 0$$

  and the generator of evolution in t, the Hamiltonian

$$H_t = \tfrac{1}{2} \left(\dot{Q}^2 + \frac{g}{Q^2} \right)$$

  follow from the dAFF action

- Absence of dimensional constants implies that the action

$$S = \tfrac{1}{2}\int dt\Big(\dot{Q}^2 - \frac{g}{Q^2}\Big)$$

is invariant under a larger group of transformations, the general conformal group

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad Q'(t') = \frac{Q(t)}{\gamma t + \delta}$$

with $\alpha\delta - \beta\gamma = 1$

- Applying Noether's theorem obtain conserved operators

i) Translations: $H_t = \frac{1}{2}\big(\dot{Q}^2 + \frac{g}{Q^2}\big)$

ii) Dilatations: $D = tH_t - \frac{1}{2}Q\dot{Q}$

iii) Special conformal transformations: $K = t^2 H_t - tQ\dot{Q} + \frac{1}{2}Q^2$

- Any combination of the generators H_t, D and K

$$G = uH_t + vD + \omega K$$

is also a constant of motion

- TIme evolution for state vector and field operator for dAFF generator G from canonical quantization $[Q(t), \dot{Q}(t)] = i$

$$G|\psi(t)\rangle = i f(t) \frac{d}{dt}|\psi(t)\rangle$$

$$i\,[G, Q(t)] = f(t)\frac{dQ(t)}{dt} - \tfrac{1}{2}\frac{df(t)}{dt}Q(t)$$

where $\quad f(t) = u + vt + wt^2$

- dAFF introduce new time variable τ and field operator $q(\tau)$

$$d\tau = \frac{dt}{u + vt + wt^2}, \qquad q(\tau) = \frac{Q(t)}{[u + vt + wt^2]^{\frac{1}{2}}}$$

- Find usual quantum mechanical evolution for time τ

$$G|\psi(\tau)\rangle = i\frac{d}{d\tau}|\psi(\tau)\rangle$$

$$i\,[G, q(\tau)] = \frac{dq(\tau)}{d\tau}$$

and usual equal-time quantization $\quad [q(t), \dot{q}(t)] = i$

- In terms of τ and $q(\tau)$

$$
\begin{aligned}
S &= \tfrac{1}{2}\int dt\Big(\dot{Q}^2 - \frac{g}{Q^2}\Big) \\
&= \tfrac{1}{2}\int d\tau\Big(\dot{q}^2 - \frac{g}{q^2} - \frac{4u\omega - v^2}{4}q^2\Big) + \mathrm{surface\,term}
\end{aligned}
$$

Action is conformal invariant invariant up to a surface term !

- The corresponding Hamiltonian

$$
H_\tau = \tfrac{1}{2}\Big(\dot{q}^2 + \frac{g}{q^2} + \frac{4u\omega - v^2}{4}q^2\Big)
$$

is a compact operator for

$$
\frac{4u\omega - v^2}{4} > 0
$$

- Scale appears in the Hamiltonian without affecting the conformal invariance of the action !

Conformal Quantum Mechanics

- The Schrödingier picture follows from the representation of q and $p = \dot{q}$ (dAFF)

$$q \to x, \;\; \dot{q} \to -i\frac{d}{dx}$$

- Schrödinger wave equation determines evolution of bound states in terms of the variable τ

$$i\frac{\partial}{\partial\tau}\psi(x,\tau) = H_\tau\Big(x, -i\frac{d}{dx}\Big)\psi(x,\tau)$$

- dAFF Hamiltonian

$$H_\tau = \tfrac{1}{2}\Big(-\frac{d^2}{dx^2} + \frac{g}{x^2} + \frac{4u\omega - v^2}{4}x^2\Big)$$

2 Light Front Dynamics

- On shell relation $P_\mu P^\mu = P^- P^+ - \mathbf{P}_\perp^2 = M^2$ leads to dispersion relation for LF Hamiltonian P^-

$$P^- = \frac{\mathbf{P}_\perp^2 + M^2}{P^+}, \quad P^+ > 0, \quad P^\pm = P^0 \pm P^3$$

- Hamiltonian equation for the relativistic bound state $(x^+ = x^0 + x^3$ light-front time)

$$i \frac{\partial}{\partial x^+} |\psi(P)\rangle = P^- |\psi(P)\rangle = \frac{M^2 + \mathbf{P}_\perp^2}{P^+} |\psi(P)\rangle$$

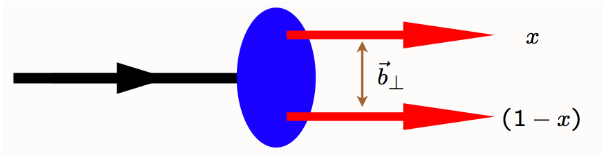
- Construct LF Lorentz invariant Hamiltonian $P^2 = P^- P^+ - \mathbf{P}_\perp^2$

$$P_\mu P^\mu |\psi(P)\rangle = M^2 |\psi(P)\rangle$$

- LF quantization allows unambiguous definition of partonic content of hadrons (wave function)
- LF Hamiltonian equation for bound states has similar structure of AdS and dAFF equations: direct connection of QCD with AdS/CFT and conformal QM (dAFF) possible !

Semiclassical Approximation to QCD in the Light Front

[GdT and S. J. Brodsky, PRL **102**, 081601 (2009)]



- Compute M^2 from hadronic matrix element

$$\langle\psi(P')|P_\mu P^\mu|\psi(P)\rangle = M^2\langle\psi(P')|\psi(P)\rangle$$

- To first approximation LF dynamics depends only on the invariant variable $\zeta^2 = x(1-x)\mathbf{b}_\perp^2$
- Factor angular φ, longitudinal $X(x)$ and transverse mode $\phi(\zeta)$

$$\psi(x,\zeta,\varphi) = e^{iL\varphi} X(x) \frac{\phi(\zeta)}{\sqrt{2\pi\zeta}}$$

- Ultra relativistic limit $m_q \to 0$ longitudinal modes $X(x)$ decouple $(L = L^z)$

$$M^2 = \int d\zeta\, \phi^*(\zeta) \sqrt{\zeta} \left(-\frac{d^2}{d\zeta^2} - \frac{1}{\zeta}\frac{d}{d\zeta} + \frac{L^2}{\zeta^2}\right) \frac{\phi(\zeta)}{\sqrt{\zeta}} + \int d\zeta\, \phi^*(\zeta)\, U(\zeta)\, \phi(\zeta)$$

where the confining forces from the interaction terms are summed up in the effective potential $U(\zeta)$

- LF eigenvalue equation $P_\mu P^\mu|\phi\rangle = M^2|\phi\rangle$ is a LF wave equation for ϕ

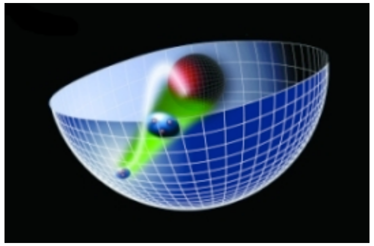
$$\left(\underbrace{-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2}}_{kinetic\ energy\ of\ partons} + \underbrace{U(\zeta)}_{confinement}\right)\phi(\zeta) = M^2\phi(\zeta)$$

- Effective relativistic and frame-independent LF Schrödinger equation, U is instantaneous in LF time
- The $SO(2)$ Casimir L^2 corresponds to group of rotations in transverse LF plane
- Semiclassical approximation to LF QCD does not account for particle creation and absorption
- Compare with dAFF Hamiltonian

$$H_\tau = \tfrac{1}{2}\left(-\frac{d^2}{dx^2} + \frac{g}{x^2} + \frac{4u\omega - v^2}{4}x^2\right)$$

- Identical with LF Hamiltonian provided x is identified with the LF variable ζ: $x = \zeta/\sqrt{2}$, g with the LF orbital angular momentum L: $g = L^2 - 1/4$ with effective LF confining interaction $U \sim \lambda^2\zeta^2$

3 Higher Integer-Spin Wave Equations in AdS Space and Light Front Holographic Mapping



$$\mathcal{R}_{NKLM} = -\frac{1}{R^2}\left(g_{NL}g_{KM} - g_{NM}g_{KL}\right)$$

- Why is AdS space important?

  AdS$_5$ is a space of maximal symmetry, negative curvature and a four-dim boundary: Minkowski space

- Isomorphism of $SO(4,2)$ group of conformal transformations with generators $P^\mu, M^{\mu\nu}, K^\mu, D$ with the group of isometries of AdS$_5$ Dim isometry group of AdS$_{d+1}$: $\frac{(d+1)(d+2)}{2}$

- AdS$_5$ metric $x^M = (x^\mu, z)$:

$$ds^2 = g_{MN}dx^M dx^N = \frac{R^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2\right)$$

- Since the AdS metric is invariant under a dilatation of all coordinates $x^\mu \to \lambda x^\mu$, $z \to \lambda z$, the variable z acts like a scaling variable in Minkowski space

- Short distances $x_\mu x^\mu \to 0$ maps to UV conformal AdS$_5$ boundary $z \to 0$

- Large confinement dimensions $x_\mu x^\mu \sim 1/\Lambda^2_{\rm QCD}$ map to large IR region of AdS$_5$, $z \sim 1/\Lambda_{\rm QCD}$, thus there is a maximum separation of quarks and a maximum value of z

Higher Spin Wave Equations in AdS Space

[GdT, H.G. Dosch and S. J. Brodsky, PRD **87**, 075004 (2013)]

- Description of higher spin modes in AdS space (Frondsal, Fradkin and Vasiliev)
- Integer spin-J fields in AdS conveniently described by tensor field $\Phi_{N_1 \cdots N_J}$ with effective action

$$S_{\textit{eff}} = \int d^d x \, dz \, \sqrt{|g|} \, e^{\varphi(z)} \, g^{N_1 N_1'} \cdots g^{N_J N_J'} \Big(g^{M M'} D_M \Phi^*_{N_1 \ldots N_J} \, D_{M'} \Phi_{N_1' \ldots N_J'} - \mu^2_{\textit{eff}}(z) \, \Phi^*_{N_1 \ldots N_J} \, \Phi_{N_1' \ldots N_J'} \Big)$$

where D_M is the covariant derivative which includes affine connection

- The z-dependent effective AdS mass $\mu_{\textit{eff}}(z)$ can absorb the contribution from different contractions in the action and is *a priori* unknown
- Effective mass $\mu_{\textit{eff}}(z)$ allows a separation of kinematical and dynamical effects and is determined by precise mapping to light-front physics
- Non-trivial geometry of pure AdS encodes the kinematics and the additional deformations of AdS encode the dynamics, including confinement

- Physical hadron has plane-wave and polarization indices along $3+1$ physical coordinates

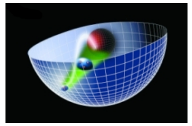
$$\Phi_P(x,z)_{\mu_1\cdots\mu_J} = e^{iP\cdot x}\Phi(z)_{\mu_1\cdots\mu_J}, \quad \Phi_{z\mu_2\cdots\mu_J} = \cdots = \Phi_{\mu_1\mu_2\cdots z} = 0$$

with four-momentum P_μ and invariant hadronic mass $P_\mu P^\mu = M^2$

- Further simplification by using a local Lorentz frame with tangent indices

- Variation of the action gives AdS wave equation for spin-J field $\Phi(z)_{\nu_1\cdots\nu_J} = \Phi_J(z)\epsilon_{\nu_1\cdots\nu_J}$

$$\left[-\frac{z^{d-1-2J}}{e^{\varphi(z)}}\partial_z\left(\frac{e^{\varphi}(z)}{z^{d-1-2J}}\partial_z\right) + \left(\frac{mR}{z}\right)^2\right]\Phi_J = M^2\Phi_J$$



with

$$(m\,R)^2 = (\mu_{\mathit{eff}}(z)R)^2 - Jz\,\varphi'(z) + J(d-J+1)$$

and the kinematical constraints

$$\eta^{\mu\nu}P_\mu\,\epsilon_{\nu\nu_2\cdots\nu_J} = 0, \quad \eta^{\mu\nu}\,\epsilon_{\mu\nu\nu_3\cdots\nu_J} = 0.$$

- Kinematical constrains in the LF imply that m must be a constant

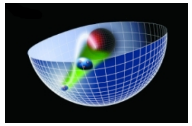
[See also: T. Gutsche, V. E. Lyubovitskij, I. Schmidt and A. Vega, Phys. Rev. D **85**, 076003 (2012)]

Light-Front Mapping

[GdT and S. J. Brodsky, PRL **102**, 081601 (2009)]

- Upon substitution $\Phi_J(z) \sim z^{(d-1)/2-J} e^{-\varphi(z)/2}\, \phi_J(z)$ and $z \to \zeta$ in AdS WE

$$\left[-\frac{z^{d-1-2J}}{e^{\varphi(z)}}\partial_z\left(\frac{e^{\varphi(z)}}{z^{d-1-2J}}\partial_z\right) + \left(\frac{mR}{z}\right)^2\right]\Phi_J(z) = M^2\Phi_J(z)$$



we find LFWE $(d = 4)$

$$\left(-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2} + U(\zeta)\right)\phi_J(\zeta) = M^2\phi_J(\zeta)$$

with

$$U(\zeta) = \tfrac{1}{2}\varphi''(\zeta) + \frac{1}{4}\varphi'(\zeta)^2 + \frac{2J-3}{2z}\varphi'(\zeta)$$

and $(mR)^2 = -(2-J)^2 + L^2$

- Unmodified AdS equations correspond to the kinetic energy terms of the partons inside a hadron
- Interaction terms in the QCD Lagrangian build the effective confining potential $U(\zeta)$ and correspond to the truncation of AdS space in an effective dual gravity approximation
- AdS Breitenlohner-Freedman bound $(mR)^2 \geq -4$ equivalent to LF QM stability condition $L^2 \geq 0$

Meson Spectrum

- Dilaton profile in the dual gravity model is also determined from conformal QM (dAFF) !

$$\varphi(z) = \lambda z^2, \qquad \lambda^2 = \frac{4u\omega - v^2}{16}$$

- Effective potential: $U = \lambda^2 \zeta^2 + 2\lambda(J-1)$
- LFWE

$$\left(-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2} + \lambda^2\zeta^2 + 2\lambda(J-1)\right)\phi_J(\zeta) = M^2 \phi_J(\zeta)$$

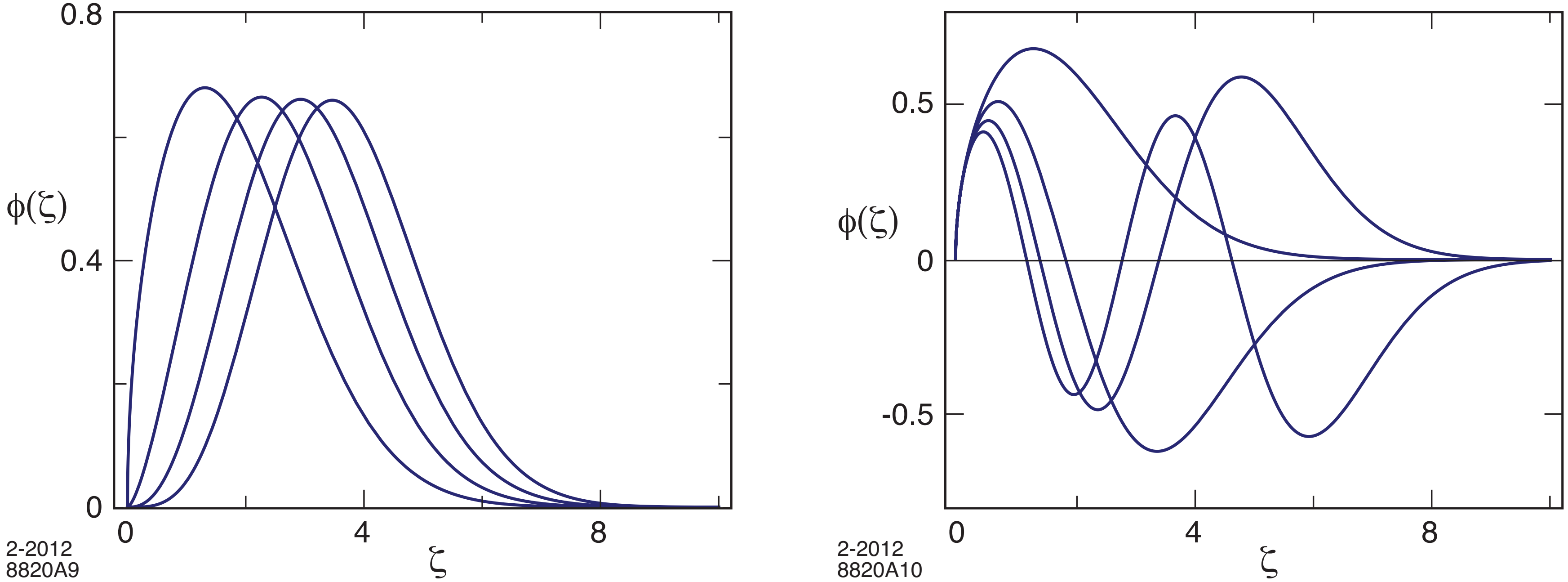
- Normalized eigenfunctions $\langle\phi|\phi\rangle = \int d\zeta\, \phi^2(z)^2 = 1$

$$\phi_{n,L}(\zeta) = |\lambda|^{(1+L)/2} \sqrt{\frac{2n!}{(n+L)!}}\, \zeta^{1/2+L} e^{-|\lambda|\zeta^2/2} L_n^L(|\lambda|\zeta^2)$$

- Eigenvalues for $\lambda > 0$

$$\mathcal{M}^2_{n,J,L} = 4\lambda\left(n + \frac{J+L}{2}\right)$$

- For $\lambda < 0$, $M^2 = -4\lambda\,(n + 1 + (L-J)/2)$, incompatible with the LF constituent interpretation of hadronic states

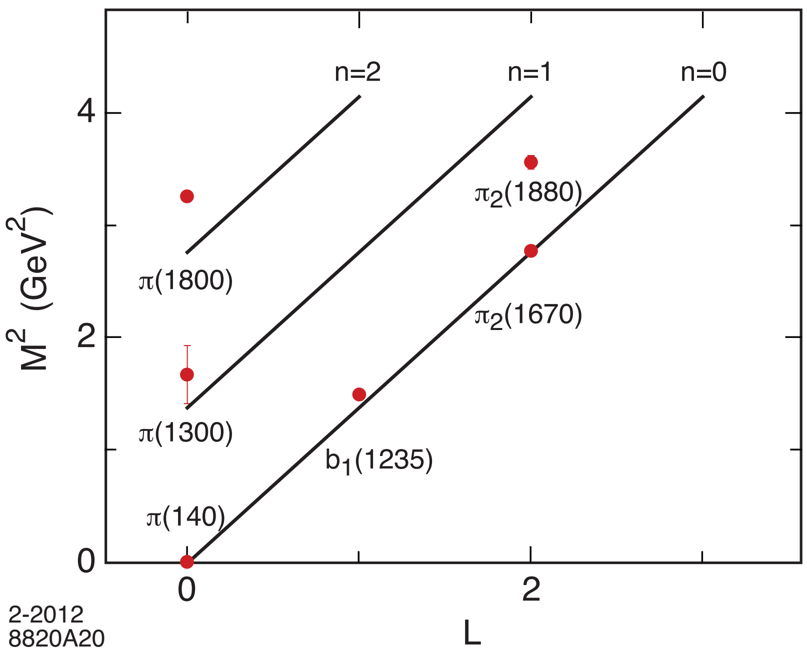


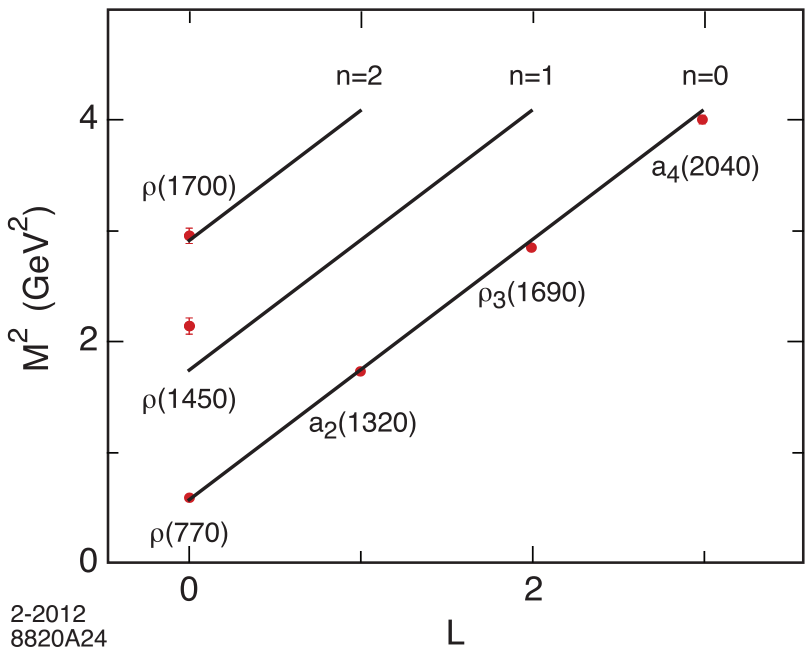
LFWFs $\phi_{n,L}(\zeta)$ in physical space-time: (L) orbital modes and (R) radial modes

Table 1: $I = 1$ mesons. For a $q\bar{q}$ state $P = (-1)^{L+1},\ C = (-1)^{L+S}$

L	S	n	J^{PC}	$I = 1$ Meson
0	0	0	0^{-+}	$\pi(140)$
0	0	1	0^{-+}	$\pi(1300)$
0	0	2	0^{-+}	$\pi(1800)$
0	1	0	1^{--}	$\rho(770)$
0	1	1	1^{--}	$\rho(1450)$
0	1	2	1^{--}	$\rho(1700)$
1	0	0	1^{+-}	$b_1(1235)$
1	1	0	0^{++}	$a_0(980)$
1	1	1	0^{++}	$a_0(1450)$
1	1	0	1^{++}	$a_1(1260)$
1	1	0	2^{++}	$a_2(1320)$
2	0	0	2^{-+}	$\pi_2(1670)$
2	0	1	2^{-+}	$\pi_2(1880)$
2	1	0	3^{--}	$\rho_3(1690)$
3	1	0	4^{++}	$a_4(2040)$

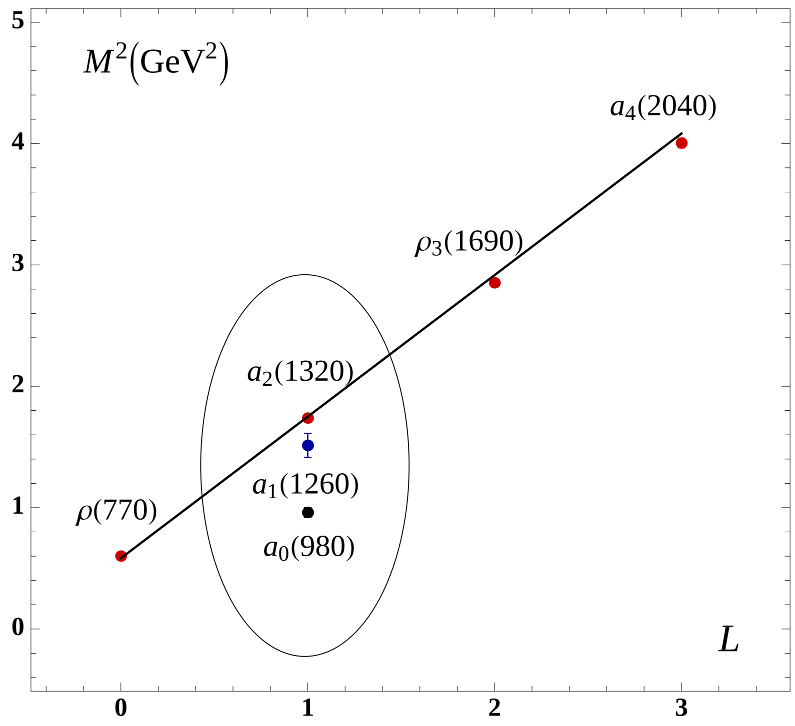
- $J = L + S, \; I = 1$ meson families $\mathcal{M}^2_{n,L,S} = 4\lambda\,(n + L + S/2)$





Orbital and radial excitations for the π ($\sqrt{\lambda} = 0.59$ GeV) and the ρ I=1meson families ($\sqrt{\lambda} = 0.54$ GeV)

$$\mathcal{M}^2_{n,J,L} = 4\lambda\left(n + \frac{J+L}{2}\right)$$



- Triplet splitting for vector meson a-states ($L = 1$, $J = 0, 1, 2$)

$$\mathcal{M}_{a_2(1320)} > \mathcal{M}_{a_1(1260)} > \mathcal{M}_{a_0(980)}$$

- Systematics of $I = 1$ light meson spectra – orbital and radial excitations as well as important $J - L$ splitting, well described by light-front harmonic confinement model
- Linear Regge trajectories, a massless pion and relation between the ρ and a_1 mass $M_{a_1}/M_\rho = \sqrt{2}$ usually obtained from Weinberg sum rules [Weinberg (1967)]

4 Higher Half-Integer Spin Wave Equations in AdS Space and Light-Front Holographic Mapping

[J. Polchinski and M. J. Strassler, JHEP **0305**, 012 (2003)]

[GdT and S. J. Brodsky, PRL **94**, 201601 (2005)]

[GdT, H.G. Dosch and S. J. Brodsky, PRD **87**, 075004 (2013)]

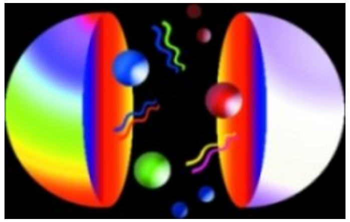
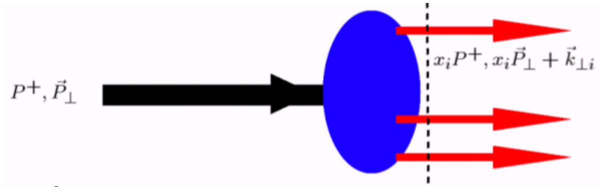


Image credit: N. Evans

- The gauge/gravity duality can give important insights into the strongly coupled dynamics of nucleons using simple analytical methods: analytical exploration of systematics of light-baryon resonances

- Extension of holographic ideas to spin-$\frac{1}{2}$ (and higher half-integral J) hadrons by considering wave equations for Rarita-Schwinger spinor fields in AdS space and their mapping to light-front physics

$$\zeta = \sqrt{\frac{x}{1-x}}\left|\sum_{j=1}^{n-1} x_j \mathbf{b}_{\perp j}\right|$$



where $x = x_n$ is the longitudinal momentum fraction of the active quark

- LF clustering decomposition: Same multiplicity of states for mesons and baryons !

- Half-integer spin $J = T + \frac{1}{2}$ conveniently represented by RS spinor $\left[\Psi_{N_1 \cdots N_T}\right]_\alpha$ with effective AdS action

$$S_{\mathit{eff}} = \frac{1}{2} \int d^d x \, dz \, \sqrt{|g|} \, g^{N_1 \, N_1'} \cdots g^{N_T \, N_T'}$$
$$\left[\overline{\Psi}_{N_1 \cdots N_T} \Big(i \, \Gamma^A \, e^M_A \, D_M - \mu - \rho(z)\Big) \Psi_{N_1' \cdots N_T'} + h.c.\right]$$

where the covariant derivative D_M includes the affine connection and the spin connection

- e^A_M is the vielbein and Γ^A tangent space Dirac matrices $\left\{\Gamma^A, \Gamma^B\right\} = \eta^{AB}$
- For fermions one cannot introduce confinement with dilaton since it can be scaled away [I. Kirsch (2006)]
- Introduce effective interaction $\rho(z)$ constrained by the condition that the square of the Dirac equation leads to harmonic confinement (dAFF)
- Linear covariant derivatives in the action prevents mixing between dynamical and kinematical effects
- In contrast with effective action for integer spin, the AdS mass μ is constant: systematics of meson and baryon spectrum different !

- Physical baryons have spinors, and T polarization indices along $3+1$ physical coordinates

$$\Psi_{\mu_1 \cdots \mu_T}, \qquad \Psi_{z \mu_2 \cdots \mu_T} = \cdots = \Psi_{\mu_1 \mu_2 \cdots z} = 0$$

- Further simplification by using a local Lorentz frame with tangent indices

- Variation of the action gives AdS wave equation for spin-J field $\Psi_{\nu_1 \cdots \nu_J}$

$$\left[i \left(z \eta^{MN} \Gamma_M \partial_N + \frac{d - 2T}{2} \Gamma_z \right) - \mu R - R \, \rho(z) \right] \Psi_{\nu_1 \ldots \nu_T} = 0$$

and the Rarita-Schwinger condition

$$\gamma^\nu \Psi_{\nu \nu_2 \ldots \nu_T} = 0$$

[See also: T. Gutsche, V. E. Lyubovitskij, I. Schmidt and A. Vega, Phys. Rev. D **85**, 076003 (2012)]

Light-Front Mapping

- A physical baryon satisfies the Rarita-Schwinger equation for spinors in physical space-time

$$(i\gamma^\mu \partial_\mu - M)\, u_{\nu_1 \cdots \nu_T}(P) = 0, \qquad \gamma^\nu u_{\nu \nu_2 \cdots \nu_T}(P) = 0$$

- Upon substitution in AdS wave equation for spin J $(u^\pm$ chiral spinors)

$$\Psi^\pm_{\nu_1 \cdots \nu_T}(x, z) = e^{iP\cdot x} \left(\frac{R}{z}\right)^{T-d/2} \psi^\pm_T(z)\, u^\pm_{\nu_1 \cdots \nu_T}(P)$$

and $z \to \zeta$ find LFWE

$$\begin{aligned} -\frac{d}{d\zeta}\psi_- - \frac{\nu+\frac{1}{2}}{\zeta}\psi_- - V(\zeta)\psi_- &= M\psi_+ \\ \frac{d}{d\zeta}\psi_+ - \frac{\nu+\frac{1}{2}}{\zeta}\psi_+ - V(\zeta)\psi_+ &= M\psi_- \end{aligned}$$

provided that $|\mu R| = \nu + \frac{1}{2}$ and $\psi^\pm_T = \psi_\pm$ with effective LF potential

$$V(\zeta) = \frac{R}{\zeta}\rho(\zeta)$$

a J-independent potential – No spin-orbit coupling along a given trajectory !

Baryon Spectrum

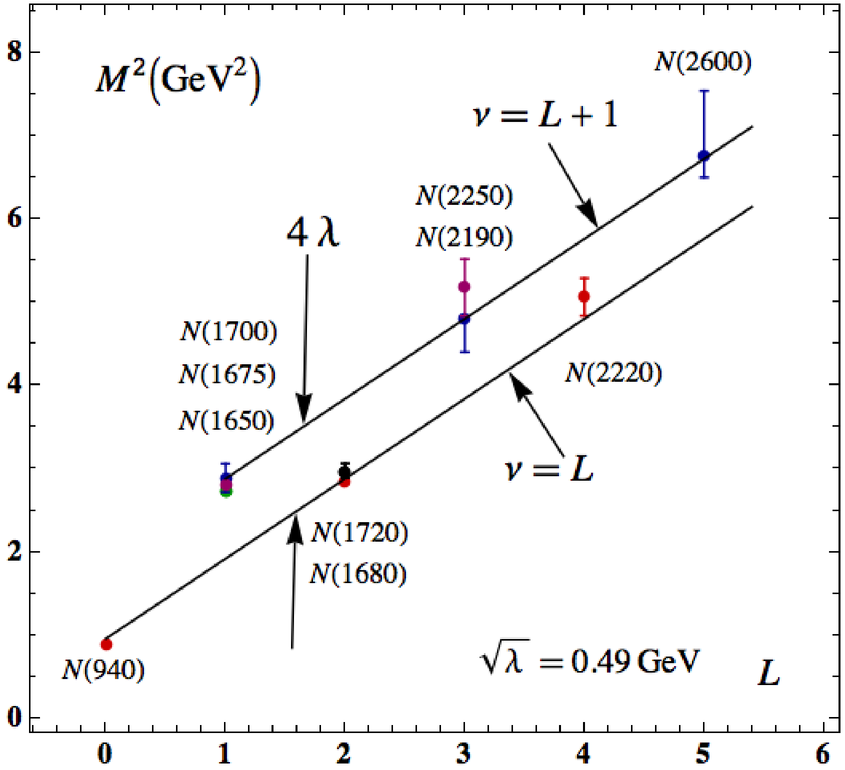
- Choose linear potential $V = \lambda\,\zeta, \quad \lambda > 0$ from underlying conformality of the theory (dAFF)
- Eigenfunctions

$$\psi_+(\zeta) \sim \zeta^{\frac{1}{2}+\nu} e^{-\lambda\zeta^2/2} L_n^{\nu}(\lambda\zeta^2), \qquad \psi_-(\zeta) \sim \zeta^{\frac{3}{2}+\nu} e^{-\lambda\zeta^2/2} L_n^{\nu+1}(\lambda\zeta^2)$$

- Eigenvalues

$$M^2 = 4\lambda(n + \nu + 1)$$

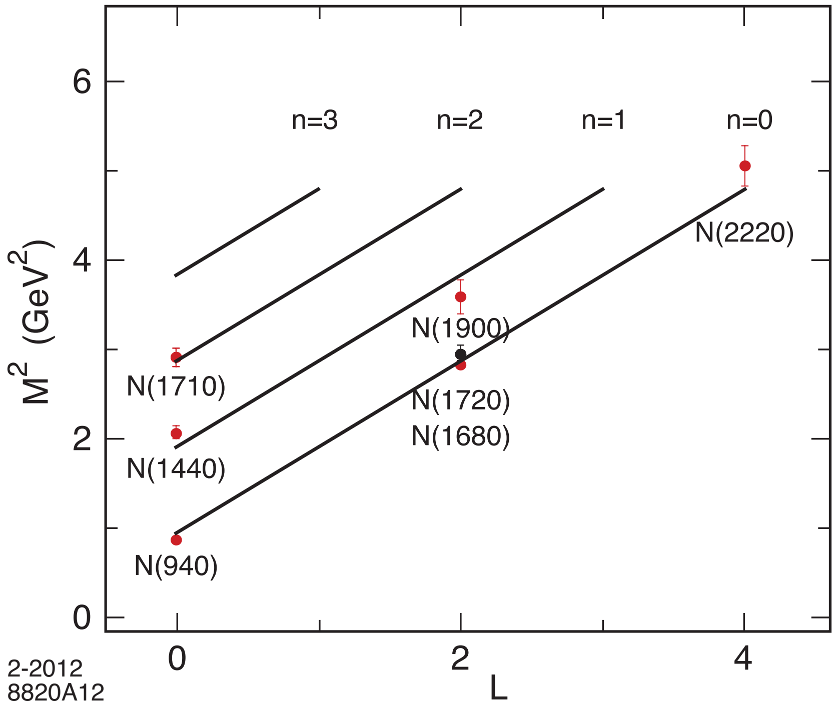
- Gap scale 4λ determines trajectory slope <u>and</u> spectrum gap between plus-parity spin-$\frac{1}{2}$ and minus-parity spin-$\frac{3}{2}$ nucleon families !
- For nucleons $\quad \nu^+_{1/2} = L, \quad \nu^-_{3/2} = L + 1,$ where L is the relative LF angular momentum between the active quark and spectator cluster
- For $\lambda < 0$ no solution possible

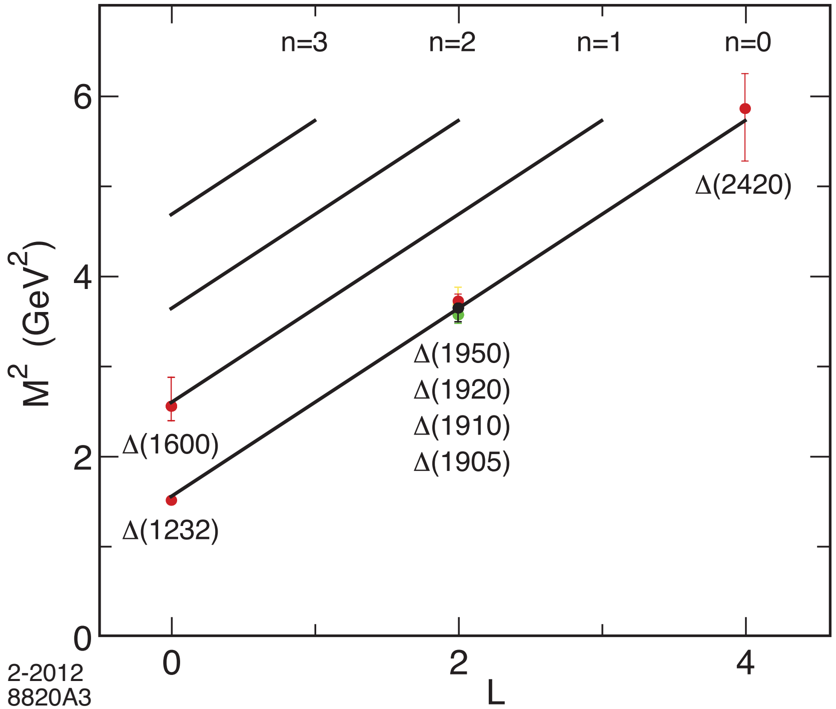


L	S	n	Baryon State
0	$\frac{1}{2}$	0	$N\frac{1}{2}^+(940)$
0	$\frac{1}{2}$	1	$N\frac{1}{2}^+(1440)$
0	$\frac{1}{2}$	2	$N\frac{1}{2}^+(1710)$
0	$\frac{3}{2}$	0	$\Delta\frac{3}{2}^+(1232)$
0	$\frac{3}{2}$	1	$\Delta\frac{3}{2}^+(1600)$
1	$\frac{1}{2}$	0	$N\frac{1}{2}^-(1535)$ $N\frac{3}{2}^-(1520)$
1	$\frac{3}{2}$	0	$N\frac{1}{2}^-(1650)$ $N\frac{3}{2}^-(1700)$ $N\frac{5}{2}^-(1675)$
1	$\frac{1}{2}$	0	$\Delta\frac{1}{2}^-(1620)$ $\Delta\frac{3}{2}^-(1700)$
2	$\frac{1}{2}$	0	$N\frac{3}{2}^+(1720)$ $N\frac{5}{2}^+(1680)$
2	$\frac{1}{2}$	1	$N\frac{5}{2}^+(1900)$
2	$\frac{3}{2}$	0	$\Delta\frac{1}{2}^+(1910)$ $\Delta\frac{3}{2}^+(1920)$ $\Delta\frac{5}{2}^+(1905)$ $\Delta\frac{7}{2}^+(1950)$
3	$\frac{1}{2}$	0	$N\frac{5}{2}^-$ $N\frac{7}{2}^-$
3	$\frac{3}{2}$	0	$N\frac{3}{2}^-$ $N\frac{5}{2}^-$ $N\frac{7}{2}^-(2190)$ $N\frac{9}{2}^-(2250)$
3	$\frac{1}{2}$	0	$\Delta\frac{5}{2}^-$ $\Delta\frac{7}{2}^-$
4	$\frac{1}{2}$	0	$N\frac{7}{2}^+$ $N\frac{9}{2}^+(2220)$
4	$\frac{3}{2}$	0	$\Delta\frac{5}{2}^+$ $\Delta\frac{7}{2}^+$ $\Delta\frac{9}{2}^+$ $\Delta\frac{11}{2}^+(2420)$
5	$\frac{1}{2}$	0	$N\frac{9}{2}^-$ $N\frac{11}{2}^-$
5	$\frac{3}{2}$	0	$N\frac{7}{2}^-$ $N\frac{9}{2}^-$ $N\frac{11}{2}^-(2600)$ $N\frac{13}{2}^-$

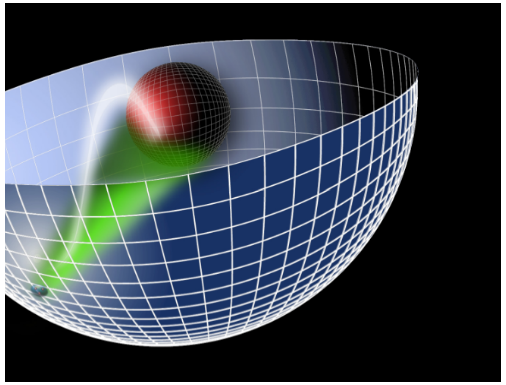
- Phenomenological identification to describe the full baryon spectrum: plus and negative sectors have internal spin $S = \frac{1}{2}$ and $S = \frac{3}{2}$

$$\nu^+_{1/2} = L, \qquad \nu^+_{3/2} = L + 1/2$$
$$\nu^-_{1/2} = L + 1/2, \qquad \nu^-_{3/2} = L + 1$$





Example: Orbital and radial excitations for positive parity N and Δ baryon families ($\sqrt{\lambda} \simeq 0.5$ GeV)



Many thanks !